A toy model of holography: sparse SYK, wormholes and chaos

by

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E.C, A. Misobuchi, R. Pimentel JHEP 11 (2021) 015

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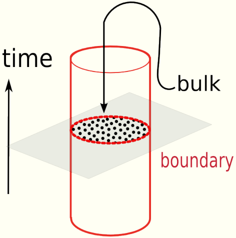
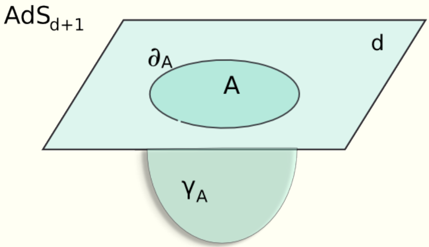
Introduction

AdS/CFT: d-dim CFT $\leftrightarrow$ d+1 AdS



AdS/CFT: Quantum $\leftrightarrow$ Geometry

- Holographic entanglement entropy. Ryu-Takayanagi (RT) 2006; Hubeny-Rangamani-Takayanagi (HRT) 2007.



- Complexity. Susskind et al. 2015
- Quantum information

How are quantum degrees of freedom encoded in gravity?

- One hurdle: not many solvable quantum models of holography

Sachdev–Ye–Kitaev model, **SYK**, 2015

- Recently, Xu, Susskind, Swingle, 2020 proposed a new class of models

Sparse SYK

- Sparse model is more computationally efficient. Numerical simulations, finite N
- This talk: what is the sparse SYK, results

Outline

- Introduction
- SYK and sparse SYK
- Two coupled sparse SYK: traversable wormhole, revivals
- Spectral form factor
- Future directions

Sachdev-Ye-Kitaev model

(all-to-all) SYK

Quantum mechanical system of N Majorana fermions χ^j with all-to-all random interactions [Kitaev '15]

$$H = i^{q/2} \sum_{1 \le j_1 < \ldots < j_q \le N} \underbrace{J_{j_1 \ldots j_q}}_{\text{Gaussian}} \underbrace{\chi^{j_1} \ldots \chi^{j_q}}_{q\text{-body}}, \qquad \langle (J_{j_1 \ldots j_q})^2 \rangle = \frac{(q-1)! J^2}{N^{q-1}}$$

- Analytically solvable at $N \to \infty$
- Emergent conformal symmetry at low energies
- Maximally chaotic $\lambda_L = \frac{2\pi}{\beta}$ [Maldacena, Shenker, Stanford 1503.01409]

- SYK is a toy model of holography when $\beta J \gg 1$. Jackiw-Teitelboim gravity.
- Generalizations: charged, supersymmetric, etc.
- Drawback: computational cost

number of terms $\sim N^q$

state of the art $N = 52$, 7 million terms

Is there an SYK modification that retains all the interesting physics but is more computationally efficient?

Sparse SYK

Sparse SYK

Sparse SYK model

Sparsity: Reduce number of terms in the Hamiltonian summation while preserving original properties, e.g., chaotic behavior

[Xu, Susskind, Su, Swingle 2008.02303]

Two ways of introducing sparseness

- Random pruning
- Hypergraphs

Random prunning

$$H = i^{q/2} \sum_{1 \le j_1 < \ldots < j_q \le N} J_{j_1 \ldots j_q} x_{j_1 \ldots j_q} \chi^{j_1} \ldots \chi^{j_q}, \qquad \langle \left(J_{j_1 \ldots j_q}\right)^2 \rangle = \frac{(q-1)!\, J^2}{p N^{q-1}}$$

where

$$x_{ijkl} = \begin{cases} 0 \text{ with probability} & 1-p \\ 1 \text{ with probability} & p \end{cases}$$

Note that

- Computational cost We want number of terms $\sim kN$, where $k \sim O(1)$

$$\binom{N}{q} p = kN$$

For $N = 52$, $k = 4$, 208 terms. We can study higher N and higher q.

- Path integral formulation. Chaos. [Xu, Susskind, Su, Swingle 2008.02303]

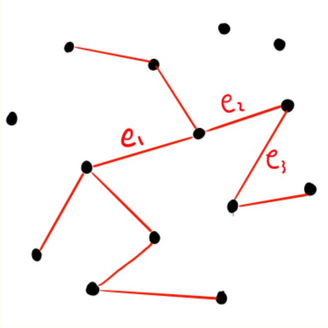
How sparse can the model be ? Something new?

Useful language: hypergraphs

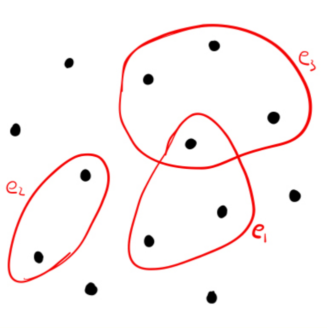
Hypergraphs

Hypergraphs: Generalization of a graph where hyperedges can connect more than two vertices

Graph $G = (V, E)$



Hypergraph $H = (V, E)$



$E = \{e_1, e_2, e_3 ....\}$
$|e_i| = 2$
pairs of vertices

$E = \{e_1, e_2, e_3 ....\}$
$|e_i| > 2$

Sparse SYK as a random hypergraphs: Majorana fermions are identified with vertices, and each interaction term correspond to a hyperedge connecting q vertices (q-uniform).

- k quantifies the degree of sparsity in the Hamiltonian

$$k = \frac{p}{N}\binom{N}{q}$$

$\Rightarrow$ Sparse Hamiltonian is a sum of exactly kN terms

- Math results for random regular hypergraphs

We want sparse hypergraphs that are highly connected, ***expanders***

Study measures of hypergraph connectivity:

- Algebraic hypergraph entropy
- Vertex expansion
- Spectral gap

$\Longrightarrow$

$$q = 4, \quad k = 4 \quad \checkmark$$
$$q = 8, \quad k = 2 \quad \checkmark$$

Traversable wormholes and sparse SYK

Two coupled sparse SYK

Eternal traversable wormhole with a global AdS_2 geometry can be realized by coupling two copies of SYK in the large N and small coupling limit [Maldacena, Qi 1804.00491]

- Solution can be obtained from JT gravity by adding coupling between boundaries

  [Gao, Jafferis, Wall 1608.05687]

- Same physics can be derived from two coupled SYKs

$$H = H_L^{\mathrm{SYK}} + H_R^{\mathrm{SYK}} + H_{\mathrm{int}}, \quad H_{\mathrm{int}} = i\mu \sum_{j=1}^{N} \chi_L^j \chi_R^j$$

→ Two coupled sparse SYKs

Properties of the two coupled SYK model

[Maldacena, Qi 1804.00491]

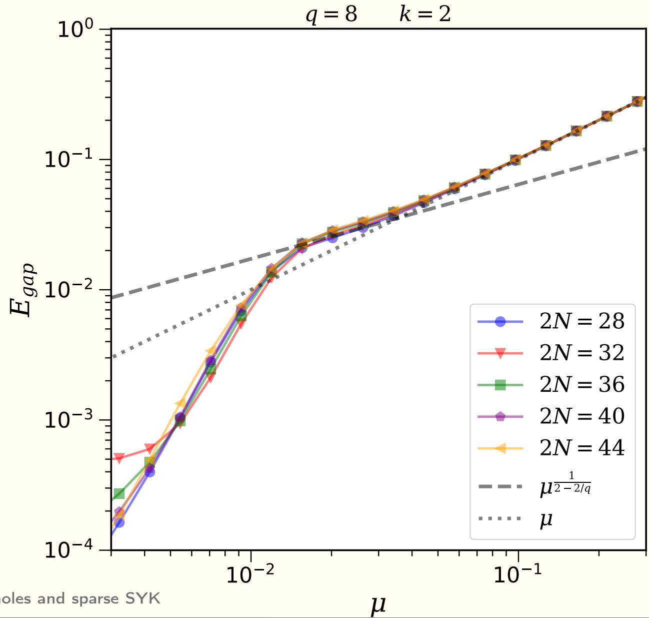
$$H = H_L^{\mathrm{SYK}} + H_R^{\mathrm{SYK}} + i\mu \sum_{j=1}^{N} \chi_L^j \chi_R^j$$

- Ground state $|\Psi_0\rangle$ approximately a TFD state (for some $\beta(\mu)$)
- Energy gap scaling. Derived in large N. Gravitational.

$$E_{\mathrm{gap}} \sim \mu^{\frac{1}{2-2/q}} \qquad \text{at weak coupling}$$

$$E_{\mathrm{gap}} \sim \mu \qquad \text{at strong coupling}$$

Energy gap


$q = 8 \qquad k = 2$
E_{gap}
μ
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
$2N = 28$
$2N = 32$
$2N = 36$
$2N = 40$
$2N = 44$
$\mu^{\frac{1}{2-2/q}}$
μ

- $q = 8$ matches scaling expected from gravity for large N and appropriate range of couplings
- Finite N effects dominate at very small couplings μ

Revival dynamics phenomena

1. Start with ground state $|\Psi_0\rangle$ of the two coupled SYK
2. Create Majorana excitation in Right system

$$|\Psi(t=0)\rangle = \chi_R |\Psi_0\rangle$$

3. Excitation gets scrambled
4. Excitation reassembles and becomes localized in Left system

$$|\Psi(t=t_{\text{rev}})\rangle = \chi_L |\Psi_0\rangle$$

5. Process is repeated with $L \leftrightarrow R \Rightarrow$ 'Revival oscillations'

   [Plugge, Lantagne-Hurtubise, Franz 2003.03914]

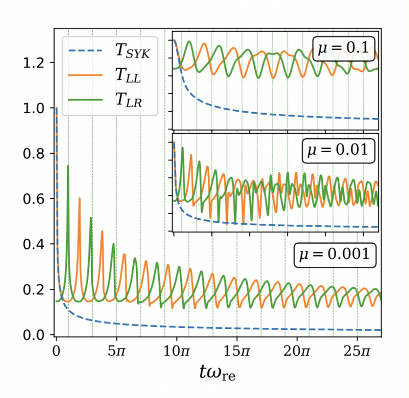
Gravity picture: Perturbation travels through the wormhole

Diagnostic of revivals

Transmission amplitude $\quad T_{ab} = 2|G^{>}_{ab}|$

$$G^{>}_{ab}(t) = -\frac{i\theta(t)}{N}\sum_j \langle \chi^j_a(t)\chi^j_b(0)\rangle = \begin{pmatrix} G^{>}_{LL}(t) & G^{>}_{LR}(t) \\ G^{>}_{RL}(t) & G^{>}_{RR}(t) \end{pmatrix}$$

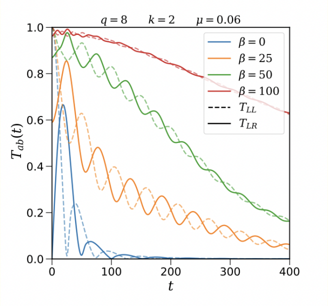
$|T_{LR}(t)|^2$: Probability of recovering χ^j_L at some time t after inserting χ^j_R at $t = 0$.



[Plugge, Lantagne-Hurtubise and Franz, 2020]

Revivals in sparse SYK

Sparse SYK with $q = 8$ is compatible for some range of couplings and temperatures



Tools:

- Dynamite: a python library that makes use of PETSc and SLEPc. Krylov subspace methods combined with massive parallelization [Github:GregDMeyer/dynamite]
- Texas Advanced Computing Center (TACC): Use of computational resources from Stampede2 supercomputer

Chaos and spectral form factor

Quantum chaos:

- Out of Time Order Correlators (OTOC)

  $$C(t) = -\langle [W(t), V(0)]^2 \rangle_\beta, \qquad V, W \text{ Hermitian operators}$$

  Basic intuition: How much an early perturbation V affects the later measurement of W. Lyapunov exponent
- Random Matrix Theory (RMT)

Random matrix theory (RMT) provides an alternative diagnostic of quantum chaos

- Quantum chaos encoded in the statistical properties of the spectrum
- Spectra of quantum chaotic systems show the same fluctuation properties as predicted by RMT

Quantity sensitive to energy level statistics: Spectral Form Factor

$$Z(\beta + it, \beta - it) = \langle Tr(e^{-(\beta+it)H})\, Tr(e^{-(\beta-it)H}) \rangle_J$$

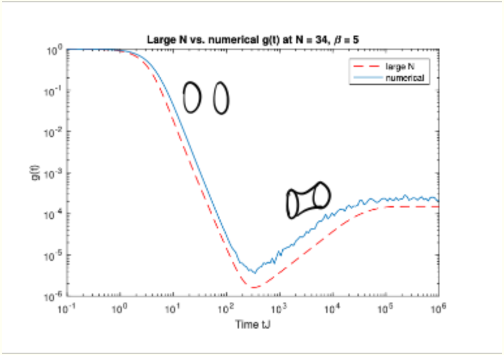
$$g(t, \beta) = \frac{Z(\beta + it, \beta - it)}{|Z(\beta)|^2}$$

$$g_d(t, \beta) = \frac{Z(\beta + it)\, Z(\beta - it)}{|Z(\beta)|^2}$$

$$g_c(t, \beta) = g(t, \beta) - g_d(t, \beta)$$

Spectral form factor

The late time behavior of the spectral form factor in the all-to-all SYK is governed by Random Matrix Theory, just as expected from a chaotic system.



Slope, dip, ramp, plateaux. [Cotler et. al.]

At early times

$$g(t,\beta) \approx |Z(\beta+iT)|^2 \qquad g_d$$

At late times

$$g(t,\beta) \approx RMT \qquad g_c$$

• We can ask, how does the connected piece go at early times? Connected contributions can dominate at early times []Berkooz et. al, 2020]

$$= \sum_{m_1, m_2 = 0}^{\infty} \langle Tr(H^{m_1})\, Tr(H^{m_2}) \rangle_J \frac{\beta_1^{m_1}}{m_1!} \frac{\beta_2^{m_2}}{m_2!} (-1)^{m_1 + m_2}$$

At early times,

$$g_c \approx \frac{\epsilon}{2}(\beta^2 + t^2) | \frac{\partial Z(\beta')}{\partial \beta'} |^2_{\beta' = \beta + it}$$

$$\epsilon = \binom{N}{q}^{-1}$$

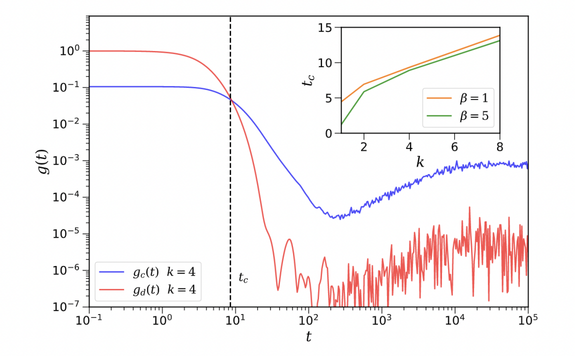
- In the all-to-all SYK we would need $N > 60$ to see this effect $t_{crit} \sim t_{dip}$

$$\rightarrow \text{Sparse SYK}$$

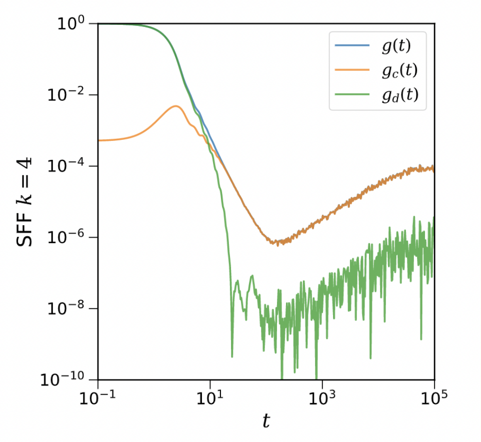
Same analysis as Berkooz et. al. but with random non-Gaussian couplings [R. Feng, G. Tian and D. Wei]

$$\epsilon = \frac{1}{2}\binom{N}{q}^{-1}\left(\frac{3}{p} - 1\right)$$

SFF in Sparse SYK



Connected and disconnected parts, exchange of dominance. $N=30$, $k=4\,q=4$



Spectral form factor $k = 4$, $N = 30$, $q = 4$

Work in progress: OTOCs

Another way of diagnosing chaos: Out of TIme Order Correlators (OTOCs)

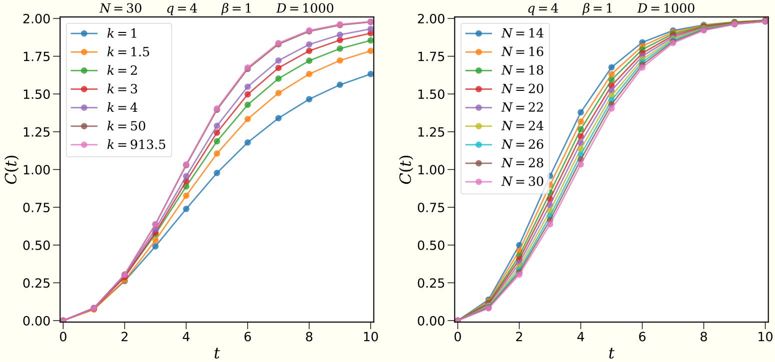
- $C(t) = \langle [W(t), V(0]^2 \rangle = 2 - 2F(t)$
- $$F(t) \equiv \langle W(t)V(0)W(t)V(0) \rangle_\beta$$
- Goal: Lyapunov exponent $\lambda_L \leq \frac{2\pi}{\beta}$
- dependence of λ_L with k?
- λ_L is difficullt to extract numerically by direct fitting.

Improved numerical method [Kobrin et. al] relies on OTOC ansatz for large N

$$F(t) = C_0 + C_1\left(\frac{e^{\lambda t}}{N}\right) + C_2\left(\frac{e^{\lambda t}}{N}\right)^2 + \ldots$$

for $t \lesssim \frac{1}{\lambda}\log N$
$F(t)$ obeys

$$N \to rN, \qquad t \to t + \frac{1}{\lambda}\log r.$$



$N = 30 \quad q = 4 \quad \beta = 1 \quad D = 1000$
$k = 1$
$k = 1.5$
$k = 2$
$k = 3$
$k = 4$
$k = 50$
$k = 913.5$
$C(t)$
t
$q = 4 \quad \beta = 1 \quad D = 1000$
$N = 14$
$N = 16$
$N = 18$
$N = 20$
$N = 22$
$N = 24$
$N = 26$
$N = 28$
$N = 30$
$C(t)$
t

Future directions

Future directions

- Collisions behind the horizon [Haehl and Zhao, 2105.12755, 2202.04661]

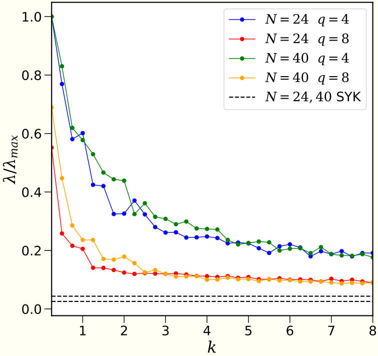
$$\mathscr{F}_6 \sim \frac{\langle W_1 W_1 \mathscr{O}_j \mathscr{O}_j W_2 W_2 \rangle}{\langle W_1 W_1 \rangle \langle \mathscr{O}_j \mathscr{O}_j \rangle \langle W_2 W_2 \rangle}$$

- More on hypergraphs, operator growth.....
- Double scaling limit, dS?
- .....

Thanks!

Adjancency matrix

$$[A]_{ij} = \begin{cases} \text{\# of hyperedges containing vertices } i \text{ and } j & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$



- Second largest eigenvalue λ → Spectral gap
- The spectral gap controls other measures of hypergraph expansion: algebraic entropy and vertex expansion

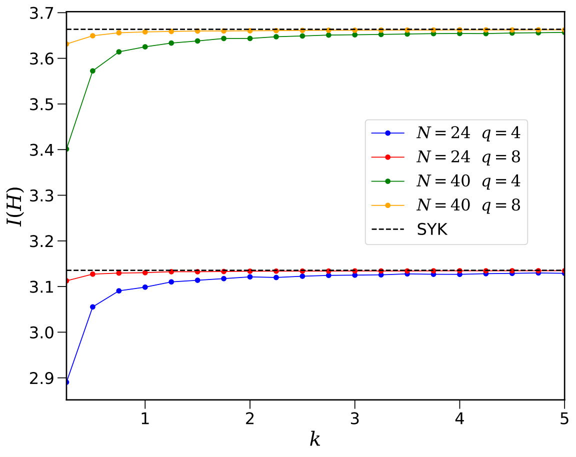
- Algebraic hypergraph entropy
  Consider a hypergraph $H = (V, E)$ and its adjancency matrix $A(H)$. Define

$$D = diag(d_1, d_2, ..., d_N)), \qquad d_i = \sum_{j \in V} A_{ij}.$$

and

$$L(H) = \frac{1}{TrD}(D - A(H)) \qquad \text{with eigenvalues } \nu_i$$

$$\boxed{I(H) = \sum \nu_i \log \nu_i}$$



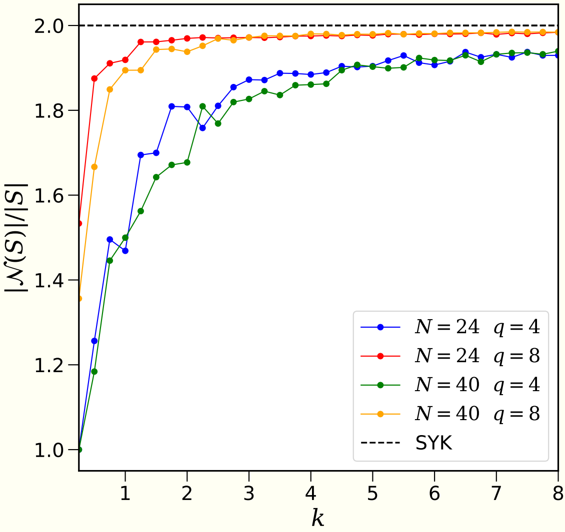
3.7
3.6
3.5
3.4
3.3
3.2
3.1
3.0
2.9
$I(H)$
1
2
3
4
5
k
$N=24 \quad q=4$
$N=24 \quad q=8$
$N=40 \quad q=4$
$N=40 \quad q=8$
SYK

- Vertex expansion
  Consider a subset $S \subset V$. Define its neighborhood

$$\mathcal{N}(S) := \{i : \exists j \in S \text{ such that } \{i,j\} \subseteq e \text{ for some } e \in E\}.$$

Lower bound on vertex expansion [Dumitriu and Zhu, 2019]

$$\boxed{\frac{|\mathcal{N}(S)|}{|S|} \geq \left[1 - \frac{1}{2}\left(1 - \frac{\lambda^2}{r^2(s-1)^2}\right)\right]^{-1}.}$$



2.0
1.8
1.6
1.4
1.2
1.0
$|\mathcal{N}(S)|/|S|$
1
2
3
4
5
6
7
8
k
$N = 24 \quad q = 4$
$N = 24 \quad q = 8$
$N = 40 \quad q = 4$
$N = 40 \quad q = 8$
SYK

Level statistics

Level spacing: $s = \frac{E_{i+1} - E_i}{\Delta}$, Δ: mean level spacing

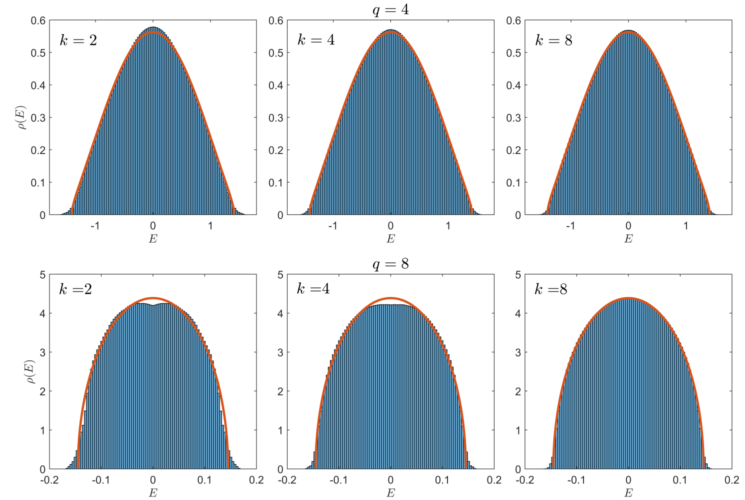
Level spacing distribution: $P(s)$, probability to find consecutive eigenvalues E_i, E_{i+1} at distance s

- For quantum chaotic system:

$$P_W(s) \simeq A_\alpha s^\alpha e^{-B_\alpha s^\alpha}, \quad \alpha = \begin{cases} 1 \text{ GOE} \\ 2 \text{ GUE} \\ 4 \text{ GSE} \end{cases} \qquad \text{(Wigner-surmise)}$$

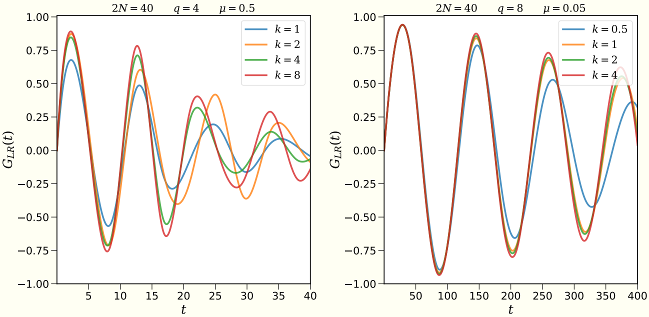
- For integrable system:

$$P_P(s) = e^{-s} \quad \text{(Poisson)}$$


$q = 4$
$k = 2$
$k = 4$
$k = 8$
$\rho(E)$
E
$q = 8$
$k = 2$
$k = 4$
$k = 8$
$\rho(E)$
E

Green's functions

$$G_{ab}(t) = \frac{1}{N} \sum_j 2\mathrm{Re}\langle \chi_a^j(t) \chi_b^j(0) \rangle, \qquad a, b = L, R.$$

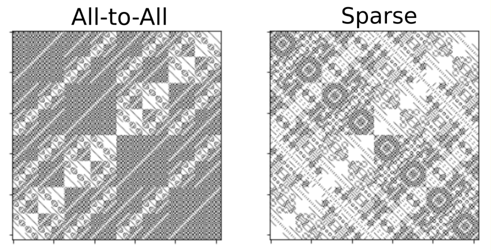


- Sparse SYK similar to original SYK using k of order 1
- Larger q allows us to choose smaller k

Numerical methods

SYK maps to $N/2$-qubit system via **Jordan-Wigner transformation**

$$\chi_{2n} = \left(\prod_{j=1}^{n-1} \sigma_j^x\right) \sigma_n^z, \quad \chi_{2n-1} = \left(\prod_{j=1}^{n-1} \sigma_j^x\right) \sigma_n^y, \qquad \{\chi_i, \chi_j\} = 2\delta_{ij}$$



Numerical methods

Krylov subspace

$$\mathcal{K}_m = \mathrm{span}\{|\psi(t)\rangle, H|\psi(t)\rangle, H^2|\psi(t)\rangle, \ldots, H^{m-1}|\psi(t)\rangle\}$$

Get approximation for time evolution

$$e^{-iH\Delta t}|\psi(t)\rangle \simeq V_m e^{-iV_m H V_m \Delta t} e_1$$

Typicality:

$$\langle \chi_a^j(t)\chi_b^j(0)\rangle = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\chi_a^j(t)\chi_b^j(0)\right] \simeq \frac{\langle\beta|\chi_a^j(t)\chi_b^j(0)|\beta\rangle}{\langle\beta|\beta\rangle}$$

$|\beta\rangle = e^{-\frac{\beta}{2}H}|\psi\rangle, \quad |\psi\rangle$ random state