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# Lattice QCD Determination of the Bjorken- $x$ dependence of PDFs at Next-to-next-leading-order

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LaMET 2021  
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**DEC 7, 2021**



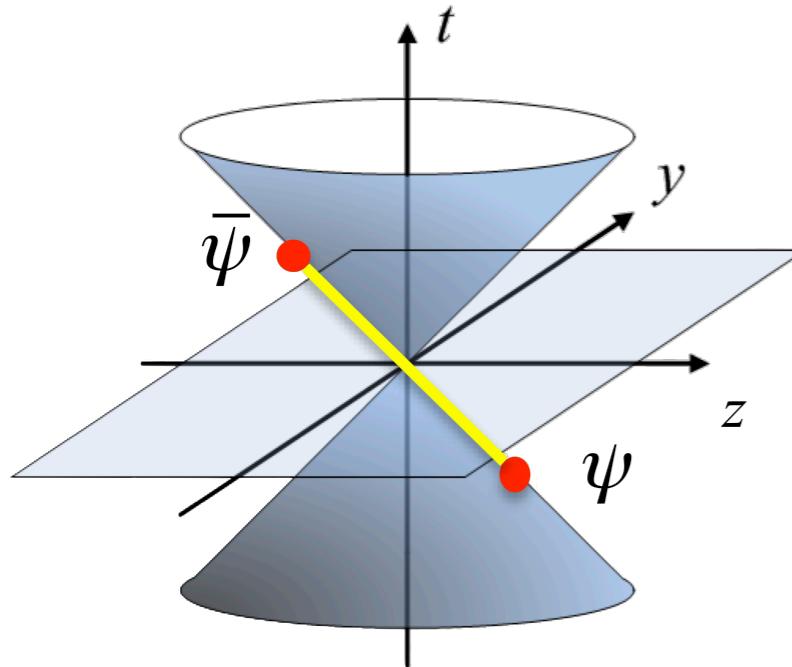
In collaboration with Xiang Gao, Andrew Hanlon, Swagato Mukherjee,  
Peter Petreczky, Philipp Scior and Sergey Syritsyn, arXiv: 2112.02208.

# Outline

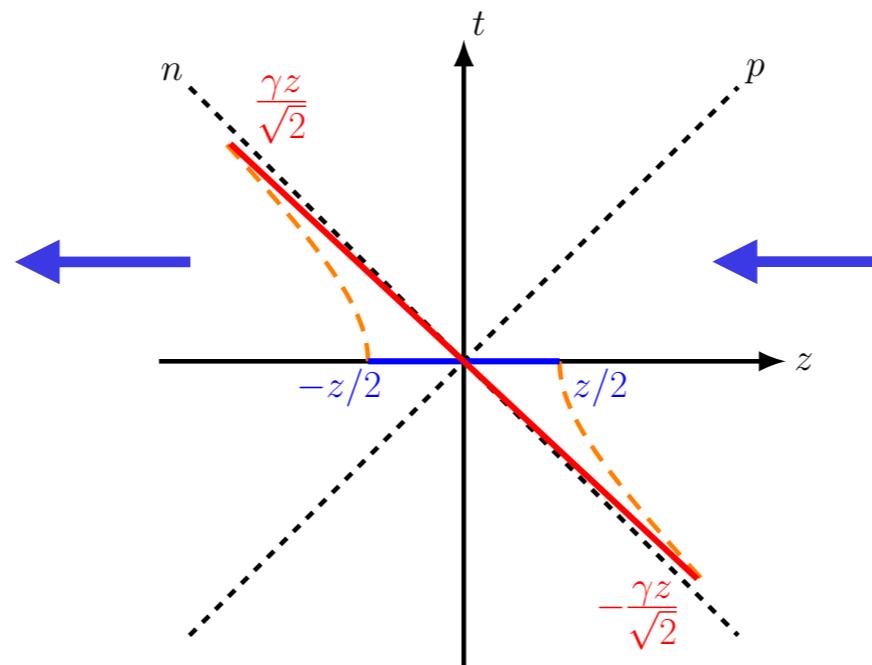
- Methodology
  - Large-momentum effective theory
  - Hybrid scheme renormalization
- Lattice calculation
  - Wilson line mass renormalization and matching
  - Fourier transform and perturbative matching
  - Final prediction

# Large-Momentum Effective Theory (LaMET)

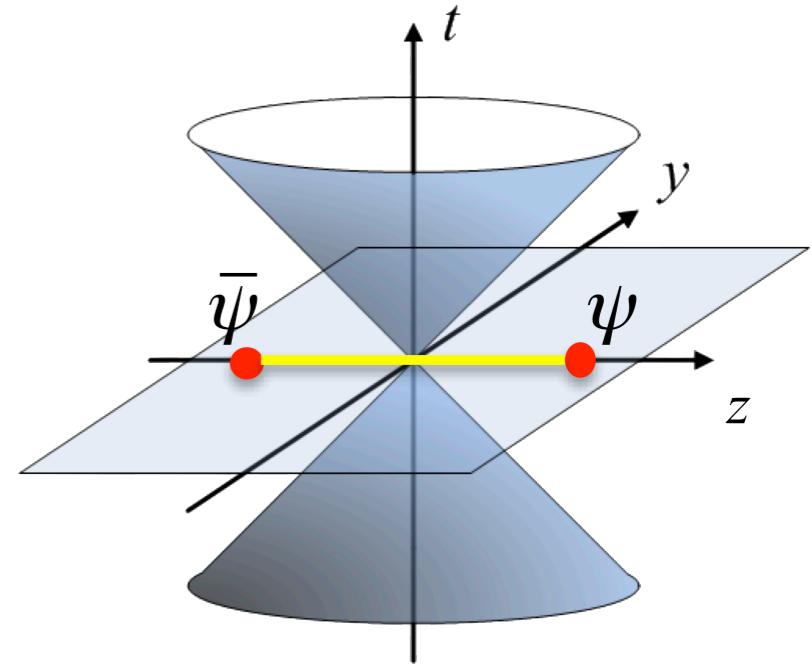
$$z + ct = 0, \quad z - ct \neq 0$$



Related by Lorentz boost



$$t = 0, \quad z \neq 0$$



PDF  $f(x)$ :  
Cannot be calculated  
on the lattice

$$\begin{aligned} f(x) &= \int \frac{dz^-}{2\pi} e^{-ib^-(xP^+)} \langle P | \bar{\psi}(z^-) \\ &\quad \times \frac{\gamma^+}{2} W[z^-, 0] \psi(0) | P \rangle \end{aligned}$$

- X. Ji, PRL 110 (2013); SCPMA57 (2014);
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, RMP 93 (2021).

Quasi-PDF  $\tilde{f}(x, P^z)$ :  
Directly calculable on  
the lattice

$$\begin{aligned} \tilde{f}(x, P^z) &= \int \frac{dz}{2\pi} e^{iz(xP^z)} \langle P | \bar{\psi}(z) \\ &\quad \times \frac{\gamma^z}{2} W[z, 0] \psi(0) | P \rangle \end{aligned}$$

# Large-Momentum Effective Theory (LaMET)

$$\tilde{f}(y, P^z, \tilde{\mu}) = \int_{-1}^1 \frac{dx}{|x|} C\left(\frac{y}{x}, \frac{\mu}{xP^z}, \frac{\tilde{\mu}}{\mu}\right) f(x, \mu) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{(yP^z)^2}, \frac{\Lambda_{\text{QCD}}^2}{((1-y)P^z)^2}\right)$$
$$\Rightarrow f(x, \mu) = \int_{-\infty}^{\infty} \frac{dy}{|y|} C^{-1}\left(\frac{x}{y}, \frac{\mu}{yP^z}, \frac{\tilde{\mu}}{\mu}\right) \tilde{f}(y, P^z, \tilde{\mu}) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{(xP^z)^2}, \frac{\Lambda_{\text{QCD}}^2}{((1-x)P^z)^2}\right)$$

## Systematic control:

- X. Xiong, X. Ji, J.-H. Zhang and YZ, PRD 90 (2014);
- Y.-Q. Ma and J. Qiu, PRD98 (2018);
- T. Izubuchi, X. Ji, L. Jin, I. Stewart, and YZ, PRD98 (2018).
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, RMP 93 (2021).

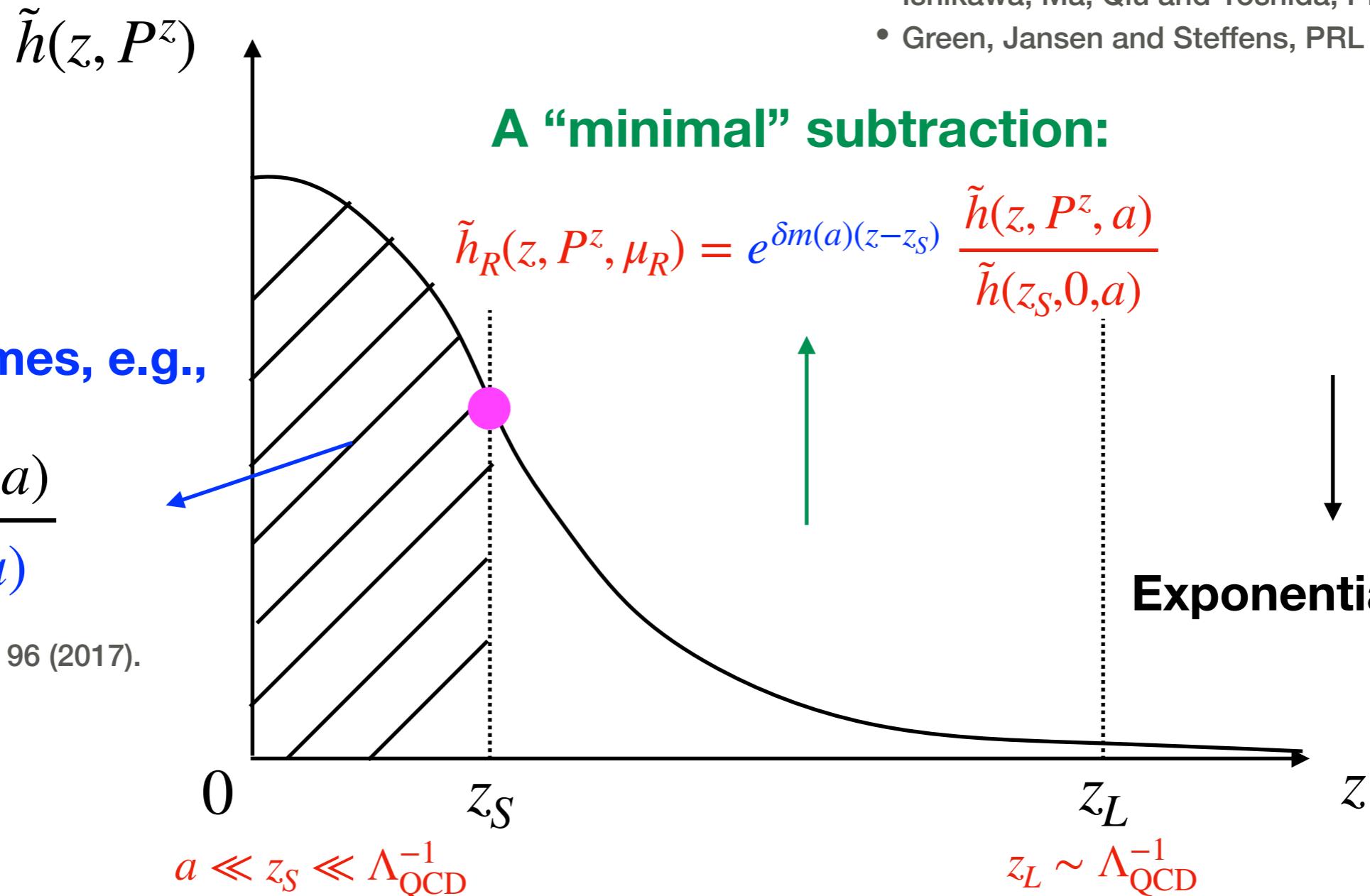
- Lattice: excited state contamination, spacing  $a \rightarrow 0$ , physical  $m_\pi$ , lattice size  $L \rightarrow \infty$ , etc.;
- Perturbative matching (currently at NNLO) and resummation (for end-point regions);
  - L.-B. Chen, R.-L. Zhu and W. Wang, PRL126 (2021);
  - Z.-Y. Li, Y.-Q. Ma and J.-W. Qiu, PRL126 (2021).
  - X. Gao, YZ, et al., PRD103 (2021).
- Power corrections, controllable within  $[x_{\min}, x_{\max}]$  at a given finite  $P^z$ .

# Hybrid renormalization scheme

$$O_B^\Gamma(z, a) = \bar{\psi}_0(z)\Gamma W_0[z, 0]\psi_0(0) = e^{-\delta m(a)|z|} Z_O(a) O_R^\Gamma(z)$$

X. Ji, YZ, et al., NPB 964 (2021).

- Ji, Zhang and YZ, PRL 120 (2018);
- Ishikawa, Ma, Qiu and Yoshida, PRD 96 (2017);
- Green, Jansen and Steffens, PRL 121 (2018).



# Lattice data for the pion valence PDF

- Wilson-clover valence fermion on 2+1 flavor HISQ gauge configurations (HotQCD).

$n_z$	$P_z$ (GeV)		$\zeta$
	$a = 0.06$ fm	$a = 0.04$ fm	
0	0	0	0
1	0.43	0.48	0
2	0.86	0.97	1
3	1.29	1.45	2/3
4	1.72	1.93	3/4
5	2.15	2.42	3/5

$48^3 \times 64$        $64^3 \times 64$

$$m_\pi = 300 \text{ MeV}$$

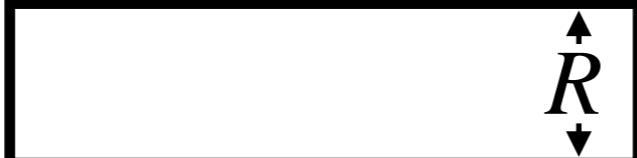
- **BNL20**, X. Gao, YZ, et al., PRD102 (2020);
- X. Gao, YZ, et al., PRD103 (2021).

# Wilson-line mass renormalization

- Polyakov loop

$$\langle \Omega | \boxed{\quad} | \Omega \rangle \propto \exp[-V(R)T]$$

$\leftarrow T \rightarrow \infty \rightarrow$



- Renormalization (subtraction) condition:

$$V^{\text{lat}}(r, a) \Big|_{r=r_0} + 2\delta m(a) = 0.95/r_0$$

$$a\delta m(a = 0.04 \text{ fm}) = 0.1508(12)$$

$$\delta m(a) = \frac{1}{a} \sum_n c_n \alpha_s^n (1/a) + m_0$$

$$a\delta m(a = 0.06 \text{ fm}) = 0.1586(8)$$

A. Bazavov et al., TUMQCD, PRD98 (2018).

$$m_0 \sim \Lambda_{\text{QCD}}$$

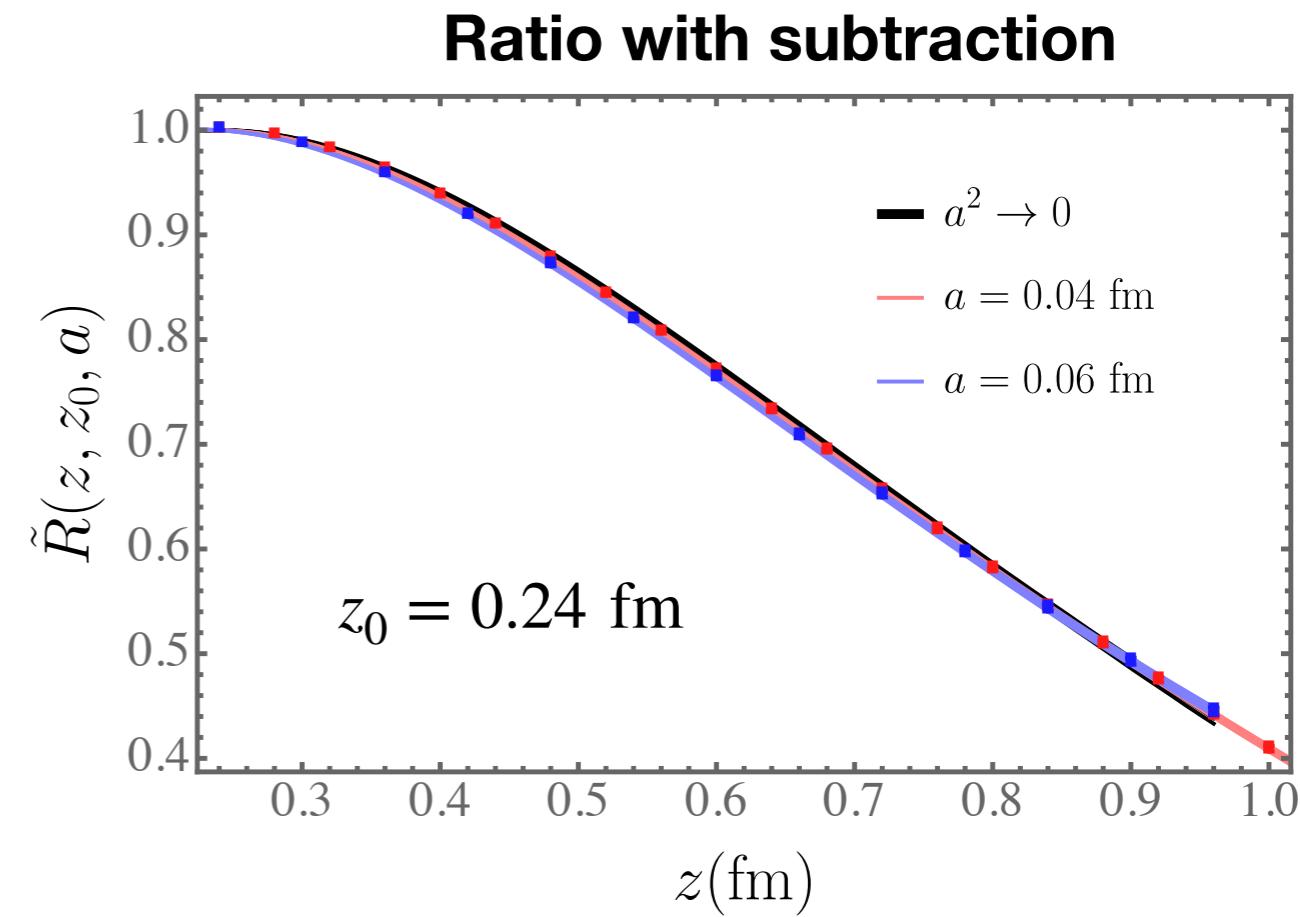
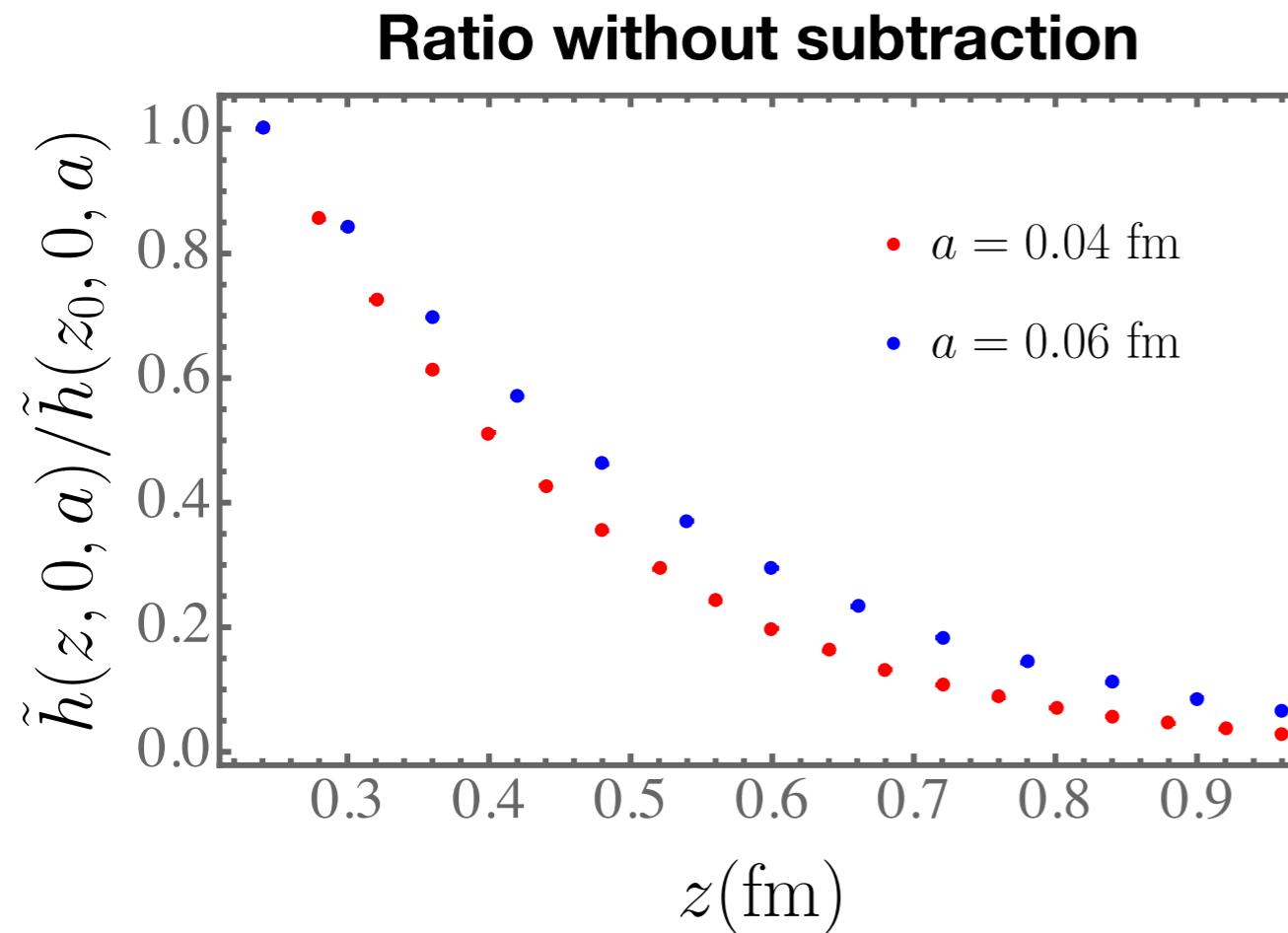
C. Bauer, G. Bali and A. Pineda, PRL108 (2012).

# Check of continuum limit

- Renormalization-group invariant ratio:

$$O_B^\Gamma(z, a) = e^{-\delta m|z|} Z_O(a) O_R^\Gamma(z)$$

$$\lim_{a \rightarrow 0} e^{-\delta m(z-z_0)} \frac{\tilde{h}(z, a, P^z = 0)}{\tilde{h}(z_0, a, P^z = 0)} = \frac{\tilde{h}(z, P^z = 0, \mu)}{\tilde{h}(z_0, P^z = 0, \mu)} \quad z, z_0 \gg a$$



# Matching to the continuum

$\bar{m}_0$ :

- Cancel lattice subtraction scheme dependence of  $\delta m(a)$ ;
- Still includes the UV renormalon uncertainty of  $C_0$  in  $\overline{\text{MS}}$  scheme;
- The same at all  $z$ .

M. Beneke and V. Braun, NPB 426 (1994).

$$\lim_{a \rightarrow 0} e^{-\delta m(z-z_0)} \frac{\tilde{h}(z, a, P^z = 0)}{\tilde{h}(z_0, a, P^z = 0)} = e^{-\bar{m}_0(z-z_0)} \frac{C_0(\alpha_s(\mu), z^2 \mu^2) + \mathcal{O}(z^2 \Lambda_{\text{QCD}}^2)}{C_0(\alpha_s(\mu), z_0^2 \mu^2) + \mathcal{O}(z_0^2 \Lambda_{\text{QCD}}^2)}$$

$a \ll z, z_0 \ll \Lambda_{\text{QCD}}^{-1}$

**Wilson coefficient:**

Known to NNLO with 3-loop anomalous dimension

**IR renormalon:**

Leading IR renormalon  $\propto z^2 \Lambda_{\text{QCD}}^2$

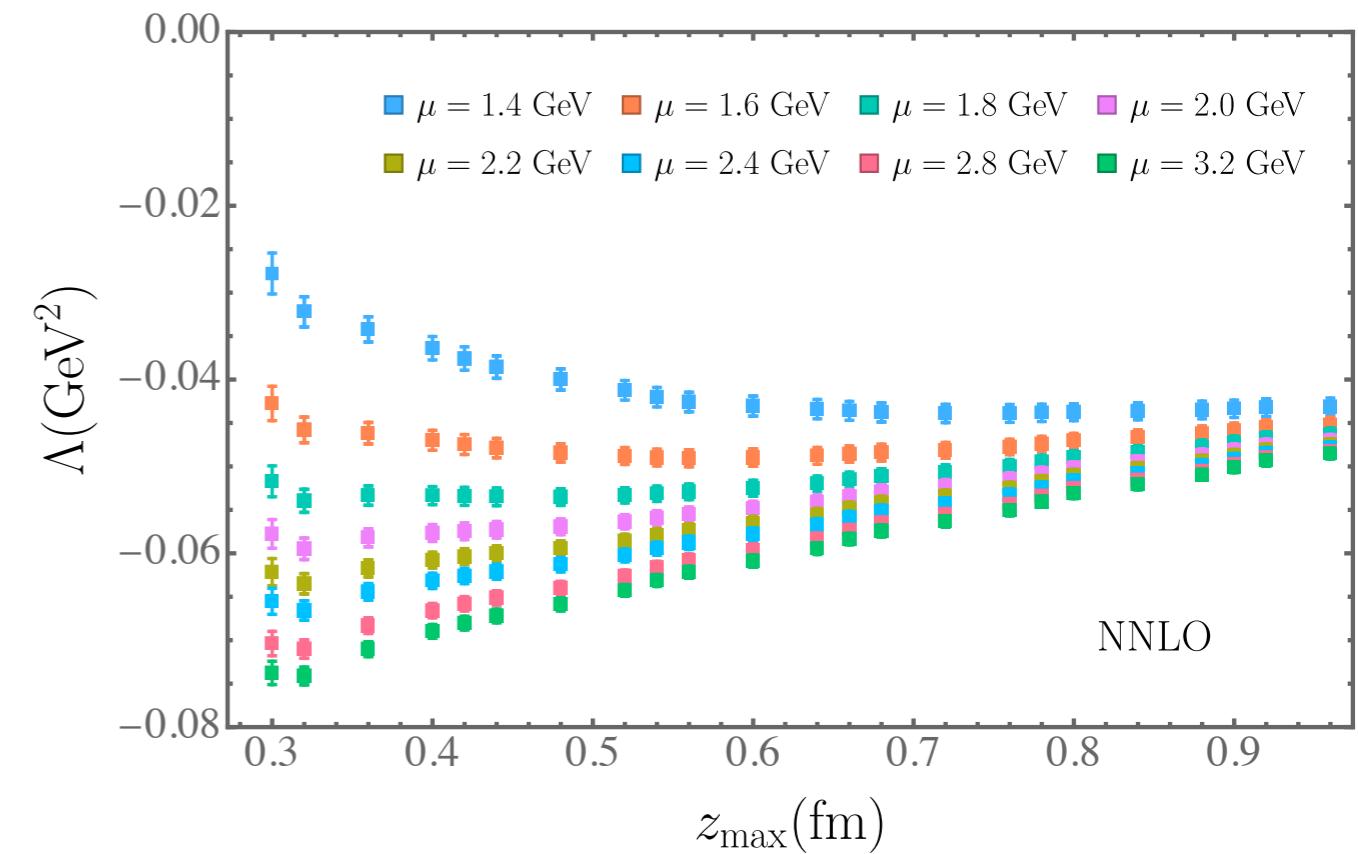
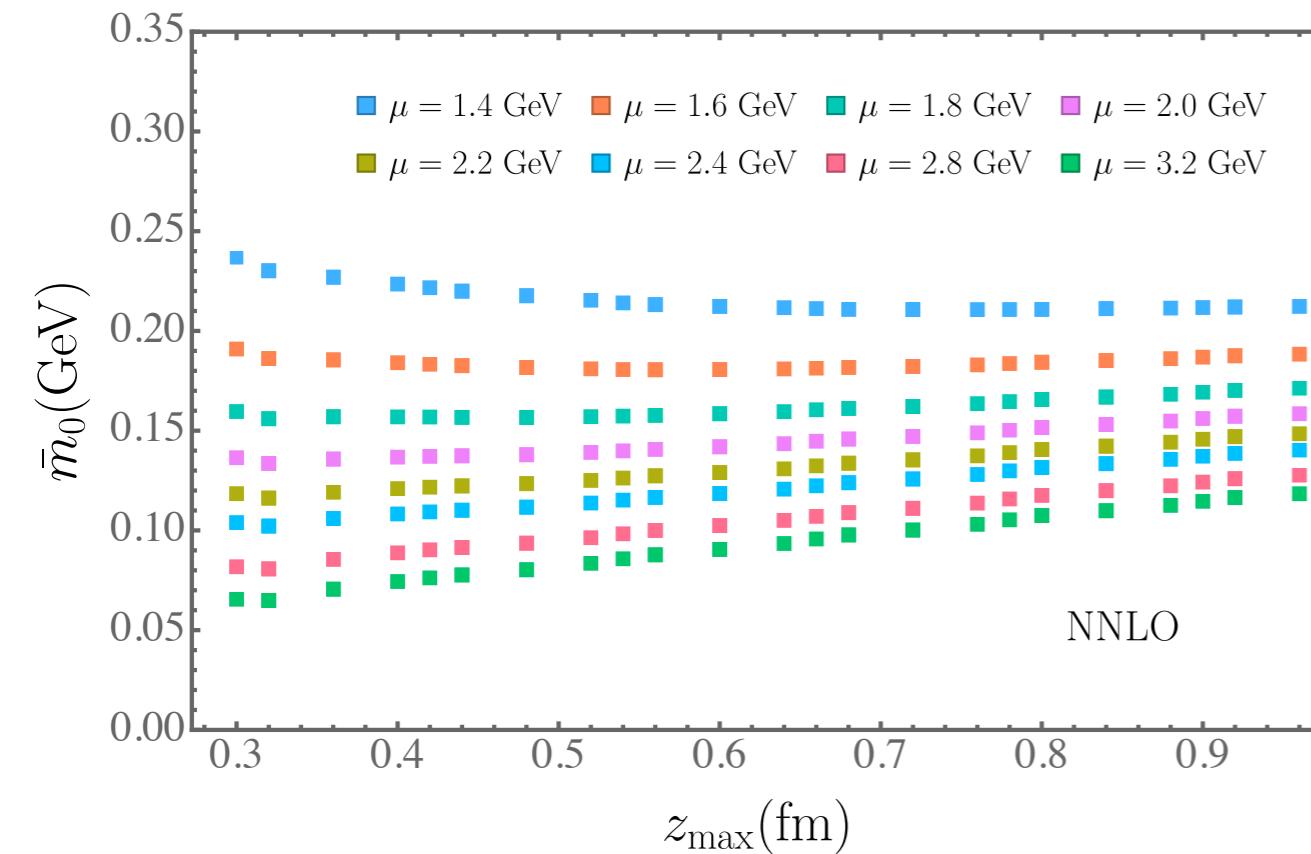
V. Braun, A. Vladimirov and J.-H. Zhang, PRD99 (2019).

- Z.-Y. Li, Y.-Q. Ma and J.-W. Qiu, PRL126 (2021);
- V. Braun and K. G. Chetyrkin, JHEP 07 (2020).

# Matching to the continuum

$$\lim_{a \rightarrow 0} e^{\delta m(a)(z-z_0)} \frac{\tilde{h}(z, P^z = 0, a)}{\tilde{h}(z_0, P^z = 0, a)} = e^{-\bar{m}_0(z-z_0)} \frac{C_0(\alpha_s(\mu), z^2 \mu^2) + \Lambda z^2}{C_0(\alpha_s(\mu), z_0^2 \mu^2) + \Lambda z_0^2}$$

**Fitting range:**  $z_0 = 0.24$  fm,  $z_0 < z \leq z_{\max}$



$\bar{m}_0(\mu)$  and  $\Lambda(\mu)$  fitted from  $z_{\max} = 0.40$  fm at  $1.4 \text{ GeV} \leq \mu \leq 3.2 \text{ GeV}$

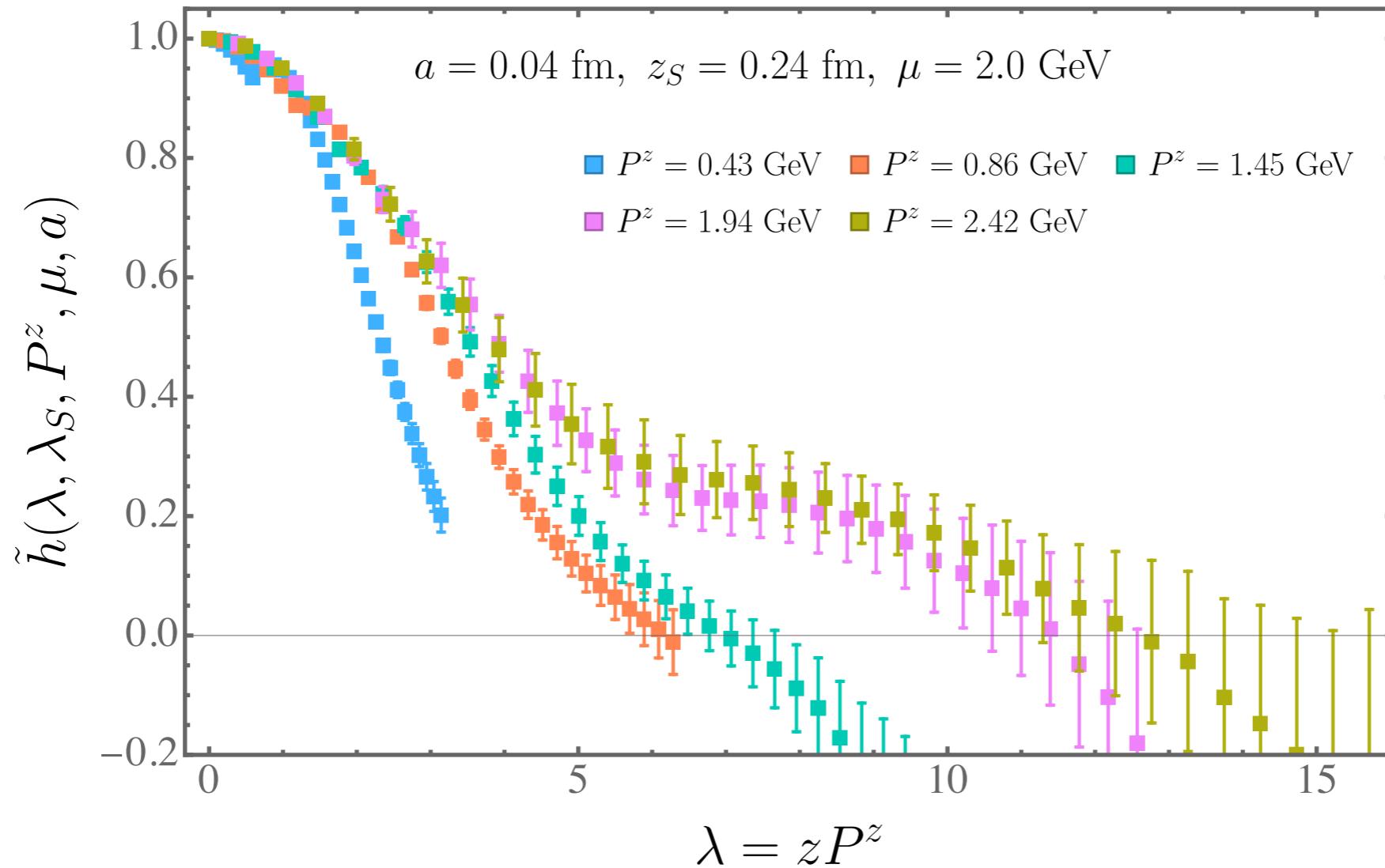
# Renormalized matrix elements

Full hybrid scheme:

$$\begin{aligned}\tilde{h}(z, z_S, P^z, \mu, a) &= \frac{\tilde{h}(z, P^z, a)}{\tilde{h}(0, a)} \frac{C_0(z^2\mu^2) + \Lambda z^2}{C_0(z^2\mu^2)} \theta(z_S - z) \\ &\quad + e^{(\delta m + \bar{m}_0)(z - z_S)} \frac{\tilde{h}(z, P^z, a)}{\tilde{h}(0, a)} \frac{C_0(z_S^2\mu^2) + \Lambda z_S^2}{C_0(z_S^2\mu^2)} \theta(z - z_S) \\ a \xrightarrow{=} 0 & \quad \frac{\tilde{h}_0^{\overline{\text{MS}}}(z, P^z, \mu)}{C_0(z^2\mu^2)} \theta(z_S - z) + \frac{\tilde{h}_0^{\overline{\text{MS}}}(z, P^z, \mu)}{C_0(z_S^2\mu^2)} \theta(z - z_S) \\ & \quad z_S \ll \Lambda_{\text{QCD}}^{-1}\end{aligned}$$

Perturbatively related to the  $\overline{\text{MS}}$  scheme at all  $z$ !

# Renormalized matrix elements



# Physical extrapolation and Fourier transform (FT)

Extrapolation with form featuring an exponential decay:

$$\langle \pi(p) | j(x)j(0) | \pi(p) \rangle \xrightarrow{|x| \rightarrow \infty} e^{-m_{\text{eff}}|x|} g[p \cdot x, \cos(p \cdot x), \sin(p \cdot x)]$$

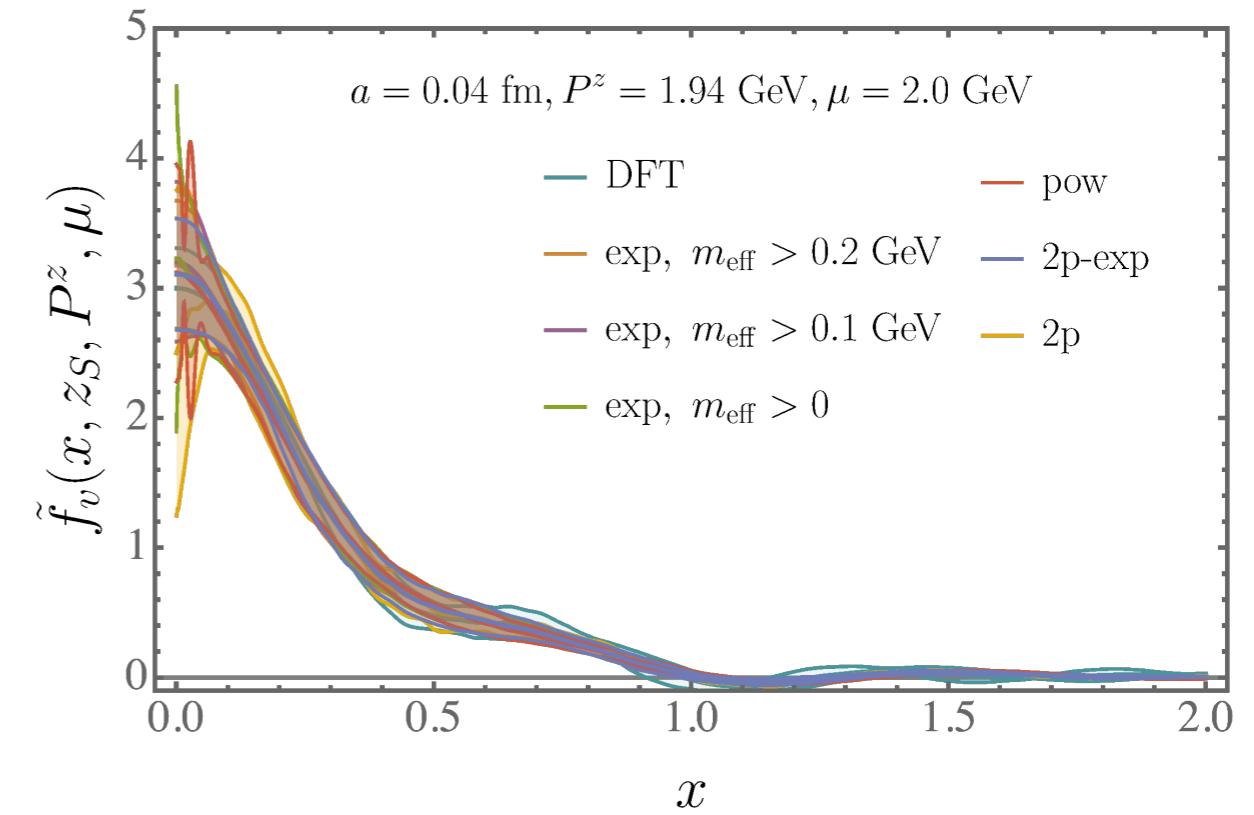
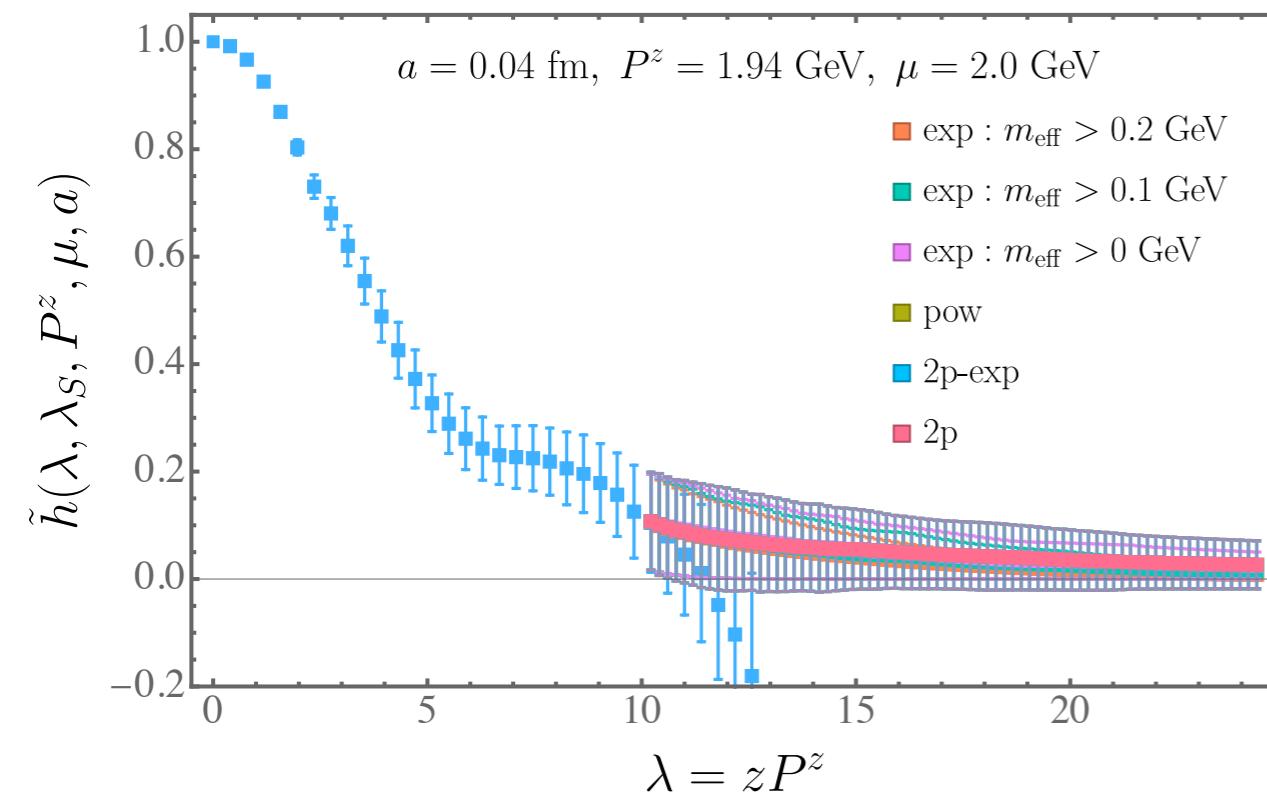
Burkardt, Grandy and Negele, Annals of Physics 238 (1995).

Models compared:

Discrete FT (DFT)

$$\begin{aligned} \text{exp : } & \frac{Ae^{-m_{\text{eff}}|\lambda|}}{\lambda^d} \\ \text{pow : } & \frac{A}{\lambda^d} \end{aligned}$$

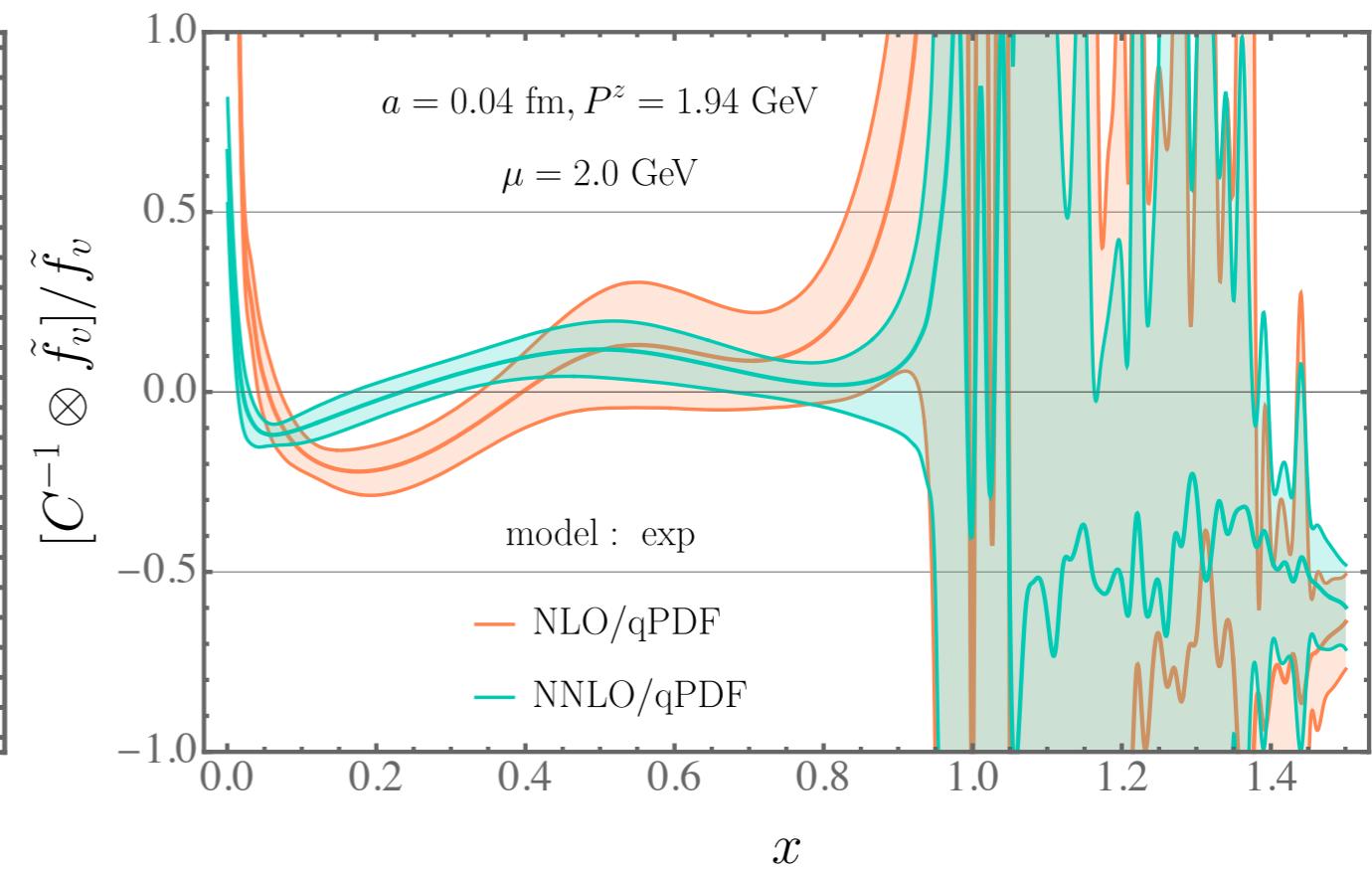
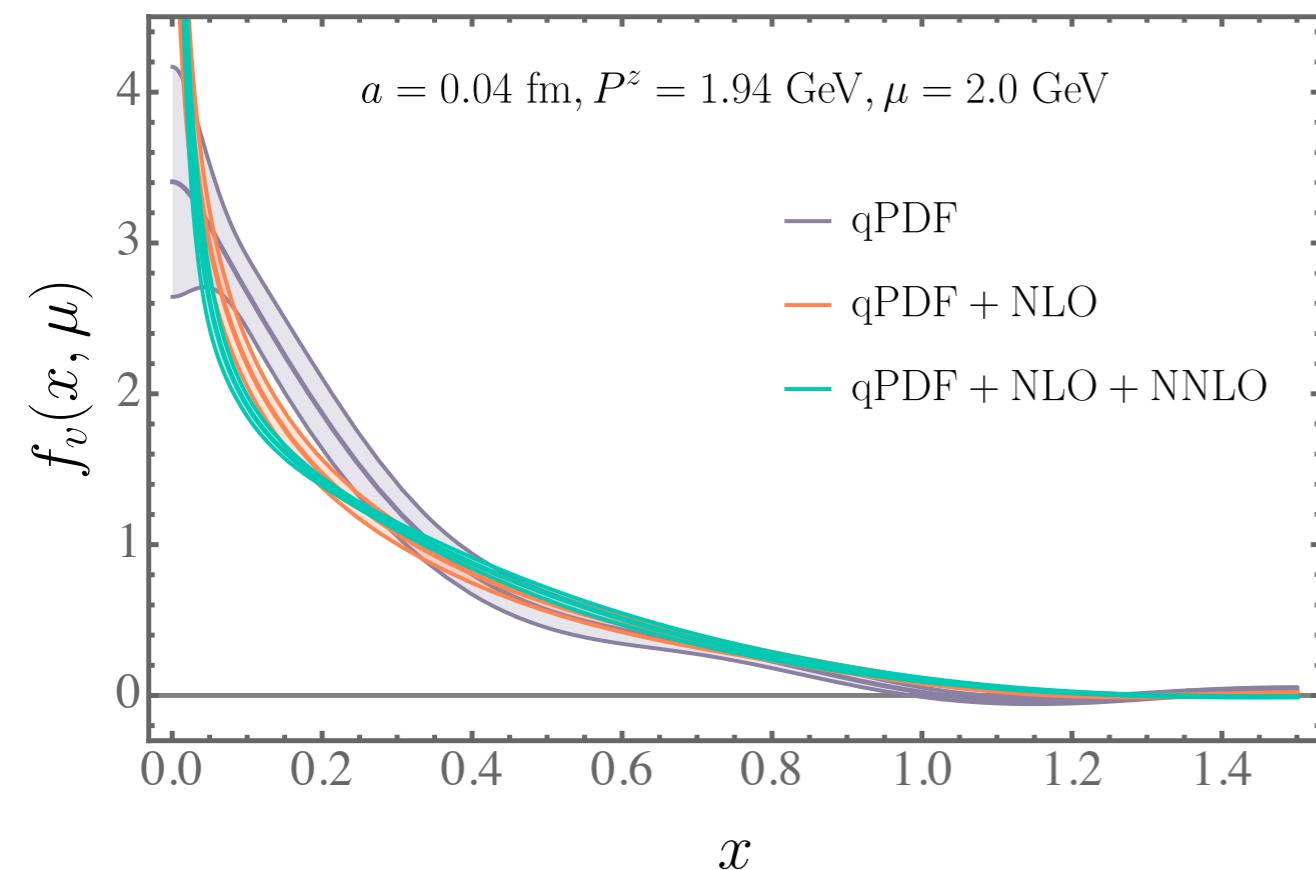
$$\begin{aligned} \text{2p-exp : } & A \operatorname{Re} \left[ \frac{\Gamma(1+a)}{(-i|\lambda|)^{a+1}} + e^{i\lambda} \frac{\Gamma(1+b)}{(i|\lambda|)^{b+1}} \right] e^{-m_{\text{eff}}|\lambda|} \\ \text{2p : } & A \operatorname{Re} \left[ \frac{\Gamma(1+a)}{(-i|\lambda|)^{a+1}} + e^{i\lambda} \frac{\Gamma(1+b)}{(i|\lambda|)^{b+1}} \right] \end{aligned}$$



**Due to exponential decay at large  $z$ , moderate to large  $x$  region is insensitive to extrapolations.**

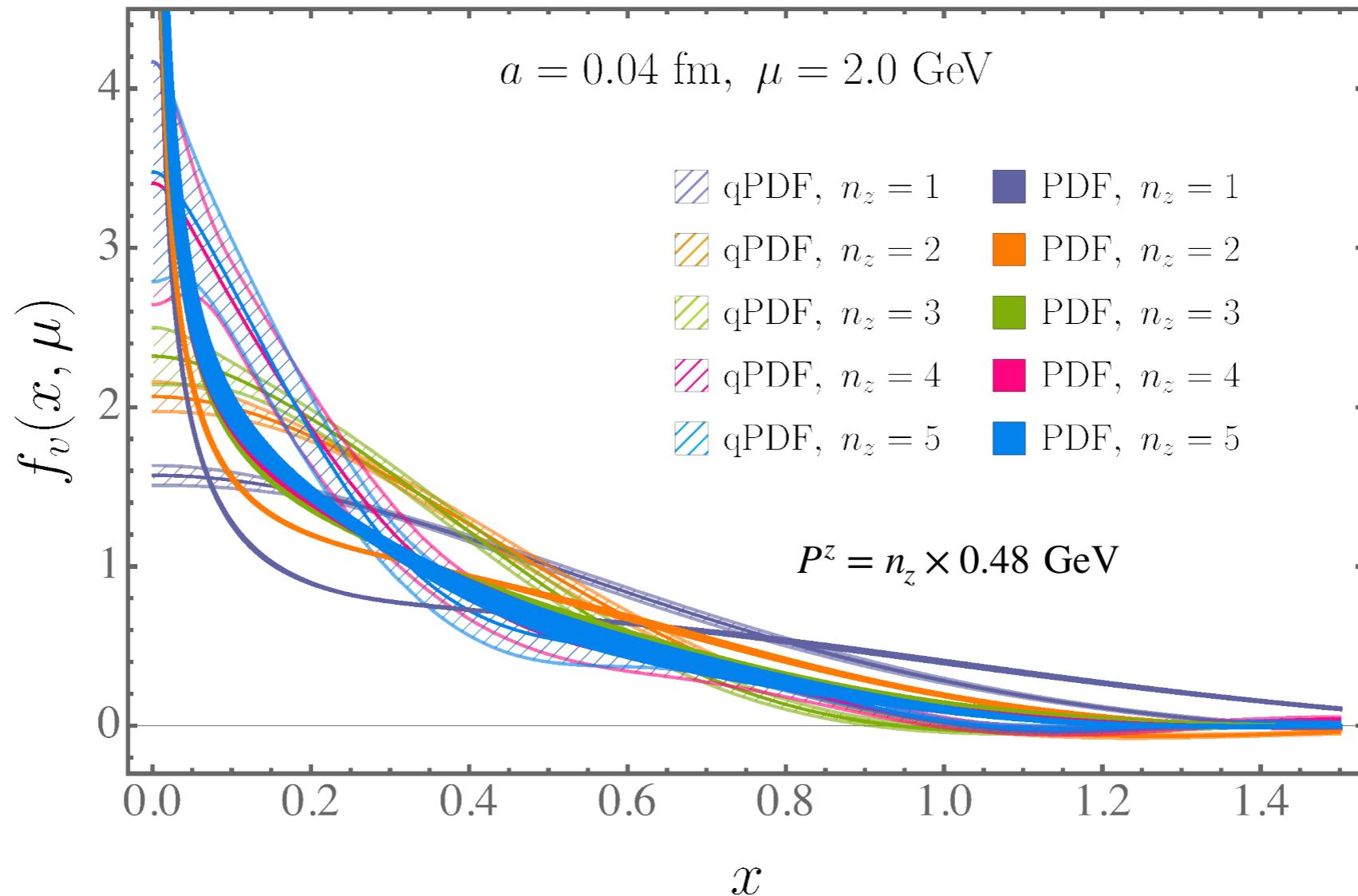
# Perturbative matching at NNLO

Perturbative correction shows good convergence at moderate  $x$ :



# Perturbative matching at NNLO

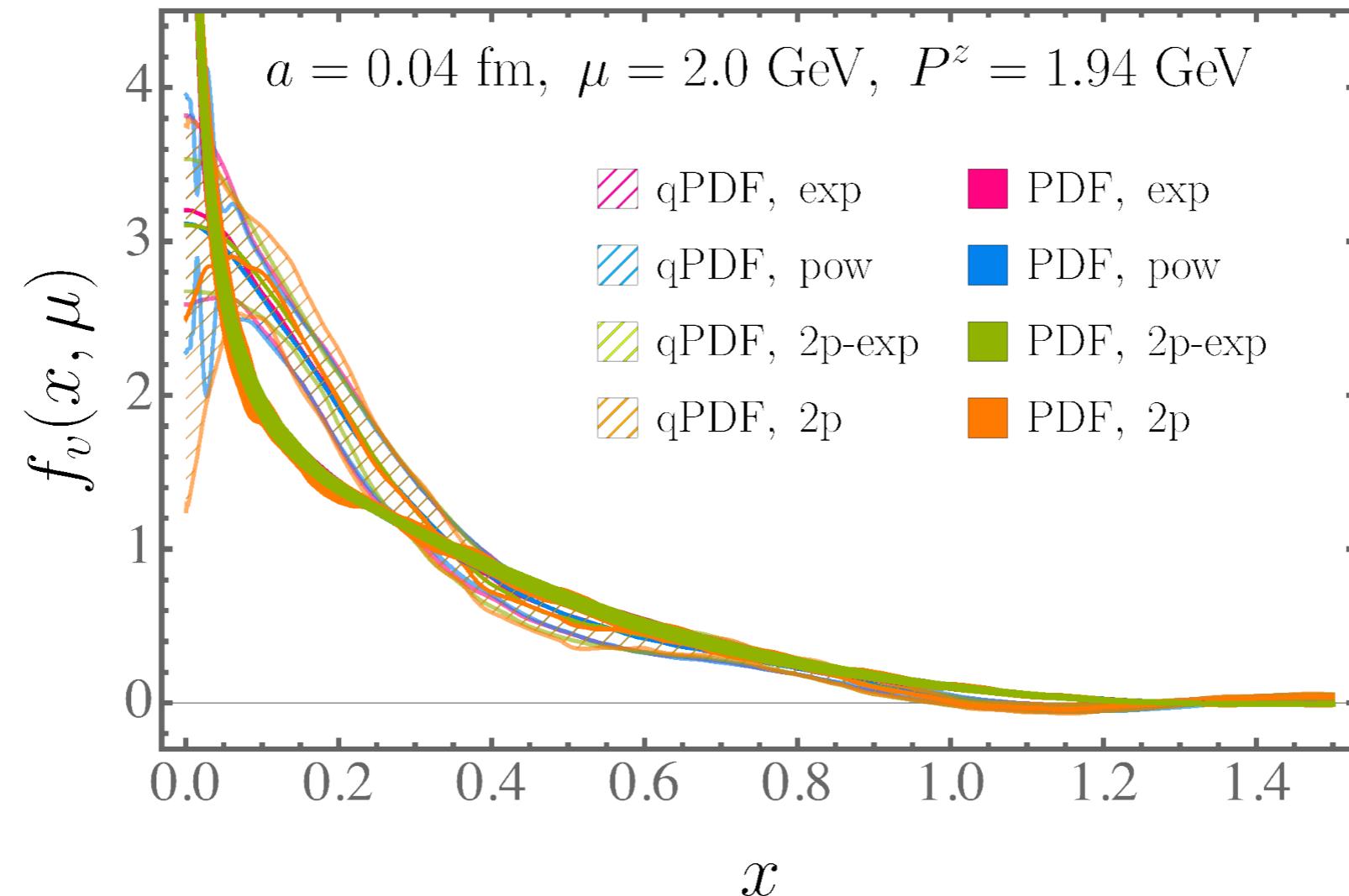
Momentum-dependence significantly reduced:



- Results show convergence at  $P^z > 1.45 \text{ GeV}$  (Lorentz boost factor > 5.0) at moderate  $x$ ;
- Non-vanishing tail at  $x \sim 1$ , which indicate power corrections, generally decreases in  $P^z$ .

# Perturbative matching at NNLO

Extrapolation-model dependence reduced:

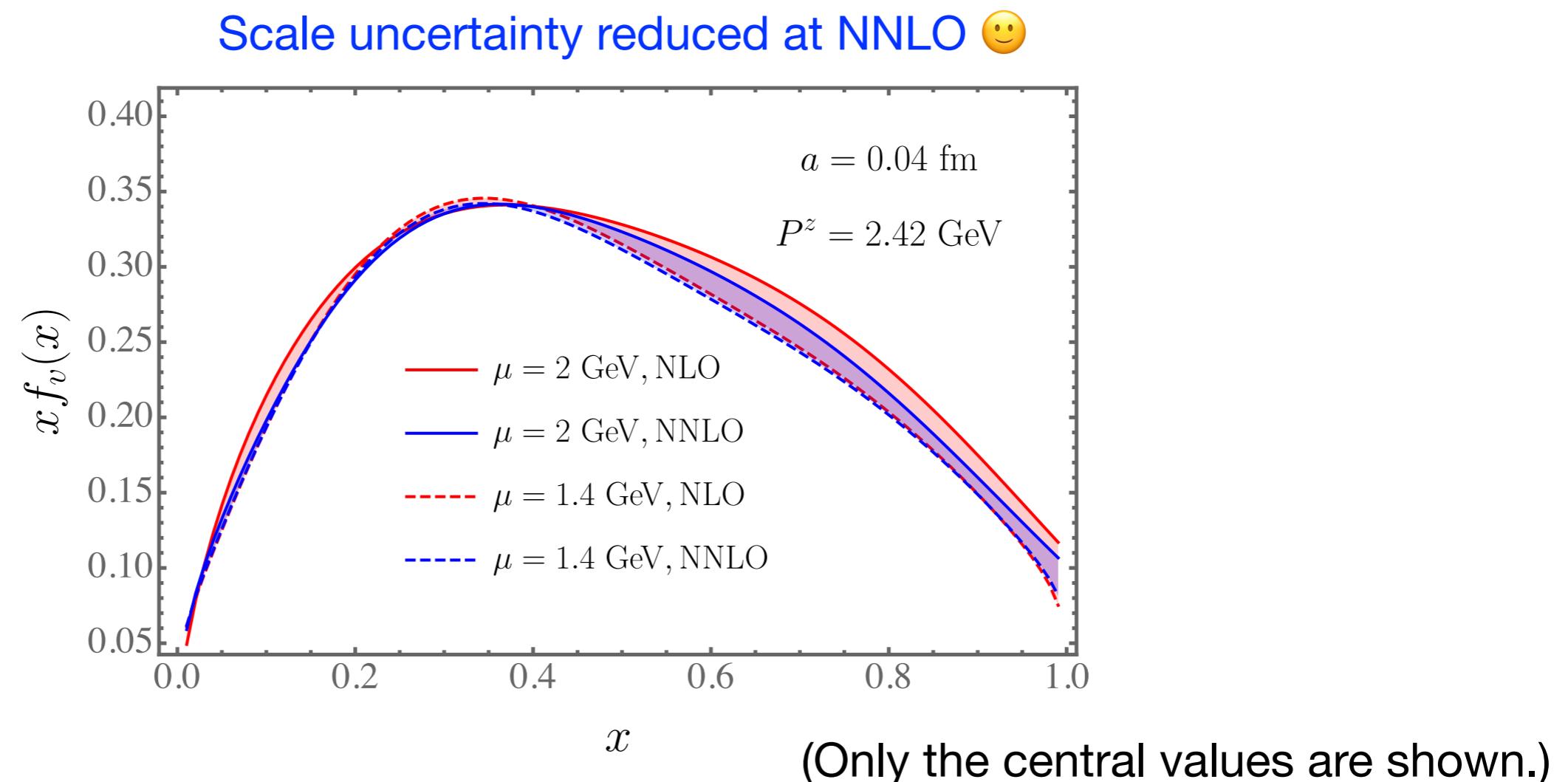


- Agreement among different extrapolation models extends to smaller  $x$  region;
- Slight oscillation in the 2p model due to the slow decay in the extrapolated region.

# Perturbative matching at NNLO

Factorization scale variation uncertainty:

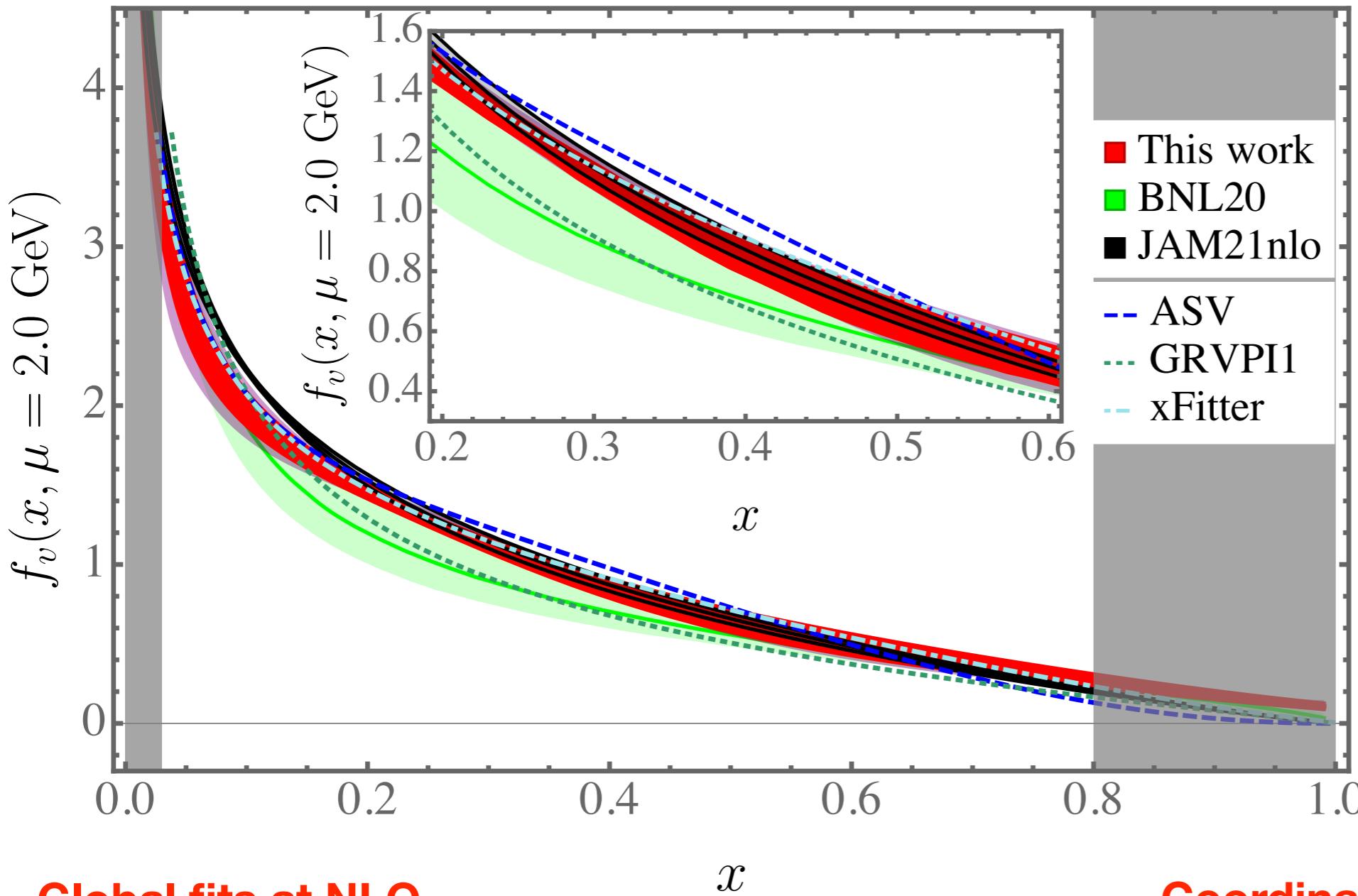
- Calculate the PDF at different  $\mu$ ;
- Evolve the results to  $\mu = 2.0$  GeV.



# Systematic uncertainties

- Statistical uncertainty: bootstrap resampling.
  - Scale variation: error band covers  $\mu = 1.4, 2.0, 2.8$  GeV.
  - Truncation point  $z_L$ : negligible.
  - Extrapolation model dependence: negligible for the  $x$  region of interest.
  - Higher-order perturbative corrections:
    - Requiring  $N^3LO/LO \leq 5\% \Rightarrow NLO/LO \leq 37\%$  and  $NNLO/LO \leq 14\%$ ;
    - $0.03 \leq x \leq 0.88$ .
  - Power corrections:
    - Use  $P_z=2.42$  GeV result as final prediction;
    - Fit  $P_z \geq 1.45$  GeV results with  $f_\nu(x) + \alpha(x)/P_z^2$  at each  $x$ ;
    - $|\alpha(x)/[P_z^2 f_\nu(x)]| \leq 10\%, \Rightarrow 0.01 \leq x \leq 0.80$ .
- 
- $0.03 \leq x \leq 0.80$
- $0.01 \leq x \leq 0.80$

# Final prediction



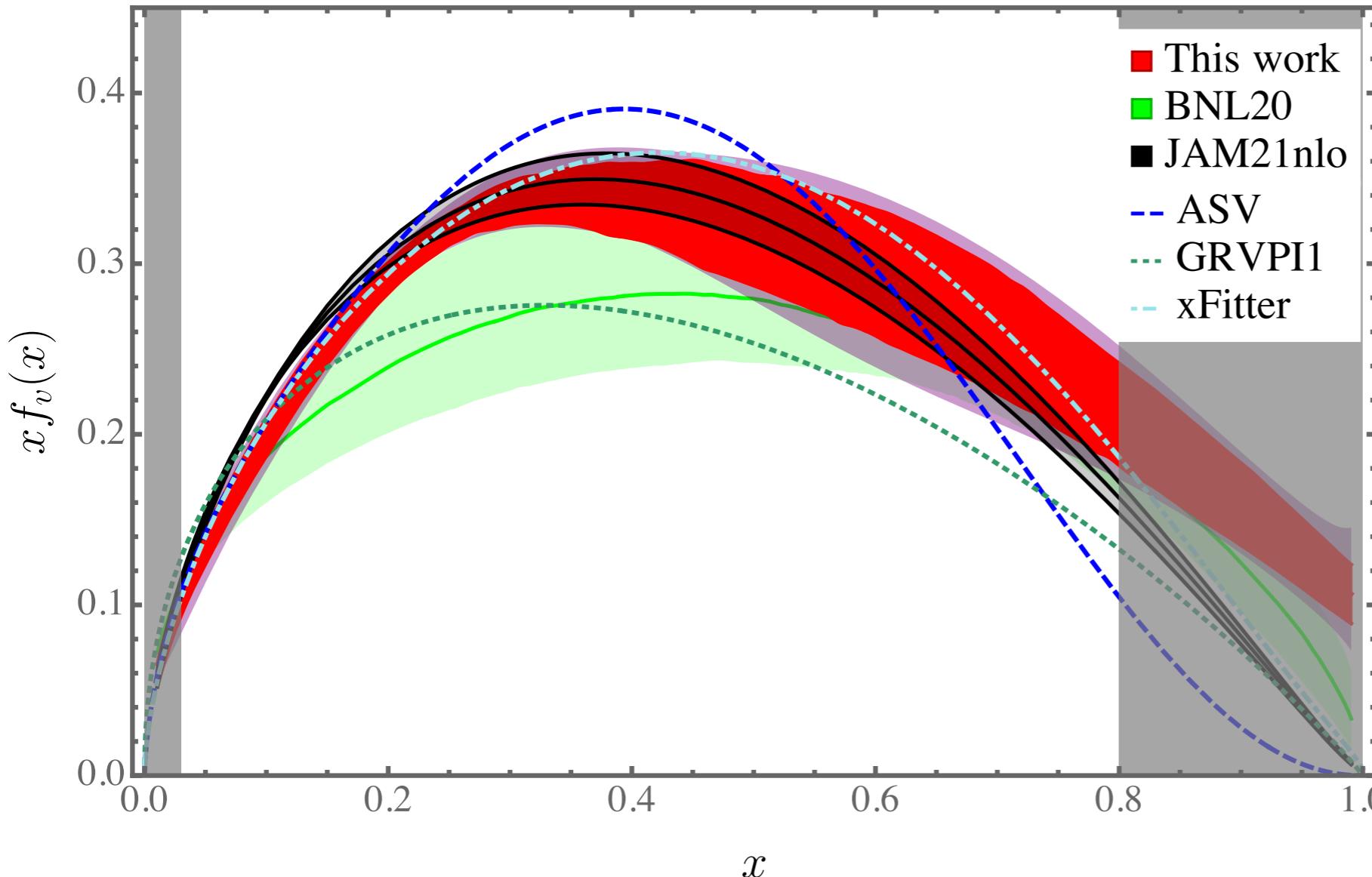
## Global fits at NLO

- **JAM21nlo**, P. C. Barry, C.-R. Ji, N. Sato, and W. Melnitchouk, PRL 127 (2021);
- **xFitter**, I. Novikov et al., PRD 102 (2020);
- **ASV**, Aicher, A. Schafer, and W. Vogelsang, PRL 105 (2010);
- **GRVPI1**, M. Gluck, E. Reya, and A. Vogt, Z. Phys. C 53 (1992).

**Coordinate-space analysis of the same lattice data with NLO OPE and parameterization of the PDF:**

**BNL20**, X. Gao, YZ, et al., PRD102 (2020).

# Final prediction



Statistical + scale-variation uncertainties: 5–20%

## Global fits at NLO

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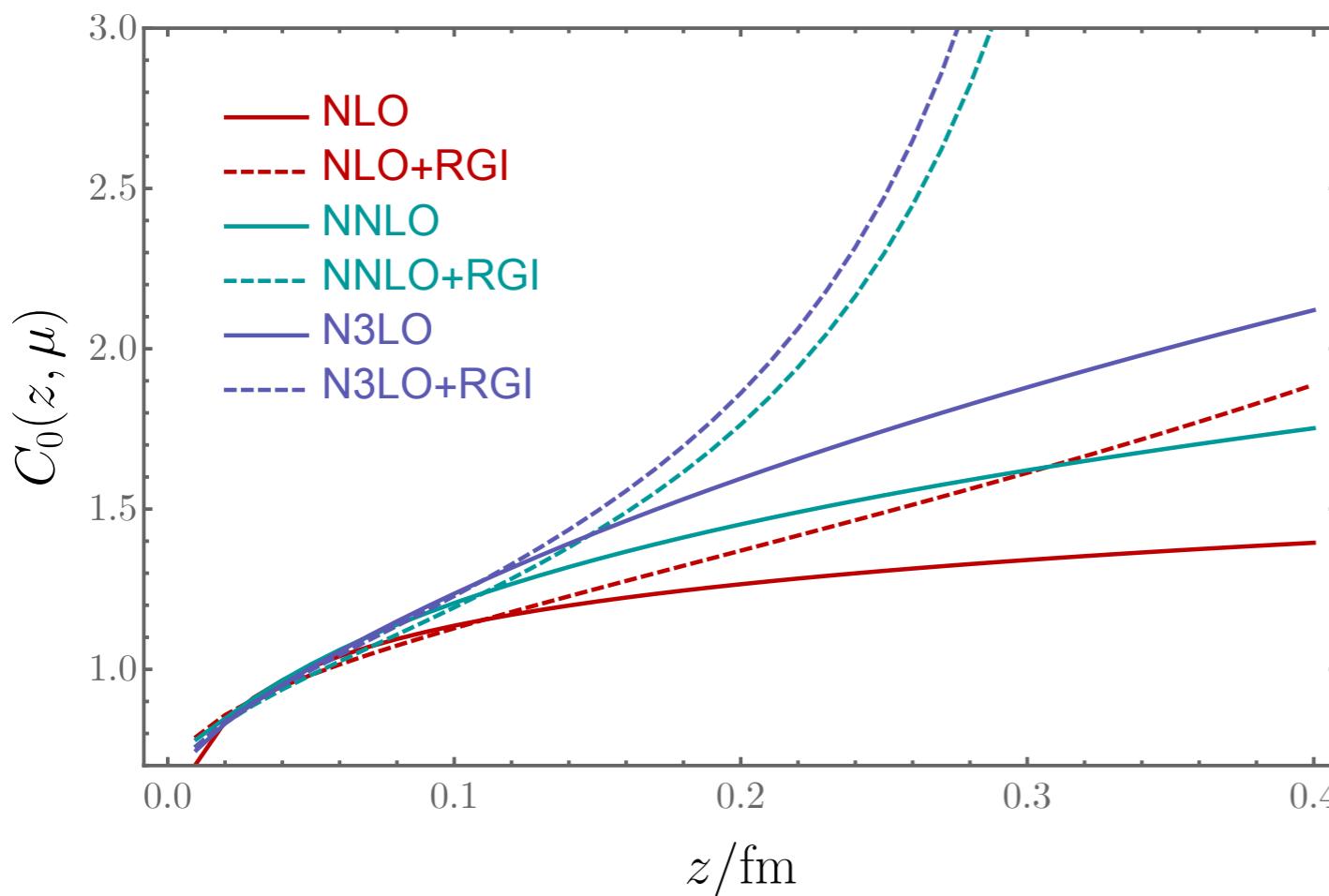
# Conclusion

- We have developed a procedure to renormalize the quasi-PDF matrix elements in the hybrid scheme and match the results to the continuum at NNLO;
- We have calculated the pion valence PDF with the state-of-the-art NNLO hybrid scheme matching coefficient;
- NNLO matching shows good perturbative convergence;
- We demonstrate that we can reliably predict the  $x$ -dependence of the pion valence PDF for  $0.03 \lesssim x \lesssim 0.80$  with 5–20% uncertainty;
- Systematics to be improved: physical pion mass, infinite volume limit, finer lattice spacings, larger boost momenta.

# Backup slides

Fixed-order and RG-improved (RGI) Wilson coefficients:

$$C_0^{\text{RGI}}(\mu^2, \bar{z}^2) = C_0(1, \alpha_s(\bar{z}^{-1})) \times \exp \left[ \int_{\bar{z}^{-1}}^{\mu} d\alpha_s(\mu') \frac{\gamma_O(\alpha_s(\mu'))}{\beta(\alpha_s(\mu'))} \right] \quad \bar{z}^2 = z^2 e^{2\gamma_E/4}$$



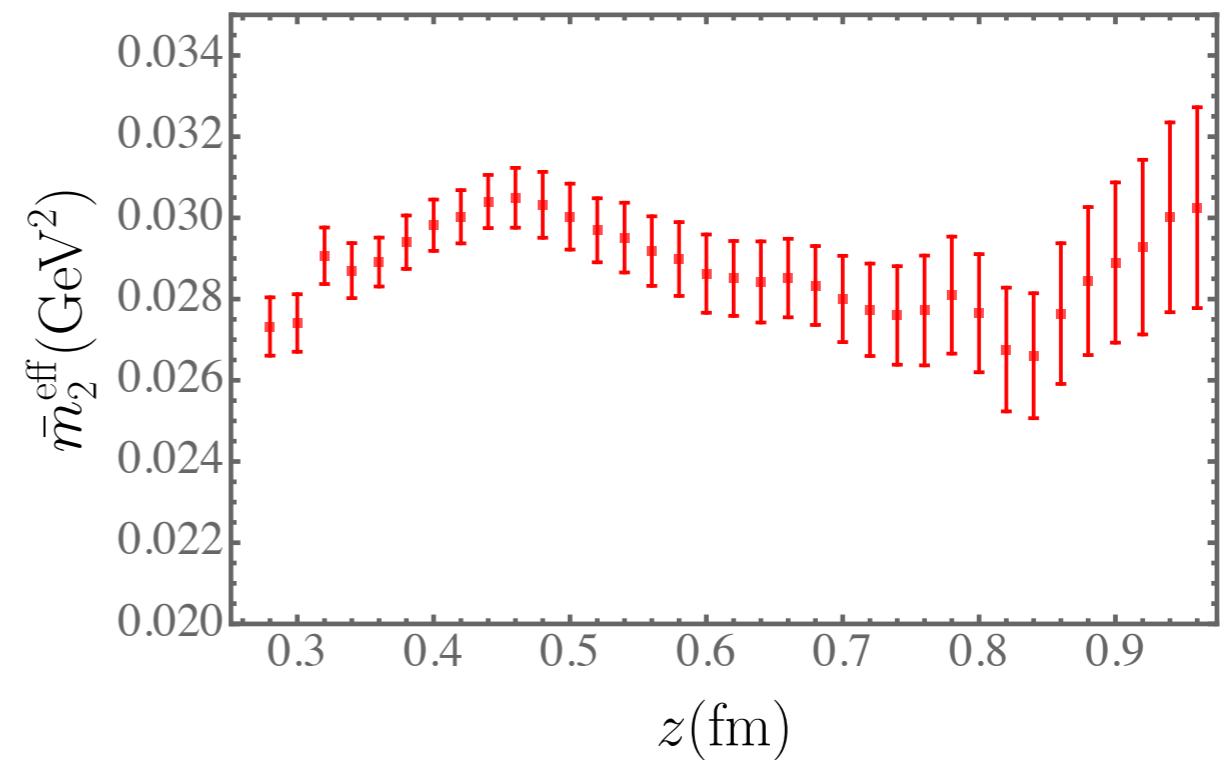
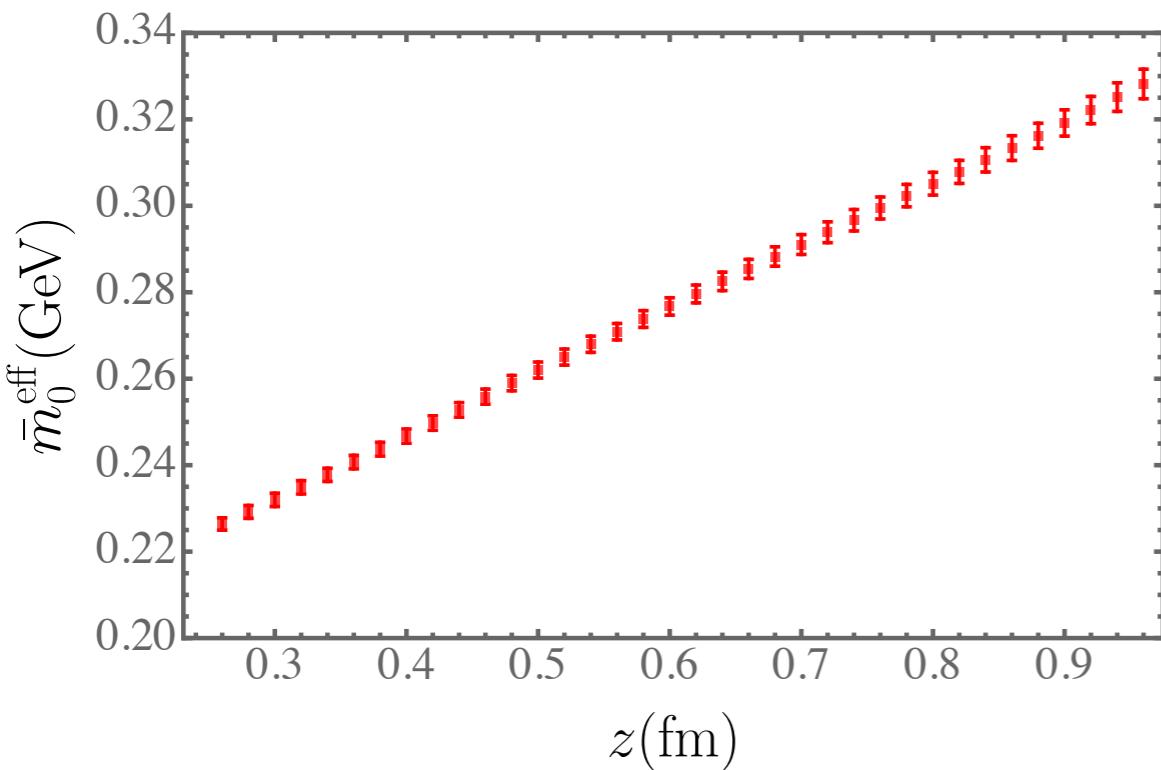
- Perturbation theory becomes unreliable beyond  $z \sim 0.2$  fm;
- To have a sufficiently large window of  $z$ , we use the fixed-order  $C_0$  and truncate at  $z_{\text{max}} = 0.4$  fm;
- Future improvement should include smaller lattice spacings.

# Fitting $\bar{m}_0$

Define effective mass and its slope in  $z$ :

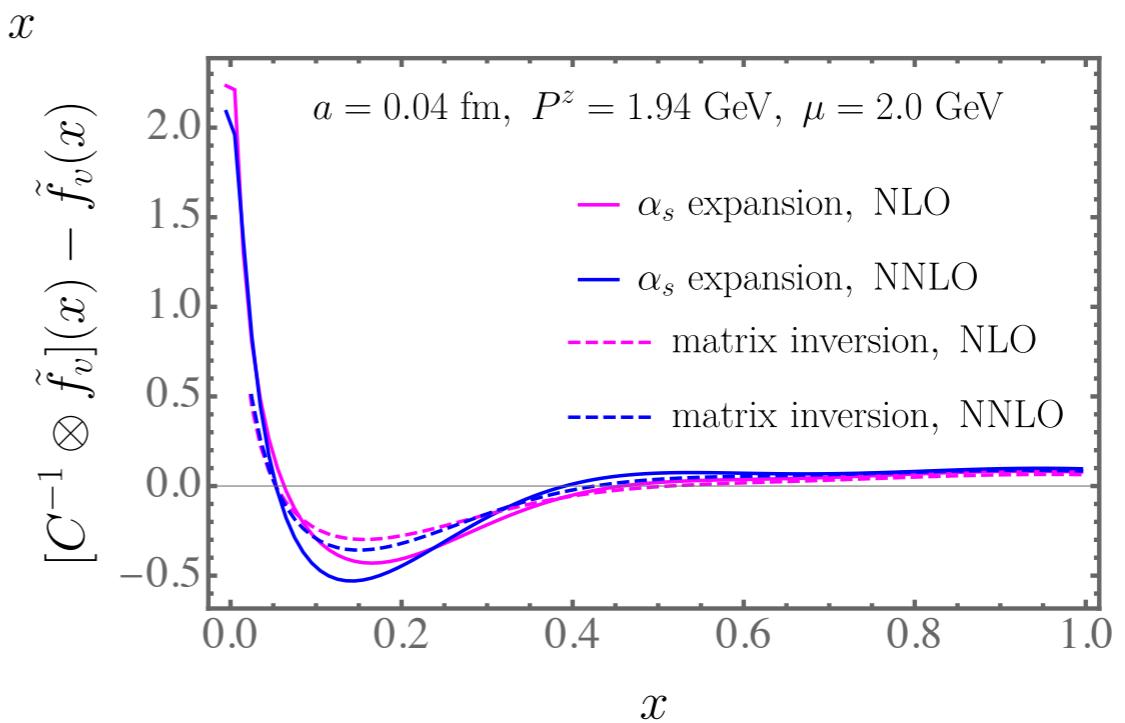
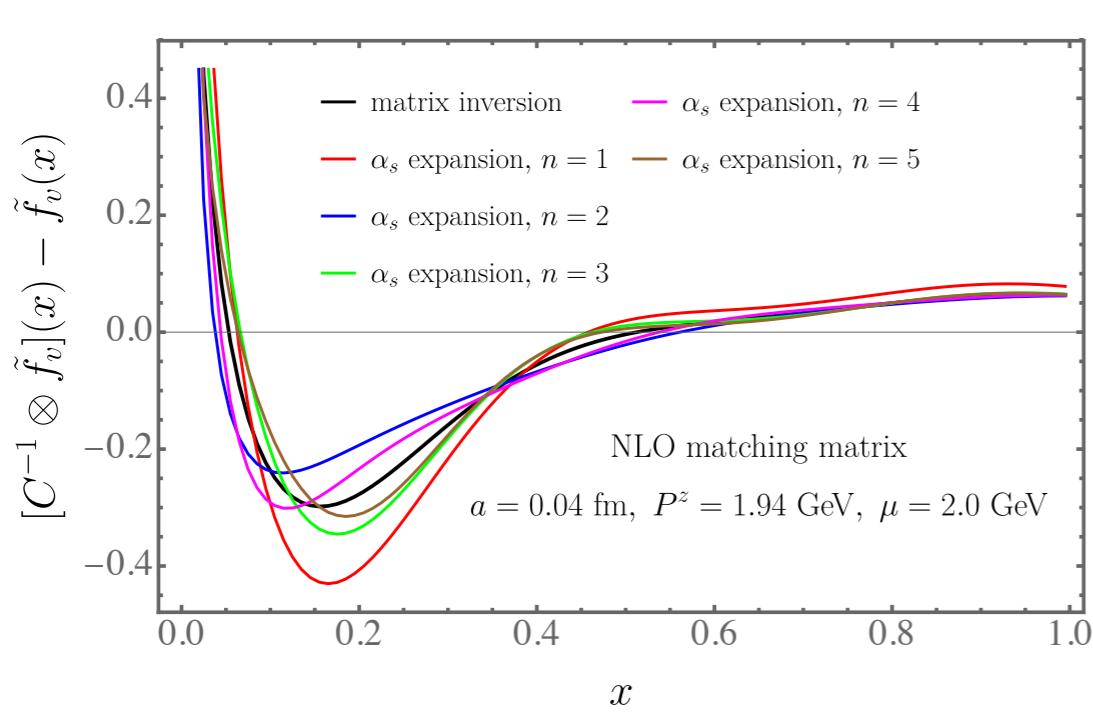
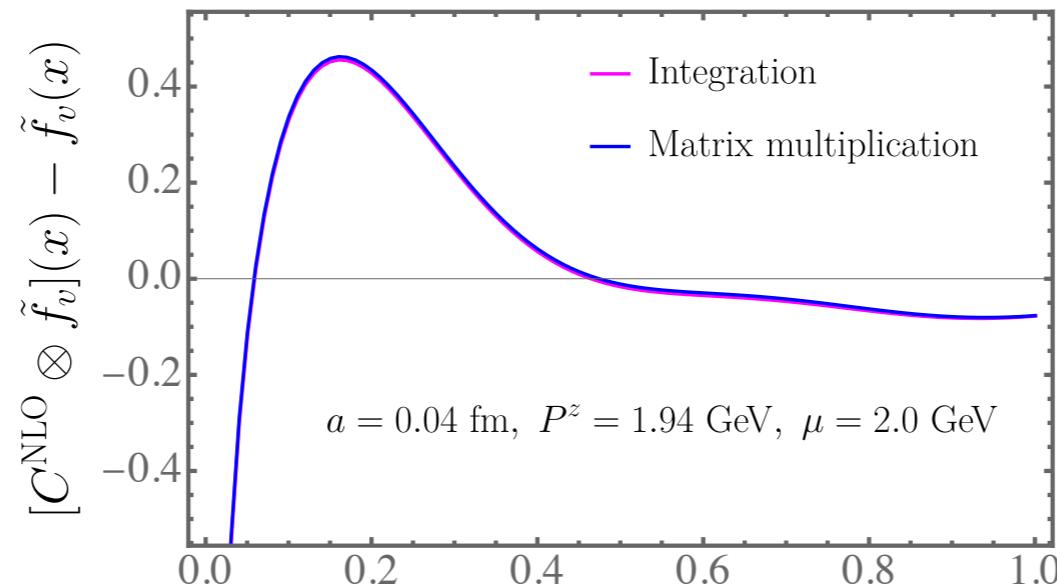
$$\bar{m}_0^{\text{eff}}(z)(z - z_0) \equiv -\ln \frac{\tilde{h}(z, 0, a)}{\tilde{h}(z_0, 0, a)} + \ln \frac{C_0^{\text{NNLO}}(z^2 \mu^2)}{C_0^{\text{NNLO}}(z_0^2 \mu^2)}$$

$$\bar{m}_2^{\text{eff}}(z) = \frac{\bar{m}_0^{\text{eff}}(z) - \bar{m}_0^{\text{eff}}(z - a)}{a}$$



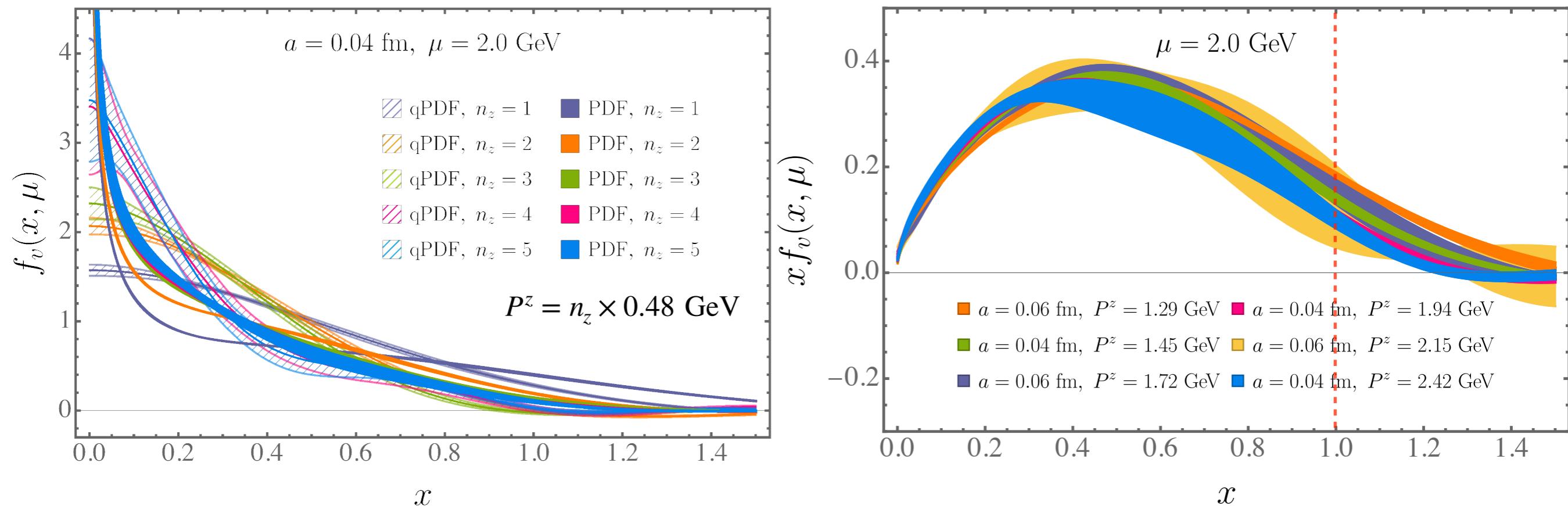
# Inverse matching

$$C^{-1}(x) = \delta(x - 1) - \frac{\alpha_S}{2\pi} C^{(1)}(x) - \left(\frac{\alpha_S}{2\pi}\right)^2 C^{(2)}(x) + \left(\frac{\alpha_S}{2\pi}\right)^2 C^{(1)} \otimes C^{(1)}(x)$$



# Perturbative matching at NNLO

Momentum-dependence significantly reduced:



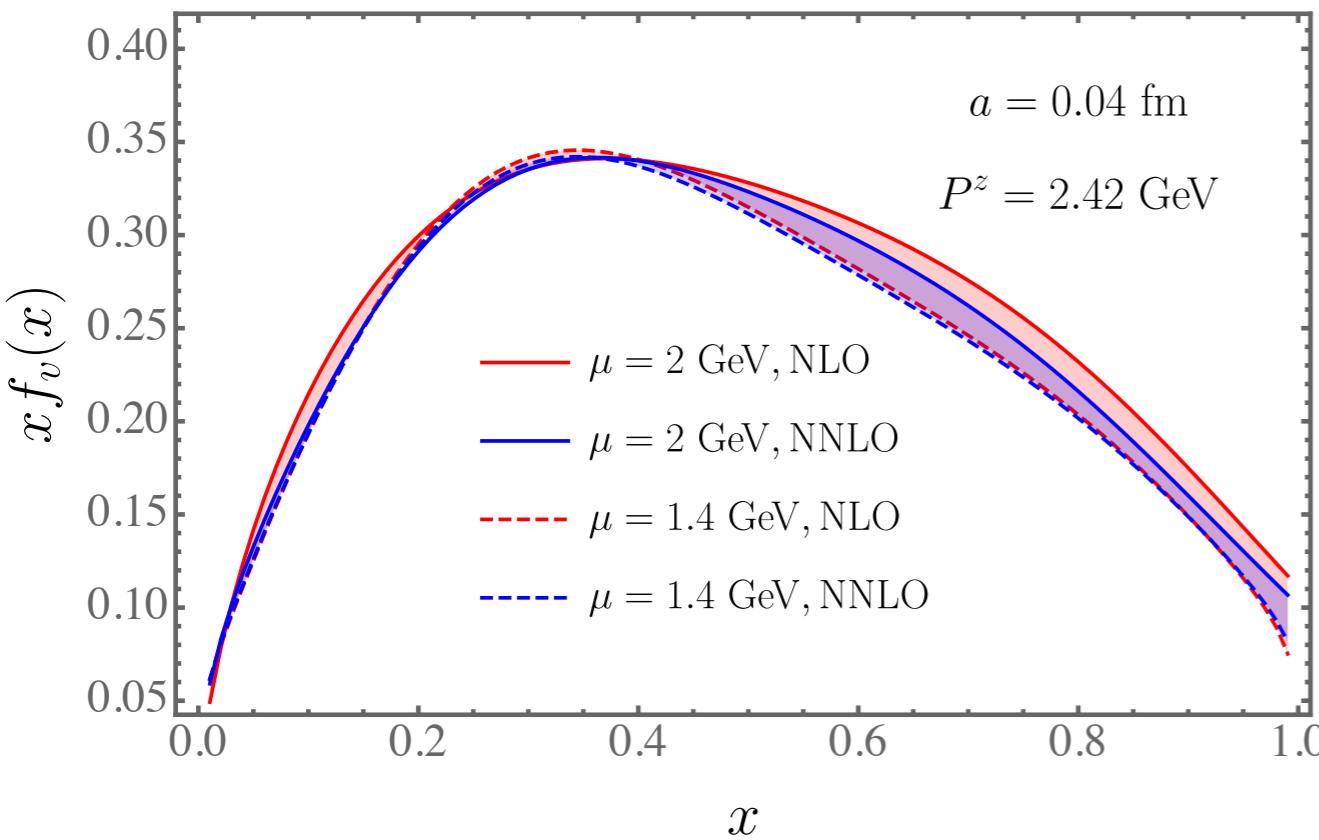
- Results show convergence at  $P^z > 1.29 \text{ GeV}$  (Lorentz boost factor > 4.0) at moderate  $x$ ;
- Non-vanishing tail at  $x \sim 1$ , which indicate power corrections, generally decreases in  $P^z$ .

# Perturbative matching at NNLO

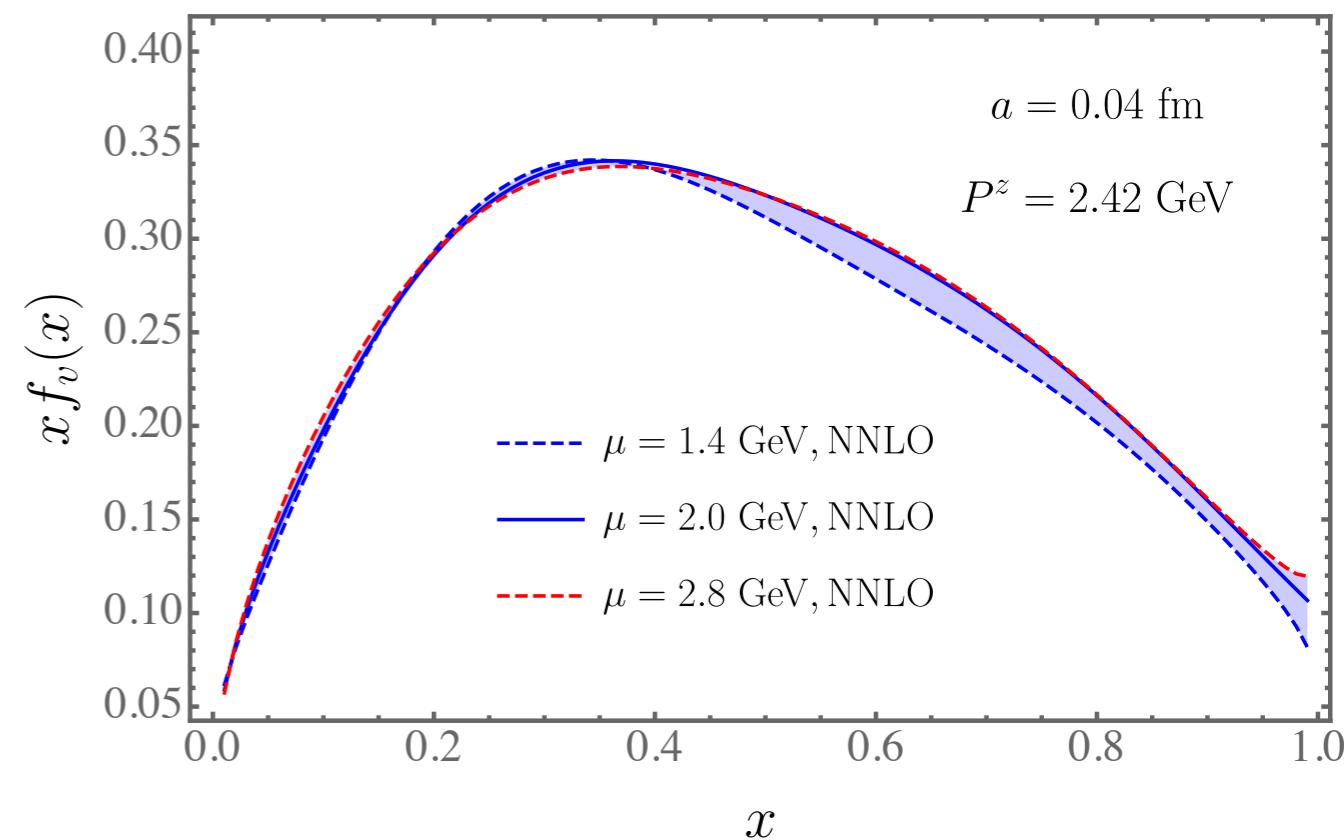
Factorization scale variation uncertainty:

- Calculate the PDF at different  $\mu$ ;
- Evolve the results to  $\mu = 2.0$  GeV.

Scale uncertainty reduced at NNLO 😊



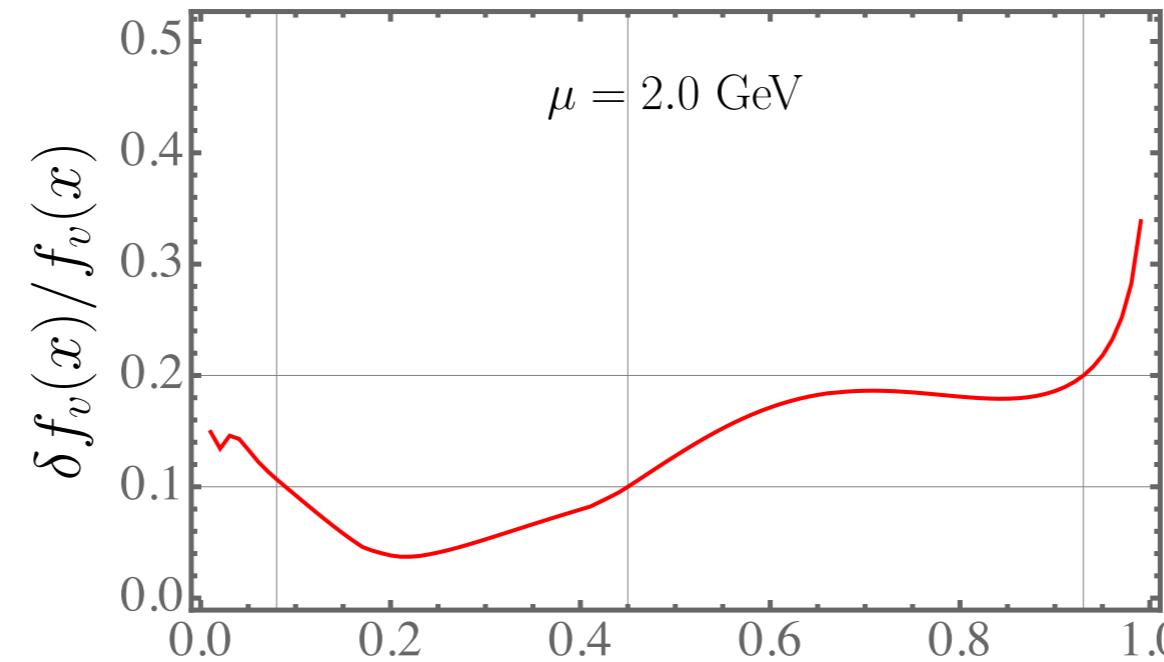
Uncertainty mainly comes from varying to lower scales.



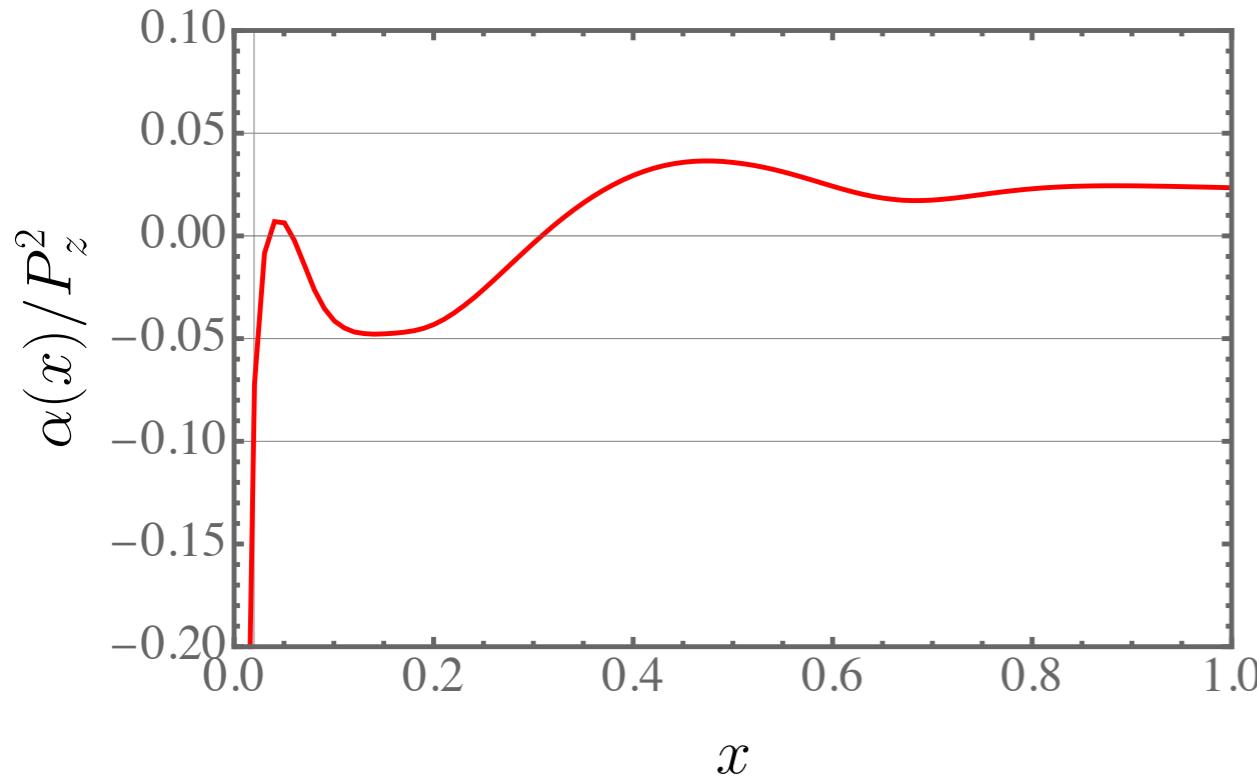
(Only the central values are shown.)

# Systematics

Statistical and scale-variation uncertainties:



Power correction absolute size:



Power correction relative size:

