

# QCD factorization for twist-three parton q(p)distributions

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based on: Vladimir M. Braun, YJ, and Alexey Vladimirov, 2103.12105, 2108.03065

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## Polarized PDFs: chiral-even sector

- **on the light-cone**  $z^2 = 0$
- **twist-2 PDF**

$$\langle p, s | \bar{q}(z) \not{z} \gamma^5 [z, 0] q(0) | p, s \rangle = 2M(s \cdot z) \int_{-1}^1 dx e^{ix\zeta} \Delta q(x) \equiv 2M(s \cdot z) \widehat{\Delta q}(\zeta),$$

- **twist-3 two-particle PDF**

$$\langle p, s | \bar{q}(z) \gamma_T^\mu \gamma^5 [z, 0] q(0) | p, s \rangle = 2Ms_T^\mu \int_{-1}^1 dx e^{ix\zeta} g_T(x) \equiv 2s_T^\mu M \widehat{g}_T(\zeta),$$

- **twist-3 three-particle PDF**

$$\langle p, s | \left[ -ig g_T^{\mu\rho} \bar{q}(az) \not{z} (F_{\rho z}(bz) \gamma_5 - i \tilde{F}_{\rho z}(bz)) q(cz) \right] | p, s \rangle = -4s_T^\mu \zeta^2 M \widehat{S}(a\zeta, b\zeta, c\zeta)$$

**with**  $\zeta = p \cdot z$ .

**Twist decomposition**  $g_T(x) = g_T^{\text{tw}2}(x) + g_T^{\text{tw}3}(x)$

- **Exact relations to all orders based on OPE**

[I. Balitsky and V. Braun, (1991)]

$$\widehat{g}_T^{\text{tw}2}(\zeta) = \int_0^1 d\alpha \widehat{\Delta q}(\alpha\zeta), \quad \text{Wandzura-Wilczek relation}$$

$$g_T^{\text{tw}2}(x) = \theta(x) \int_x^1 \frac{dy}{y} \Delta q(y) - \theta(-x) \int_{-1}^x \frac{dy}{y} \Delta q(y)$$

$$\widehat{g}_T^{\text{tw}3}(\zeta) = 2\zeta^2 \int_0^1 d\alpha \int_\alpha^1 d\beta \bar{\beta} \widehat{S}(\zeta, \beta\zeta, \alpha\zeta),$$

$$g_T^{\text{tw}3}(x) = 2 \int [dx] \int_0^1 d\alpha \left( \frac{\delta(x + \alpha x_1)}{x_1 x_3} + \frac{\delta(x + x_1 + \alpha x_2)}{x_2 x_3} + \frac{\delta(x + x_1)}{x_1 x_2} \right) S(x_1, x_2, x_3)$$

$$\int [dx] = \int_{-1}^1 \prod_{i=1}^3 dx_i \delta(x_1 + x_2 + x_3)$$

## Ioffe-time distributions

[V. Braun, P. Gornicki and L. Mankiewicz, (1994)]

- off-light-cone separation  $z^2 \neq 0$

$$\langle p, s | \bar{q}(z) \not z \gamma^5 [z, 0] q(0) | p, s \rangle = 2M(s \cdot z) \widehat{\mathcal{G}}_1(\zeta, z^2),$$

$$\langle p, s | \bar{q}(z) \gamma_T^\mu \gamma^5 [z, 0] q(0) | p, s \rangle = 2Ms_T^\mu \widehat{\mathcal{G}}_T(\zeta, z^2)$$

**tree-level**       $\widehat{\mathcal{G}}_1(\zeta) = \widehat{\Delta q}(\zeta) + \mathcal{O}(z^2)$        $\widehat{\mathcal{G}}_T(\zeta) = \widehat{g}_T(\zeta) + \mathcal{O}(z^2)$

- OPE  $\implies$  Factorization Theorem

$$\mathcal{G}_1(\zeta, z^2) = \int_0^1 d\alpha C_1(\alpha, \mu_F^2 z^2) \widehat{\Delta q}(\alpha\zeta, \mu_F) + \mathcal{O}(z^2)$$

$$\mathcal{G}_T(\zeta, z^2) = \int_0^1 d\alpha C_2(\alpha, \mu_F^2 z^2) \widehat{\Delta q}(\alpha\zeta, \mu_F)$$

$$+ 2\zeta^2 \int_0^1 d\alpha \int_0^1 d\beta C_3(\alpha, \beta, \mu_F^2 z^2) \widehat{S}(\zeta, \beta\zeta, \alpha\zeta, \mu_F) + \mathcal{O}(z^2)$$

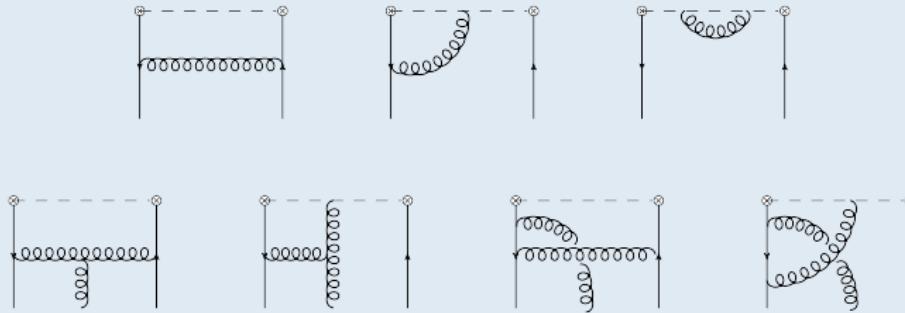
## Twist decomposition for ITDs

### Alternative representation

$$\begin{aligned} \mathcal{G}_T(\zeta, z^2) = & \int_0^1 d\alpha \mathbf{C}_T(\alpha, \mu_F^2 z^2) \hat{g}_T^{\text{tw2}}(\alpha\zeta, \mu_F) + \int_0^1 d\alpha \mathbf{C}_{2\text{pt}}(\alpha, \mu_F^2 z^2) \hat{g}_T^{\text{tw3}}(\alpha\zeta, \mu_F) \\ & + 2\zeta^2 \int_0^1 d\alpha \int_0^1 d\beta \mathbf{C}_{3\text{pt}}(\alpha, \beta, \mu_F^2 z^2) \hat{S}(\zeta, \beta\zeta, \alpha\zeta, \mu_F) \end{aligned}$$

- $\hat{S}$  is indispensable!
- separation between  $\mathbf{C}_{2\text{pt}}$  and  $\mathbf{C}_{3\text{pt}}$  is ambiguous  $g_T^{\text{tw3}} \leftrightarrow \hat{S}$

## One-loop matching



- **Background field technique** [L. Abbott (1981); I. Balitsky, V. Braun (1989)] **in axial and Feynman gauge**
- **Evolution known, used for renormalization** [V. Braun, A. Manashov, J. Rohrwild, (2009); YJ, A. Belitsky, (2014)]

## One-loop results in position space: twist-2 part

$$\mathcal{G}_1(\zeta, z^2, \mu) = \widehat{\Delta q}(\zeta, \mu) + a_s \int_0^1 d\alpha \mathbf{C}_1^{(1)}(\alpha, L_z, \mu) \widehat{\Delta q}(\alpha\zeta, \mu),$$

$$\mathbf{C}_1^{(1)}(\alpha, L_z) = 2C_F \left[ \left( -L_z \frac{1+\alpha^2}{1-\alpha} + \frac{3-8\alpha+3\alpha^2-4\ln\bar{\alpha}}{1-\alpha} \right)_+ + \delta(\bar{\alpha}) \left( \frac{3}{2}L_z + \frac{7}{2} \right) \right]$$

$$\mathcal{G}_T^{\text{tw}2}(\zeta, z^2, \mu) = \widehat{g}_T^{\text{tw}2}(\zeta, \mu) + a_s \int_0^1 d\alpha \mathbf{C}_T^{(1)}(\alpha, L_z, \mu) \widehat{g}_T^{\text{tw}2}(\alpha\zeta, \mu),$$

$$\mathbf{C}_T^{(1)}(\alpha, L_z; \mu) = 2C_F \left[ \left( -L_z \frac{1+\alpha^2}{1-\alpha} + \frac{1-4\alpha+\alpha^2-4\ln\bar{\alpha}}{1-\alpha} \right)_+ + \delta(\bar{\alpha}) \left( \frac{3}{2}L_z + \frac{5}{2} \right) \right],$$

$$L_z = \ln \left( \frac{-\mu^2 z^2}{4e^{-2\gamma_E}} \right)$$

- $\mathbf{C}_1^{(1)} \neq \mathbf{C}_T^{(1)}$   $\implies$  **WW relation**  $\mathcal{G}_T^{\text{tw}2} = \int_0^1 d\alpha \mathcal{G}_1$  **is violated:**  
**additional information ( $\mathbf{C}_T^{(1)}$ ) required for  $\mathcal{G}_T^{\text{tw}2}$  even if  $\mathcal{G}_1$  is known**

## One-loop results in position space: twist-3 part

$$\begin{aligned} \mathcal{G}_T^{\text{tw3}} &= \widehat{g}_T^{\text{tw3}}(\zeta, \mu) + a_s \int_0^1 d\alpha \mathbf{C}_T^{(1)}(\alpha, L_z, \mu) \widehat{g}_T^{\text{tw3}}(\alpha\zeta, \mu) \\ &\quad + 2a_s N_c \zeta^2 \int_0^1 d\alpha \int_\alpha^1 d\beta \left[ \bar{\beta} \left( \bar{\alpha} L_z + 2 + \alpha \right) + 2 \ln \beta \right] \widehat{S}(\zeta, \beta\zeta, \alpha\zeta) + \mathcal{O}(1/N_c) \end{aligned}$$

or equivalently

$$\begin{aligned} \mathcal{G}_T^{\text{tw3}} &= \widehat{g}_T^{\text{tw3}}(\zeta, \mu) + a_s \int_0^1 d\alpha \left[ \mathbf{C}_T^{(1)}(\alpha, L_z, \mu) + N_c (L_z (\delta(\bar{\alpha}) - \alpha) + \alpha + 2\delta(\bar{\alpha})) \right] \widehat{g}_T^{\text{tw3}}(\alpha\zeta, \mu) \\ &\quad + 4a_s N_c \zeta^2 \int_0^1 d\alpha \int_\alpha^1 d\beta \ln \beta \widehat{S}(\zeta, \beta\zeta, \alpha\zeta) + \mathcal{O}(1/N_c) \end{aligned}$$

- $\mathcal{O}(1/N_c)$  given in [2103.12105](#)

## qPDFs at NLO

- Fourier transform position-space result (ITD)  $\mapsto$  quasi-PDF

[X. Ji, (2013)]

$$\begin{aligned} g_1(x, p_v) &= p_v \int \frac{dz}{2\pi} e^{-ixzp_v} \mathcal{G}_1(zp_v, z^2) \\ g_T(x, p_v) &= p_v \int \frac{dz}{2\pi} e^{-ixzp_v} \mathcal{G}_T(zp_v, z^2) \end{aligned}$$

$$p_v = v \cdot p$$

- Factorization theorem: support  $|x| \in \mathbb{R}$

[T. Izubuchi, X. Ji, L. Jin, I. W. Stewart and Y. Zhao, (2018)]

$$\begin{aligned} g_1(x, p_v) &= \Delta q(x) + a_s \int_{-1}^1 \frac{dy}{|y|} C_1^{(1)}\left(\frac{x}{y}, L_p\right) \Delta q(y), \\ g_T(x, p_v) &= g_T(x) + a_s \int_{-1}^1 \frac{dy}{|y|} \left( C_T^{(1)}\left(\frac{x}{y}, L_p\right) g_T^{\text{tw2}}(y) + C_{2\text{pt}}^{(1)}\left(\frac{x}{y}, L_p\right) g_T^{\text{tw3}}(y) \right) \\ &\quad + 2a_s C_{3\text{pt}}^{(1)} \otimes S, \end{aligned}$$

$$L_p \equiv \ln \left( \frac{\mu^2}{4y^2 p_v^2} \right)$$

- explicit one-loop matching coefficient functions  $C_i^{(1)}$  given in 2103.12105

first explored in [S. Bhattacharya, K. Cichy, M. Constantinou, A. Metz, A. Scapellato and F. Steffens (2020),(2020)]

## pPDFs at NLO

- Fourier transform position-space result (ITD)  $\mapsto$  pseudo-PDF

[ A. Radyushkin (2017) ]

$$\begin{aligned}\mathfrak{g}_1(x, z^2) &= \int \frac{d\zeta}{2\pi} e^{-ix\zeta} \mathcal{G}_1(\zeta, z^2) \\ \mathfrak{g}_T(x, z^2) &= \int \frac{d\zeta}{2\pi} e^{-ix\zeta} \mathcal{G}_T(\zeta, z^2)\end{aligned}$$

$$p_v = v \cdot p$$

- Factorization theorem: support  $|x| < 1$

[ A. Radyushkin (2017) ]

$$\begin{aligned}\mathfrak{g}_1(x, z^2) &= \Delta q(x) + a_s \int_{|x|}^1 \frac{d\alpha}{\alpha} \mathfrak{C}_1^{(1)}(\alpha, L_z) \Delta q\left(\frac{x}{\alpha}\right), \\ \mathfrak{g}_T(x, z^2) &= g_T(x) + a_s \int_{|x|}^1 \frac{d\alpha}{\alpha} \left( \mathfrak{C}_T(\alpha, L_z) g_T^{\text{tw2}}\left(\frac{x}{\alpha}\right) + \mathfrak{C}_{2\text{pt}}^{(1)}(\alpha, L_z) g_T^{\text{tw3}}\left(\frac{x}{\alpha}\right) \right) \\ &\quad + 2a_s \mathfrak{C}_{3\text{pt}}^{(1)} \otimes S,\end{aligned}$$

$$L_p \equiv \ln \left( \frac{\mu^2}{4y^2 p_v^2} \right)$$

- explicit one-loop matching coefficient functions  $\mathfrak{C}_i^{(1)}$  given in 2103.12105



## Breaking of WW relation at NLO

- “How much” WW-relation is violated?

$$\begin{aligned} \mathcal{G}_T^{\text{tw2}}(\zeta, z^2) - \int_0^1 d\alpha \mathcal{G}_1(\alpha\zeta, \alpha^2 z^2) &= \\ = 8a_s C_F \int_0^1 d\alpha \left( \text{Li}_2(\bar{\alpha}) + \ln \bar{\alpha} \ln \alpha - \frac{\ln^2 \alpha}{4} \right) \widehat{\Delta q}(\alpha\zeta) + \mathcal{O}(a_s^2). \end{aligned}$$

- Local expansion: how much twist-2 contamination from WW subtraction?

$$\begin{aligned} \mathcal{G}_T(\zeta, z^2) - \int_0^1 d\alpha \mathcal{G}_1(\alpha\zeta, \alpha^2 z^2) &= 4a_s C_F a_0 + i\zeta \frac{5}{2} a_s C_F a_1 \\ + \frac{\zeta^2}{3} \left\{ \tilde{d}_2 - \frac{20}{9} C_F a_s a_2 + a_s \tilde{d}_2 \left[ L_z \left( \frac{13}{3} N_c - \frac{4}{3N_c} \right) - 8C_F \right] \right\} + \mathcal{O}(\zeta^3, z^2), \end{aligned}$$

$$a_n = \int_{-1}^1 dx x^n \Delta q(x), \quad \tilde{d}_2 = \int_0^1 dx x^2 [3g_T(x) - \Delta q(x)] = \int [dx] S^-(x_1, x_2, x_3).$$

- genuine twist-2**  $\frac{20}{9} C_F a_s a_2 \sim 3.6 \cdot 10^{-3}$  **vs.** **genuine twist-3**  $|\tilde{d}_2| \sim (1-5) \cdot 10^{-3}$ : **noisy** 

## chiral-odd PDFs

- Chiral-odd PDFs:

$$\begin{aligned} g_T^{\mu\nu} \langle p, s | \bar{q}(z) i\sigma_{\nu z} \gamma^5 q(0) | p, s \rangle &= 2s_T^\mu \zeta \int_{-1}^1 dx e^{ix\zeta} \delta q(x) \equiv 2s_T^\mu \zeta \widehat{\delta q}(\zeta), \\ \langle p, s | \bar{q}(z) i\sigma^{pz} \gamma^5 q(0) | p, s \rangle &= -2M\lambda_z \zeta \int_{-1}^1 dx e^{ix\zeta} h_L(x) \equiv -2M\lambda_z \zeta \widehat{h}_L(\zeta), \\ \langle p, s | \bar{q}(z) q(0) | p, s \rangle &= 2M \int_{-1}^1 dx e^{ix\zeta} e(x) \equiv 2M \widehat{e}(\zeta). \end{aligned}$$

$$\lambda_z = M(s \cdot z)/(p \cdot z)$$

- Decomposition

$$\begin{aligned} \widehat{h}_L(\zeta) &= \widehat{h}_L^{\text{tw2}}(\zeta) + \widehat{h}_L^{\text{tw3}}(\zeta) = 2 \int_0^1 d\alpha \alpha \widehat{\delta q}(\alpha\zeta) + \zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta \alpha (2\beta - \alpha) \widehat{H}(\alpha\zeta, \beta\zeta, 0) \\ \widehat{e}(\zeta) &= \Sigma_q + \widehat{e}_{\text{nl}}(\zeta) = \Sigma_q + \zeta^2 \int_0^1 d\alpha \int_0^\alpha d\beta \widehat{E}(\alpha\zeta, \beta\zeta, 0), \quad \Sigma_q : \text{nucleon } \sigma\text{-term} \end{aligned}$$

$$\begin{aligned} \langle p, s | \bar{q}(az) \sigma^{\mu z} \gamma^5 gF_{\mu z}(bz) q(cz) | p, s \rangle &= 4\lambda_z \zeta^2 M \widehat{H}(a\zeta, b\zeta, c\zeta) \\ \langle p, s | \bar{q}(az) \sigma^{\mu z} gF_{\mu z}(bz) q(cz) | p, s \rangle &= 4\zeta^2 M \widehat{E}(a\zeta, b\zeta, c\zeta) \end{aligned}$$

## Chiral-odd ITDs

- **Definitions:**

$$\langle p, s | \bar{q}(z) i\sigma^{\mu z} \gamma^5 q(0) | p, s \rangle = 2s_T^\mu (p \cdot z) \mathcal{H}_1(p \cdot z, z^2) - 2\lambda_z \left( z^\mu - \frac{z^2}{(p \cdot z)} p^\mu \right) M \mathcal{H}_L(p \cdot z, z^2),$$

$$\langle p, s | \bar{q}(z) q(0) | p, s \rangle = 2M \mathcal{E}(p \cdot z, z^2)$$

- **Decomposition (to  $\mathcal{O}(z^2)$ )**

$$\mathcal{H}_L(\zeta, z^2) = \mathcal{H}_L^{\text{tw2}}(\zeta, z^2) + \mathcal{H}_L^{\text{tw3}}(\zeta, z^2),$$

$$\mathcal{E}(\zeta, z^2) = C_\Sigma \Sigma_q + \mathcal{E}_{\text{nl}}(\zeta, z^2)$$

- **One-loop matching**

$$\mathcal{H}_L^{\text{tw2}}(\zeta, z^2) = \hat{h}_L^{\text{tw2}}(\zeta; \mu) + a_s \int_0^1 d\alpha \mathbf{C}_{L, \text{tw2}}^{(1)}(\alpha, L_z) \hat{h}_L^{\text{tw2}}(\alpha\zeta; \mu),$$

$$\mathcal{H}_L^{\text{tw3}}(\zeta, z^2) = \hat{h}_L^{\text{tw3}}(\zeta; \mu) + a_s \int_0^1 d\alpha \mathbf{C}_{L, 2\text{pt}}^{(1)}(\alpha, L_z) \hat{h}_L^{\text{tw3}}(\alpha\zeta; \mu) + 2\zeta^2 a_s \mathbf{C}_{L, 3\text{pt}}^{(1)} \otimes \hat{H},$$

$$C_\Sigma \Sigma_q = \left( 1 + C_\Sigma^{(1)} \right) \Sigma_q,$$

$$\mathcal{E}_{\text{nl}}(\zeta, z^2) = \hat{e}_{\text{nl}}(\zeta; \mu) + a_s \int_0^1 d\alpha \mathbf{C}_{S, 2\text{pt}}^{(1)}(\alpha, L_z) \hat{e}_{\text{nl}}(\alpha\zeta; \mu) + 2\zeta^2 a_s \mathbf{C}_{S, 3\text{pt}}^{(1)} \otimes \hat{E},$$



## One-loop matching: $\mathcal{H}_1$ and $\mathcal{H}_L$

$$\mathcal{H}_1(\zeta, z^2; \mu) = \widehat{\delta}q(\zeta; \mu) + a_s \int_0^1 d\alpha \mathbf{C}_1^{(1)}(\alpha, L_z) \widehat{\delta}q(\alpha\zeta; \mu),$$

$$\mathcal{H}_L^{\text{tw2}}(\zeta, z^2) = \widehat{h}_L^{\text{tw2}}(\zeta; \mu) + a_s \int_0^1 d\alpha \mathbf{C}_{L,\text{tw2}}^{(1)}(\alpha, L_z) \widehat{h}_L^{\text{tw2}}(\alpha\zeta; \mu),$$

$$\mathcal{H}_L^{\text{tw3}}(\zeta, z^2) = \widehat{h}_L^{\text{tw3}}(\zeta; \mu) + a_s \int_0^1 d\alpha \mathbf{C}_{L,2\text{pt}}^{(1)}(\alpha, L_z) \widehat{h}_L^{\text{tw3}}(\alpha\zeta; \mu) + 2\zeta^2 a_s \mathbf{C}_{L,3\text{pt}}^{(1)} \otimes \widehat{H},$$

$$\mathbf{C}_1^{(1)}(\alpha, L_z) = 4C_F \left[ \left( -L_z \frac{\alpha}{1-\alpha} - \frac{\alpha + 2 \ln \bar{\alpha}}{1-\alpha} \right)_+ + \delta(\bar{\alpha})(L_z + 1) \right], \quad \text{twist-2 PDF}$$

$$\mathbf{C}_{L,\text{tw2}}^{(1)}(\alpha, L_z) = \mathbf{C}_1^{(1)}(\alpha, L_z),$$

$$\mathbf{C}_{L,2\text{pt}}^{(1)}(\alpha, L_z) = \mathbf{C}_1^{(1)}(\alpha, L_z) + N_c \left[ L_z (1 + \delta(\bar{\alpha})) + 1 + 2\delta(\bar{\alpha}) \right] \neq \mathbf{C}_1^{(1)}(\alpha, L_z),$$

$$\begin{aligned} \mathbf{C}_{L,3\text{pt}}^{(1)} \otimes \widehat{H} &= -L_z P^{\text{tw3}} \otimes \widehat{H} + \int_0^1 d\alpha \left\{ 2N_c \int_0^\alpha d\beta \ln \bar{\beta} \widehat{H}(\alpha\zeta, \beta\zeta, 0) \right. \\ &\quad \left. + \frac{1}{N_c} \int_\alpha^1 d\beta \left[ \frac{2\bar{\beta}(\alpha + \ln \bar{\alpha})}{1-\alpha} - \beta\bar{\beta} \right] \widehat{H}(\alpha\zeta, \beta\zeta, 0) \right\}, \end{aligned}$$

- Jaffe-Ji relation preserved: to all orders!

## One-loop matching: $\mathcal{E}$

$$C_\Sigma \Sigma_q = \left(1 + C_\Sigma^{(1)}\right) \Sigma_q ,$$

$$\mathcal{E}_{\text{nl}}(\zeta, z^2) = \widehat{e}_{\text{nl}}(\zeta; \mu) + a_s \int_0^1 d\alpha \mathbf{C}_{S,2\text{pt}}^{(1)}(\alpha, L_z) \widehat{e}_{\text{nl}}(\alpha\zeta; \mu) + 2\zeta^2 a_s \mathbf{C}_{S,3\text{pt}}^{(1)} \otimes \widehat{E} ,$$

$$C_\Sigma^{(1)} = 6C_F a_s , \quad \text{no scale dependence!}$$

$$\mathbf{C}_{S,2\text{pt}}^{(1)} = 4C_F \left[ \left( -\frac{L_z}{\bar{\alpha}} - \frac{\alpha + 2 \ln \bar{\alpha}}{1 - \alpha} \right)_+ + \delta(\bar{\alpha}) \right] + N_c \left[ -L_z(1 - \delta(\bar{\alpha})) - 1 + 2\delta(\bar{\alpha}) \right] ,$$

$$\begin{aligned} \mathbf{C}_{S,3\text{pt}}^{(1)} \otimes \widehat{E} &= -L_z P^{\text{tw3}} \otimes \widehat{E} + \int_0^1 d\alpha \left\{ 2N_c \int_0^\alpha d\beta \ln \bar{\beta} \widehat{E}(\alpha\zeta, \beta\zeta, 0) \right. \\ &\quad \left. + \frac{1}{N_c} \int_\alpha^1 d\beta \left[ \frac{2\bar{\beta}(\alpha + \ln \bar{\alpha})}{1 - \alpha} + \bar{\beta} \right] \widehat{E}(\alpha\zeta, \beta\zeta, 0) \right\} \end{aligned}$$

$$P^{\text{tw3}} \otimes \widehat{E} = \frac{1}{N_c} \int_0^1 d\alpha \int_\alpha^1 d\beta \frac{-\alpha\bar{\beta}}{1 - \alpha} \widehat{E}(\alpha\zeta, \beta\zeta, 0)$$

## Quasi-PDFs: one-loop matching

- **quasi-PDFs with support**  $x \in \mathbb{R}$        $c_1^{(1)}$  agrees with [Y.-S. Liu, J.-W. Chen, L. Jin, R. Li, H.-W. Lin, Y.-B. Yang et al. (2018)]

$$\begin{aligned}
 h_1(x, p_v) &= \delta q(x) + a_s \int_{-1}^1 \frac{dy}{|y|} C_1^{(1)}\left(\frac{x}{y}, L_p\right) \delta q(y), \\
 h_L(x, p_v) &= h_L(x) + a_s \int_{-1}^1 \frac{dy}{|y|} \left( C_1^{(1)}\left(\frac{x}{y}, L_p\right) h_L^{\text{tw2}}(y) + C_{L,2\text{pt}}^{(1)}\left(\frac{x}{y}, L_p\right) h_L^{\text{tw3}}(y) \right) \\
 &\quad + 2a_s C_{L,3\text{pt}}^{(1)} \otimes H, \\
 e(x, p_v) &= e(x) + 6C_F a_s \delta(x) \Sigma_{\bar{q}q} + a_s \int_{-1}^1 \frac{dy}{|y|} C_{S,2\text{pt}}^{(1)}\left(\frac{x}{y}, L_p\right) e_{\text{nl}}(y) \\
 &\quad + 2a_s C_{S,3\text{pt}}^{(1)} \otimes E.
 \end{aligned}$$

$$P^{\text{tw3}} \otimes \hat{H} = \frac{1}{N_c} \int_0^1 d\alpha \int_\alpha^1 d\beta \frac{-\alpha\bar{\beta}}{1-\alpha} \hat{H}(\alpha\zeta, \beta\zeta, 0)$$

- **explicit expressions given in Eqs.(4.44) - (4.51) in 2108.03065**

first explored in [S. Bhattacharya, K. Cichy, M. Constantinou, A. Metz, A. Scapellato and F. Steffens (2020),(2021)]

## Pseudo-PDFs: one-loop matching

- **pseudo-PDFs with support  $|x| < 1$**

$$\begin{aligned}
 \mathfrak{h}_1(x, z^2) &= \delta q(x) + a_s \int_{|x|}^1 \frac{d\alpha}{\alpha} \mathfrak{C}_1^{(1)}(\alpha, L_z) \delta q\left(\frac{x}{\alpha}\right), \\
 \mathfrak{h}_L(x, z^2) &= h_L(x) + a_s \int_{|x|}^1 \frac{d\alpha}{\alpha} \left( \mathfrak{C}_1^{(1)}(\alpha, L_z) h_L^{\text{tw}2}\left(\frac{x}{\alpha}\right) + \mathfrak{C}_{L,2\text{pt}}^{(1)}(\alpha, L_z) h_L^{\text{tw}3}\left(\frac{x}{\alpha}\right) \right) \\
 &\quad + 2a_s \mathfrak{C}_{L,3\text{pt}}^{(1)} \otimes H, \\
 \mathfrak{e}(x, \zeta^2) &= e(x) + 6C_F a_s \delta(x) \Sigma_{\bar{q}q} + a_s \int_{|x|}^1 \frac{d\alpha}{\alpha} \mathfrak{C}_{S,2\text{pt}}^{(1)}(\alpha, L_z) e_{\text{nl}}\left(\frac{x}{\alpha}\right) \\
 &\quad + 2a_s \mathfrak{C}_{S,3\text{pt}}^{(1)} \otimes E,
 \end{aligned}$$

- **explicit expressions given in Eqs.(4.31) - (4.34) in 2108.03065**

## Local expansion of ITDs

- **twist-3 extraction:**

$$\mathcal{H}_L^{\text{tw3}}(\zeta, z^2) = \mathcal{H}_L(\zeta, z^2) - 2 \int_0^1 d\alpha \alpha \mathcal{H}_1(\alpha\zeta, z^2), \quad \text{much simpler!}$$

- **first twist-3 nontrivial moment**

$$\mathcal{H}_L^{\text{tw3}}(\zeta, z^2) = -\frac{i\zeta^3 h_3}{6} \left\{ 1 + a_s \left[ L_z \left( \frac{65N_c}{12} - \frac{25}{12N_c} \right) - \frac{289N_c}{36} + \frac{29}{36N_c} \right] \right\} + \mathcal{O}(\zeta^4, z^2),$$

$$\begin{aligned} \mathcal{E}(\zeta, z^2) &= (1 + 6C_F a_s) \Sigma_q \\ &\quad - \frac{\zeta^2 e_2}{2} \left\{ 1 + a_s \left[ L_z \left( \frac{11}{3} N_c - \frac{8}{3N_c} \right) - \frac{11N_c}{3} + \frac{11}{3N_c} \right] \right\} + \mathcal{O}(\zeta^3, z^2), \end{aligned}$$

$$h_3 = \frac{1}{5} \int [dx] x_2 H(x_1, x_2, x_3), \quad e_2 = - \int [dx] E(x_1, x_2, x_3)$$

## Summary

- QCD factorization applies to ITDs/qPDFs/pPDFs from OPE
- Complete NLO matching coefficient function available for twist-3 PDFs
- Wandzura-Wilczek relation is violated ☺; Jaffe-Ji relation is preserved to all order ☺
- No simple factorization for  $g_T$  and  $h_L$ , three-particle PDFs indispensable, ambiguity exists