Curvature invariants for accelerating, rotating and charged black holes with $\Lambda \neq 0$

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- These black hole metrics belong to the most general type D solution of the Einstein-Maxwell equations with a cosmological constant.
- Detailed plotting of the curvature invariants reveal a rich structure of the spacetime geometry surrounding the singularity of a rotating, electrically charged and accelerating black hole. These graphs also help us in an exact mathematical way to explore the interior of these black holes-a *terra incognita*.

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- Computation of explicit algebraic expression for the ZM invariants for accelerating Kerr-Newman black holes in (anti-)de Sitter spacetime .
- The electromagnetic duality anomaly in these curved backgrounds

The Christoffel symbols of the second kind are expressed in the coordinate basis in the form:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}), \tag{1}$$

where the summation convention is adopted and a comma denotes a partial derivative. The Riemann curvature tensor is given by:

$$R^{\kappa}{}_{\lambda\mu\nu} = \Gamma^{\kappa}{}_{\lambda\nu,\mu} - \Gamma^{\kappa}{}_{\lambda\mu,\nu} + \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\kappa}{}_{\alpha\mu} - \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\kappa}{}_{\alpha\nu}. \tag{2}$$

The symmetric Ricci tensor and the Ricci scalar are defined by:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}, \quad R = g^{\alpha\beta} R_{\alpha\beta}, \tag{3}$$

while the Weyl tensor $C_{\kappa\lambda\mu\nu}$ (the trace-free part of the curvature tensor) is given explicitly in terms of the curvature tensor and the metric:

$$\begin{aligned} C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} &+ \frac{1}{2}(R_{\lambda\mu}g_{\kappa\nu} + R_{\kappa\nu}g_{\lambda\mu} - R_{\lambda\nu}g_{\kappa\mu} - R_{\kappa\mu}g_{\lambda\nu}) \\ &+ \frac{1}{6}R(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}). \end{aligned} \tag{4}$$

The definitions of the Zahkary and McIntosh invariants fall into three groups Zahkary & McIntosh, GERG 1997:

Weyl invariants :

$$I_1 = C_{\alpha\beta\gamma\lambda} C^{\alpha\beta\gamma\lambda} = C_{\alpha\beta}^{\kappa\lambda} C_{\kappa\lambda}^{\alpha\beta}, \tag{5}$$

$$I_2 = -C_{\alpha\beta}^{\ \ \mu\nu} C^{*\ \alpha\beta}_{\mu\nu} = -K_2, \tag{6}$$

$$I_3 = C_{\alpha\beta}^{\ \ \mu\nu} C_{\mu\nu}^{\ \ o\rho} C_{o\rho}^{\ \ \alpha\beta}, \tag{7}$$

$$I_4 = -C_{\alpha\beta}^{\ \ \kappa\lambda} C^{*\ o\rho}_{\kappa\lambda} C_{o\rho}^{\ \ \alpha\beta} \tag{8}$$

Ricci invariants :

$$I_5 = R = g_{\alpha\beta} R^{\alpha\beta}, \tag{9}$$

$$I_6 = R_{\alpha\beta} R^{\alpha\beta} = R_{\alpha\beta} g^{\mu\alpha} g^{\lambda\beta} R_{\mu\lambda}, \tag{10}$$

$$I_7 = R_\mu^{\ \nu} R_\nu^{\ \rho} R_\rho^{\ \mu}, \tag{11}$$

$$I_8 = R_\mu^{\ \nu} R_\nu^{\ \rho} R_\rho^{\ \lambda} R_\lambda^{\ \mu} \tag{12}$$

Mixed invariants :

$$I_9 = C_{\alpha\beta\mu}{}^{\nu} R^{\beta\mu} R_{\nu}^{\alpha} = -C^{\nu}{}_{\mu\alpha\beta} R^{\beta\mu} R_{\nu}^{\alpha}, \tag{13}$$

$$I_{10} = -C^{*}_{\alpha\beta\lambda}{}^{\gamma} R^{\beta\lambda} R_{\gamma}^{\alpha}, \tag{14}$$

$$I_{11} = R^{\alpha\beta} R^{\mu\nu} (C_{o\alpha\beta}^{\ \ \ \ \rho} C_{\rho\mu\nu}^{\ \ \ \ o} - C^{*\ \rho}_{o\alpha\beta} C^{*\ o}_{\rho\mu\nu}), \tag{15}$$

$$I_{12} = -R^{\alpha\beta} R^{\mu\nu} (C^{*\ \rho}_{o\alpha\beta} C_{\rho\mu\nu}^{\ \ \ \ o} + C_{o\alpha\beta}^{\ \ \ \ \rho} C^{*\ o}_{\rho\mu\nu}), \tag{16}$$

$$I_{15} = \frac{1}{16} R^{\alpha\beta} R^{\mu\nu} (C_{o\alpha\beta\rho} C^{o}{}_{\mu\nu}{}^{\rho} + C^{*}_{o\alpha\beta\rho} C^{*o}{}_{\mu\nu}{}^{\rho}), \tag{17}$$

$$I_{16} = -\frac{1}{32} R^{\alpha\beta} R^{\mu\nu} \Bigg(C_{o\eta\sigma\rho} C^{o}_{\alpha\beta}{}^{\rho} C^{\eta}{}_{\mu\nu}{}^{\sigma} + C_{o\eta\sigma\rho} C^{*o}{}_{\alpha\beta}{}^{\rho} C^{*\eta}{}_{\mu\nu}{}^{\sigma}$$

$$- C^{*}_{o\eta\sigma\rho} C^{*o}{}_{\alpha\beta}{}^{\rho} C^{\eta}{}_{\mu\nu}{}^{\sigma} + C^{*}_{o\eta\sigma\rho} C^{o}_{\alpha\beta}{}^{\rho} C^{*\eta}{}_{\mu\nu}{}^{\sigma} \Bigg), \tag{18}$$

and

$$I_{17} = \frac{1}{32} R^{\alpha\beta} R^{\mu\nu} \left(C^{*}_{o\kappa\lambda\rho} C^{o}{}_{\alpha\beta}{}^{\rho} C^{\kappa}{}_{\mu\nu}{}^{\lambda} + C^{*}_{o\kappa\lambda\rho} C^{*o}{}_{\alpha\beta}{}^{\rho} C^{*\kappa}{}_{\mu\nu}{}^{\lambda} \right.$$
$$\left. - C_{o\kappa\lambda\rho} C^{*o}{}_{\alpha\beta}{}^{\rho} C^{\kappa}{}_{\mu\nu}{}^{\lambda} + C_{o\kappa\lambda\rho} C^{o}{}_{\alpha\beta}{}^{\rho} C^{*\kappa}{}_{\mu\nu}{}^{\lambda} \right) \tag{19}$$

Here $C^{*}_{\alpha\beta\gamma\delta}$ is the dual of the Weyl tensor, defined by:

$$C^{*}_{\alpha\beta\gamma\delta} = \frac{1}{2} E_{\alpha\beta\kappa\lambda} C^{\kappa\lambda}{}_{\gamma\delta}, \tag{20}$$

where $E_{\alpha\beta\kappa\lambda}$ is the Levi-Civita pseudotensor.
Historically, the first invariant studied was the Kretschmann's scalar:

$$K := R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}. \tag{21}$$

We calculated the ZM curvature invariants in tensorial representation for the specific black holes with Maple™2021.

The Kerr-Newman-de Sitter black hole metric

The spacetime interval for the Kerr-Newman-de Sitter black hole solution (which is of Petrov type D and Segre type [(11)(1,1)]) in Boyer-Lindquist coordinates is ($G = c = 1$):

$$\boxed{\begin{aligned} ds^2 &= \frac{\Delta_r^{KN}}{\Xi^2\rho^2}(dt - a\sin^2\theta d\phi)^2 - \frac{\rho^2}{\Delta_r^{KN}}dr^2 - \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &- \frac{\Delta_\theta\sin^2\theta}{\Xi^2\rho^2}(a dt - (r^2+a^2)d\phi)^2 \end{aligned}} \tag{22}$$

$$\Delta_\theta := 1 + \frac{a^2\Lambda}{3}\cos^2\theta, \quad \Xi := 1 + \frac{a^2\Lambda}{3}, \tag{23}$$

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{24}$$

$$\Delta_r^{KN} := \left(1 - \frac{\Lambda}{3}r^2\right)(r^2+a^2) - 2mr + q^2, \tag{25}$$

This is accompanied by a non-zero electromagnetic field $F = dA$ with vector potential :

$$A = -\frac{qr}{\Xi(r^2+a^2\cos^2\theta)}(dt - a\sin^2\theta d\phi). \tag{26}$$

Result for the Chern-Pontryagin invariant K_2 for the KN(a)dS BH

The Chern-Pontryagin invariant K_2 is also equal to the invariant built from the dual of the Riemmann tensor:

$$K_2 = C^{*}_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} = \frac{1}{2} E^{\alpha\beta\sigma\rho} R_{\sigma\rho}^{\ \ \mu\nu} R_{\alpha\beta\mu\nu} \equiv^{*} \mathbf{R}\cdot\mathbf{R}. \tag{27}$$

$^{*}\mathbf{R}\cdot\mathbf{R}$ has been proposed by Ciufolini to characterise the spacetime geometry and curvature generated by mass-energy currents and by the intrinsic angular momentum of a central body. We computed the invariant $^{*}\mathbf{R}\cdot\mathbf{R} = \frac{1}{2} E^{\alpha\beta\sigma\rho} R_{\sigma\rho}^{\ \ \mu\nu} R_{\alpha\beta\mu\nu}$-the Hirzebruch signature density in closed form for the KN(a)dS BH Kraniotis Class.Quantum Grav. 39 (2022) 145002:

$$\begin{aligned} K_2 = {} & \frac{96a}{\left(r^2 + a^2\cos(\theta)^2\right)^6} \left(\cos(\theta)^3 a^2 m - 3\cos(\theta)\, m\, r^2 + 2\cos(\theta)\, q^2 r\right) \\ & \times \left(-3a^2\cos(\theta)^2\, mr + a^2\cos(\theta)^2\, q^2 + m\, r^3 - q^2 r^2\right). \end{aligned} \tag{28}$$

The Euler-Poincare invariant for the Kerr-Newman-(anti-)de Sitter black hole

The Hirzebruch signature density, K_2, is of course an example of a topological invariant. Another interesting topological invariant, besides K_2, is the quantity constructed from the doubly dual curvature tensor:

$$K_{\mathrm{Euler}} = \frac{1}{4} E^{\alpha\beta\gamma\delta} E^{\mu\nu\rho\sigma} R_{\alpha\beta\mu\nu} R_{\gamma\delta\rho\sigma}. \tag{29}$$

The topological invariant K_{Euler} is essentially Euler's density whose integral over spacetime measure gives the so called Euler-Poincare characteristic χ Indeed, the Euler-Poincare characteristic in four dimensions is: $\chi = \int \frac{-1}{128\pi^2} \sqrt{-g} K_{\mathrm{Euler}} \mathrm{d}^4 x$.

The topological Euler invariant has been studied in relation to Weyl conformal anomaly in four derivative theories such as conformal gravity and conformal supergravity Duff 2020 as well as in the context of boundary conformal invariants in five dimensions Astaneh & Solodukhin 2021

The Euler-Poincare invariant for the Kerr-Newman-(anti-)de Sitter black hole

We calculated the invariant $K_{\rm Euler}$ for the Kerr-Newman-(anti-)de Sitter black hole spacetime. The novel explicit algebraic expression of the Euler invariant we computed is:

$$
\begin{aligned}
K_{\rm Euler} = {} & \frac{1}{3\left(r^2+a^2\cos(\theta)^2\right)^6}\Bigg(-8\Lambda^2\cos(\theta)^{12}a^{12}-48\Lambda^2\cos(\theta)^{10}a^{10}r^2 \\
& -120\Lambda^2\cos(\theta)^8a^8r^4+\left(-160a^6\Lambda^2r^6+144a^6m^2\right)\cos(\theta)^6 \\
& +\left(-120\Lambda^2a^4r^8-2160a^4m^2r^2+1440a^4m\,q^2r-120q^4a^4\right)\cos(\theta)^4 \\
& -2160\left(\frac{1}{45}\Lambda^2a^2r^8-a^2m^2r^2+\frac{4}{3}a^2m\,q^2r-\frac{19}{45}a^2q^4\right)r^2\cos(\theta)^2 \\
& -8r^{12}\Lambda^2-144m^2r^6+288m\,q^2r^5-120q^4r^4\Bigg).
\end{aligned}
\tag{30}
$$

Results for the ZM invariants for the KN(a)dS BH

$$I_1 = \frac{48}{\left(r^2+a^2\cos(\theta)^2\right)^6}\Big(-a^3 m\cos(\theta)^3 + \left(-3a^2 mr + a^2q^2\right)\cos(\theta)^2 + \left(3am\,r^2 - 2a\,q^2 r\right)\cos(\theta)$$

$$+\left(mr - q^2\right)r^2\Big)$$

$$\times\left(a^3 m\cos(\theta)^3 + \left(-3a^2 mr + a^2q^2\right)\cos(\theta)^2 + \left(-3am\,r^2 + 2a\,q^2 r\right)\cos(\theta) - \left(-mr + q^2\right)r^2\right), \tag{31}$$

$$I_3 = \frac{96}{\left(r^2+a^2\cos(\theta)^2\right)^9}\Big(-3\cos(\theta)^6 a^6 m^2 + \left(27a^4m^2r^2 - 18a^4 m\,q^2 r + a^4q^4\right)\cos(\theta)^4$$

$$+\left(-33a^2m^2r^4 + 44a^2 m\,q^2 r^3 - 14a^2q^4r^2\right)\cos(\theta)^2 + r^6m^2 - 2m\,q^2r^5 + q^4r^4\Big)$$

$$\times\left(\left(-3a^2 mr + a^2q^2\right)\cos(\theta)^2 + m\,r^3 - q^2r^2\right) \tag{32}$$

$$I_4 = -\frac{864a}{\left(r^2+a^2\cos(\theta)^2\right)^9}\left(\frac{\cos(\theta)^3 a^2 m}{3} + \left(-m\,r^2 + \frac{2}{3}q^2 r\right)\cos(\theta)\right)$$

$$\times\Bigg[-\frac{\cos(\theta)^6 a^6 m^2}{3} + \left(11a^4m^2r^2 - \frac{22}{3}a^4 m\,q^2 r + a^4q^4\right)\cos(\theta)^4$$

$$+\left(-9a^2m^2r^4 + 12a^2 m\,q^2r^3 - \frac{10}{3}a^2q^4r^2\right)\cos(\theta)^2 - \frac{r^4\left(-3m^2r^2 + 6m\,q^2 r - 3q^4\right)}{3}\Bigg], \tag{33}$$

$$I_5 = R = 4\Lambda, \tag{34}$$

$$I_7 = \frac{4\left(\Lambda^2\cos(\theta)^8 a^8 + 4\Lambda^2\cos(\theta)^6 a^6 r^2 + 6\Lambda^2\cos(\theta)^4 a^4 r^4 + 4\Lambda^2\cos(\theta)^2 a^2 r^6 + \Lambda^2 r^8 + 3q^4\right)\Lambda}{\left(r^2 + a^2\cos(\theta)^2\right)^4}, \tag{35}$$

$$\begin{aligned} I_8 = {} & \frac{1}{\left(r^2 + a^2\cos(\theta)^2\right)^8}\Bigg(4\Lambda^4\cos(\theta)^{16} a^{16} + 32\Lambda^4\cos(\theta)^{14} a^{14} r^2 + 112\Lambda^4\cos(\theta)^{12} a^{12} r^4 \\ & + 224\Lambda^4\cos(\theta)^{10} a^{10} r^6 + \left(280\Lambda^4 a^8 r^8 + 24\Lambda^2 a^8 q^4\right)\cos(\theta)^8 + 96r^2\left(\frac{7}{3}a^6\Lambda^4 r^8 + a^6 q^4\Lambda^2\right)\cos(\theta)^6 \\ & + \left(112\Lambda^4 a^4 r^{12} + 144\Lambda^2 a^4 q^4 r^4\right)\cos(\theta)^4 + 96r^2\left(\frac{1}{3}\Lambda^4 a^2 r^{12} + \Lambda^2 a^2 q^4 r^4\right)\cos(\theta)^2 \\ & + 4\Lambda^4 r^{16} + 24\Lambda^2 q^4 r^8 + 4q^8\Bigg), \end{aligned} \tag{36}$$

$$I_6 = \frac{4q^4}{\left(r^2 + a^2\cos(\theta)^2\right)^4} + 4\Lambda^2, \tag{37}$$

$$I_9 = -\frac{16\left(-3a^2\cos(\theta)^2 mr + a^2\cos(\theta)^2 q^2 + m\,r^3 - q^2 r^2\right)q^4}{\left(r^2 + a^2\cos(\theta)^2\right)^7}, \tag{38}$$

$$I_{10} = \frac{16aq^4}{\left(r^2 + a^2\cos(\theta)^2\right)^7}\left(\cos(\theta)^3 a^2 m - 3\cos(\theta)\, m\, r^2 + 2\cos(\theta)\, q^2 r\right), \tag{39}$$

$$\begin{aligned} I_{11} = {} & \frac{64q^4}{\left(r^2 + a^2\cos(\theta)^2\right)^{10}}\Big[\cos(\theta)^3 a^3 m + \left(-3a^2 mr + a^2 q^2\right)\cos(\theta)^2 + \left(-3am\, r^2 + 2a\, q^2 r\right)\cos(\theta) \\ & - r^2\left(-mr + q^2\right)\Big] \\ & \times \left(-\cos(\theta)^3 a^3 m + \left(-3a^2 mr + a^2 q^2\right)\cos(\theta)^2 + \left(3am\, r^2 - 2a\, q^2 r\right)\cos(\theta) + \left(mr - q^2\right) r^2\right), \end{aligned} \tag{40}$$

$$\begin{aligned} I_{12} = {} & -\frac{128aq^4}{\left(r^2 + a^2\cos(\theta)^2\right)^{10}}\left(\cos(\theta)^3 a^2 m - 3\cos(\theta)\, m\, r^2 + 2\cos(\theta)\, q^2 r\right) \\ & \times \left(-3a^2\cos(\theta)^2\, mr + a^2\cos(\theta)^2\, q^2 + m\, r^3 - q^2 r^2\right), \end{aligned} \tag{41}$$

$$I_{15} = \frac{4q^4\left(\cos(\theta)^2 a^2 m^2 + r^2 m^2 - 2m\, q^2 r + q^4\right)}{\left(r^2 + a^2\cos(\theta)^2\right)^8}, \tag{42}$$

$$I_{16} = -\frac{24q^4\left(\cos(\theta)^2 a^2 m^2 + \left(mr - q^2\right)^2\right)\left(a^2\left(mr - \frac{q^2}{3}\right)\cos(\theta)^2 - \frac{r^2\left(mr - q^2\right)}{3}\right)}{\left(r^2 + a^2\cos(\theta)^2\right)^{11}}, \tag{43}$$

$$I_{17} = -\frac{24 a\, q^4}{\left(r^2 + a^2 \cos(\theta)^2\right)^{11}} \left(\frac{\cos(\theta)^3 a^2 m}{3} + \left(-m\, r^2 + \frac{2}{3} q^2 r\right)\cos(\theta)\right)\left(\cos(\theta)^2 a^2 m^2 + (mr - q^2)^2\right). \tag{44}$$

We summarise our results as follows:

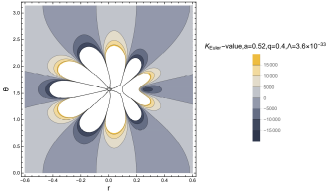
Theorem

The exact algebraic expressions for the curvature invariants calculated for the Kerr-Newman-(anti-)de Sitter metric are given in Equations (31)-(44) and (28).

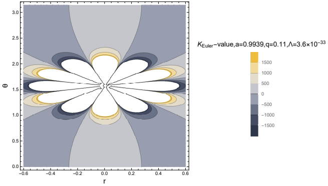
Theorem

The Kretschmann invariant K for the KN(a)dS black hole is given by the expression:

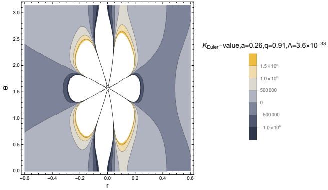
$$K^{KN(a)dS} = \frac{1}{3\left(r^2 + a^2\cos(\theta)^2\right)^6}\Bigg(8\Lambda^2\cos(\theta)^{12} a^{12} + 48\Lambda^2\cos(\theta)^{10} a^{10} r^2 + 120\Lambda^2 \cos(\theta)^8 a^8 r^4$$
$$+ \left(160\Lambda^2 a^6 r^6 - 144 a^6 m^2\right)\cos(\theta)^6 + 2160\left(\frac{1}{18} r^8\Lambda^2 + m^2 r^2 - \frac{2}{3} m\, q^2 r + \frac{7}{90} q^4\right) a^4 \cos(\theta)^4$$
$$- 2160\left(-\frac{1}{45} r^8 \Lambda^2 + m^2 r^2 - \frac{4}{3} m\, q^2 r + \frac{17}{45} q^4\right) a^2 r^2 \cos(\theta)^2 + 8\Lambda^2 r^{12} + 144 m^2 r^6 - 288 m\, q^2 r^5$$
$$+ 168 q^4 r^4\Bigg). \tag{45}$$



(a) Contour plot of K_{Euler}.

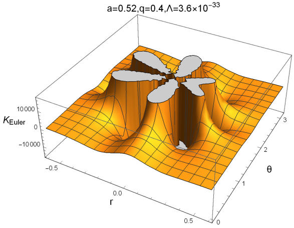


(b) Contour plot of K_{Euler}

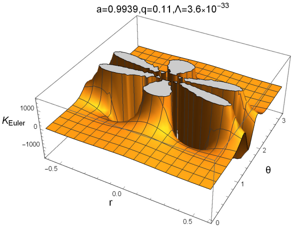


(c) Contour Plot of K_{Euler}.

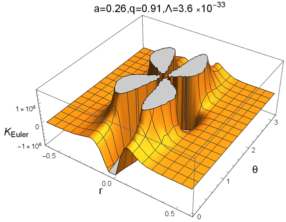
Figure: Contour plots of K_{Euler}. (a) for spin parameter $a = 0.52$, charge $q = 0.4$,dimensionless cosmological parameter $\Lambda = 3.6 \times 10^{-33}$, $m = 1$. (b) For spin $a = 0.9939$, charge $q = 0.11$,dimensionless cosmological parameter $\Lambda = 3.6 \times 10^{-33}$, $m = 1$. (c) For low spin $a = 0.26$, electric charge $q = 0.91$,dimensionless cosmological parameter $\Lambda = 3.6 \times 10^{-33}$, $m = 1$.



(a) 3D plot of K_{Euler}.

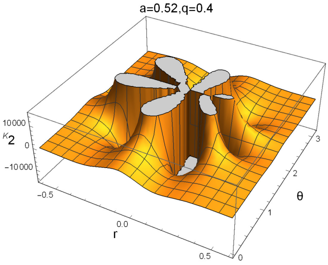


(b) 3D plot of K_{Euler}

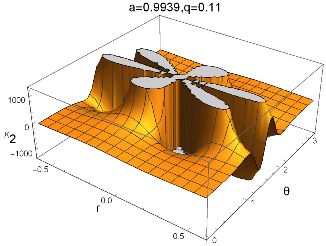


(c) 3D Plot of K_{Euler}.

Figure: 3D plots of the Euler invariant, K_{Euler}, plotted as a function of the Boyer-Lindquist coordinates r and θ . (a) for spin parameter $a = 0.52$, charge $q = 0.4$,dimensionless cosmological parameter $\Lambda = 3.6 \times 10^{-33}$, $m = 1$. (b) For spin $a = 0.9939$, charge $q = 0.11$,dimensionless cosmological parameter $\Lambda = 3.6 \times 10^{-33}$, $m = 1$. (c) For low spin $a = 0.26$, electric charge $q = 0.91$,dimensionless cosmological parameter $\Lambda = 3.6 \times 10^{-33}$ and mass $m = 1$.

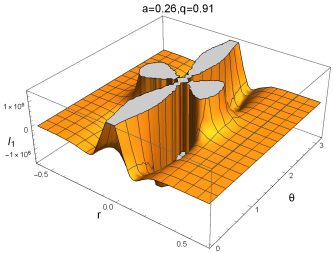


(a) 3D plot of K_2.

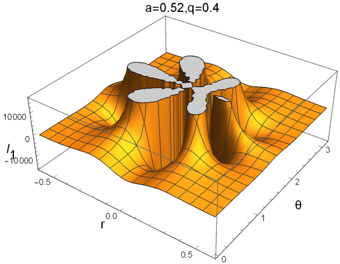


(b) 3D Plot of K_2.

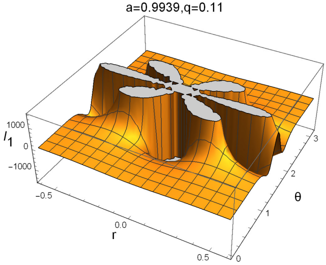
Figure: 3D plots of the Chern-Pontryagin invariant, K_2, plotted as a function of the Boyer-Lindquist coordinates r and θ . (a) for spin parameter $a = 0.52$, charge $q = 0.4$, mass $m = 1$. (b) For spin $a = 0.9939$, charge $q = 0.11$, mass $m = 1$.



(a) 3D plot of I_1.



(b) 3D plot of I_1



(c) 3D Plot of I_1.

The Newman-Penrose (NP) formalism is a tetrad formalism with a special choice of the tetrad in terms the null vectors $\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}$. In the NP formalism, the ten independent components of the Weyl tensor are determined by the five complex scalar functions defined as:

$$\boxed{\begin{aligned}\Psi_0 &= C_{\mu\nu\lambda\sigma} l^\mu m^\nu l^\lambda m^\sigma, \\ \Psi_1 &= C_{\mu\nu\lambda\sigma} l^\mu n^\nu l^\lambda m^\sigma, \\ \Psi_2 &= C_{\mu\nu\lambda\sigma} \overline{m}^\mu n^\nu l^\lambda m^\sigma, \\ \Psi_3 &= C_{\mu\nu\lambda\sigma} \overline{m}^\mu n^\nu l^\lambda n^\sigma, \\ \Psi_4 &= C_{\mu\nu\lambda\sigma} \overline{m}^\mu n^\nu \overline{m}^\lambda n^\sigma.\end{aligned}} \tag{46}$$

Two particularly useful complex scalar polynomial invariants for a vacuum spacetime are given in terms of the Weyl tensor components by Podolský & Griffiths 2009:

$$\mathbf{I} = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2, \;\; \mathbf{J} = \det\Psi, \;\; \Psi = \begin{bmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{bmatrix}. \tag{47}$$

For our computations we use the generalised Kinnersley null tetrad used in Kraniotis 2019:

$$l^{\mu} = \left[\frac{(r^2+a^2)\Xi}{\Delta_r^{KN}}, 1, 0, \frac{a\Xi}{\Delta_r^{KN}} \right], \;\; n^{\mu} = \left[\frac{\Xi(r^2+a^2)}{2\rho^2}, -\frac{\Delta_r^{KN}}{2\rho^2}, 0, \frac{a\Xi}{2\rho^2} \right]$$

$$m^{\mu} = \frac{1}{(r+ia\cos\theta)\sqrt{2\Delta_\theta}} \left[ia\Xi\sin\theta, 0, \Delta_\theta, \frac{i\Xi}{\sin\theta} \right]$$

$$\overline{m}^{\mu} = \frac{-1}{(r-ia\cos\theta)\sqrt{2\Delta_\theta}} \left[ia\Xi\sin\theta, 0, -\Delta_\theta, \frac{i\Xi}{\sin\theta} \right] \tag{48}$$

where we computed for the Ricci scalars:

$$\Phi_{00} \equiv \frac{1}{2} R_{\mu\nu} l^{\mu} l^{\nu} = 0, \;\; \Phi_{01} \equiv \frac{1}{2} l^{\mu} m^{\nu} = \overline{\Phi}_{10} = 0, \tag{49}$$

$$\Phi_{02} \equiv \frac{1}{2} R_{\mu\nu} m^{\mu} m^{\nu} = \overline{\Phi}_{20} = 0, \;\; \Phi_{22} \equiv \frac{1}{2} R_{\mu\nu} n^{\mu} n^{\nu} = 0, \tag{50}$$

$$\Phi_{12} \equiv \frac{1}{2} R_{\mu\nu} n^{\mu} m^{\nu} = \overline{\Phi}_{21} = 0, \tag{51}$$

$$\Phi_{11} \equiv \frac{1}{4} R_{\mu\nu} (l^{\mu} n^{\nu} + m^{\mu} \overline{m}^{\nu}) = \frac{q^2}{2(r^2+a^2\cos(\theta)^2)^2} \tag{52}$$

The only non-zero Weyl scalar is Ψ_2:

$$\boxed{\Psi_2 = -\frac{i\cos(\theta)ma + mr - q^2}{(r - ia\cos(\theta))^3(r + ia\cos(\theta))}.} \tag{53}$$

A non-trivial check of our analytic computations performed in the tensorial representation of the ZM invariants, provided with the aid of the NP formalism, is the following equation that relates the Weyl invariant I_1 and the Chern-Potryagin invariant K_2 with the Weyl scalar Ψ_2 computed in eqn(53):

$$\mathbb{I} := I_1 - iK_2 = 16.3\Psi_2^2 = 16\mathbf{I}. \tag{54}$$

We also derived the following relation:

$$\mathbb{J} := I_3 + iI_4 = 96(-\Psi_2)^3 = 6.16(-\Psi_2)^3 = 96\mathbf{J}. \tag{55}$$

The metric for an accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime

The Plebański-Demiański metric covers a large family of solutions which include the physically most significant case: that of an accelerating, rotating and charged black hole with $\Lambda \neq 0$. We focus on the following metric that describes an accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime Podolský & Griffiths 2006:

$$\boxed{\begin{aligned} \mathrm{d}s^2 = \frac{1}{\Omega^2}\Bigg\{ & -\frac{Q}{\rho^2}\left[\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi\right]^2 + \frac{\rho^2}{Q}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 \\ & + \frac{P}{\rho^2}\sin^2\theta\left[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\right]^2\Bigg\}, \end{aligned}} \tag{56}$$

$$\Omega = 1 - \alpha r\cos\theta, \tag{57}$$

$$P = 1 - 2\alpha m\cos\theta + (\alpha^2(a^2+q^2) + \frac{1}{3}\Lambda a^2)\cos^2\theta, \tag{58}$$

$$Q = ((a^2+q^2) - 2mr + r^2)(1-\alpha^2 r^2) - \frac{1}{3}\Lambda(a^2+r^2)r^2, \tag{59}$$

and α is the acceleration of the black hole.

We first compute the Chern-Pontryagin invariant K_2 for an accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime. The analytic explicit result for this fundamental invariant is:

$$K_2 = \frac{96a\left(\alpha r\cos(\theta)-1\right)^6}{\left(r^2+a^2\cos(\theta)^2\right)^6}\Big(\cos(\theta)^3 a^4\alpha m+\cos(\theta)^3 a^2\alpha q^2 r$$
$$-3\cos(\theta)a^2\alpha m r^2-\cos(\theta)\alpha q^2 r^3-3a^2\cos(\theta)^2 mr+a^2\cos(\theta)^2 q^2+m r^3-q^2 r^2\Big)$$
$$\times\left(3\cos(\theta)^2 a^2\alpha mr+2\cos(\theta)^2\alpha q^2 r^2+\cos(\theta)^3 a^2 m-\alpha r^3 m-3\cos(\theta) m r^2+2\cos(\theta) q^2 r\right) \tag{60}$$

$$I_1 = \frac{1}{\left(r^2+a^2\cos(\theta)^2\right)^6} 48 \Big(\left(q^2 r\alpha + am\left(a\alpha-1\right)\right) a^2\cos(\theta)^3 + \left(-2a\alpha q^2 r^2 - 3a^2 m\left(a\alpha+1\right) r + a^2 q^2\right)\cos(\theta)^2$$

$$+ \left(-\alpha q^2 r^3 - 3am\left(a\alpha-1\right) r^2 - 2a\, q^2 r\right)\cos(\theta) + \left(m\left(a\alpha+1\right) r - q^2\right) r^2 \Big)$$

$$\times \Big(a^2\left(q^2 r\alpha + am\left(a\alpha+1\right)\right)\cos(\theta)^3 + \left(2a\alpha q^2 r^2 + 3a^2 m\left(a\alpha-1\right) r + a^2 q^2\right)\cos(\theta)^2$$

$$+ \left(-\alpha q^2 r^3 - 3am\left(a\alpha+1\right) r^2 + 2a\, q^2 r\right)\cos(\theta) - \left(m\left(a\alpha-1\right) r + q^2\right) r^2 \Big) \left(\alpha r\cos(\theta)-1\right)^6, \tag{61}$$

$$I_3 = -\frac{96\left(\alpha r\cos(\theta)-1\right)^9}{\left(r^2+a^2\cos(\theta)^2\right)^9} \Big(a^4\left(q^4 r^2\alpha^2 + 2a^2 m\, q^2 r\alpha^2 + a^2 m^2\left(a^2\alpha^2-3\right)\right)\cos(\theta)^6$$

$$-24\left(\frac{3m\, q^2 r^2}{4} + \left(a^2 m^2 - \frac{q^4}{12}\right) r - \frac{a^2 m\, q^2}{12}\right) a^4\alpha\cos(\theta)^5 + \Big[-14a^2\alpha^2 q^4 r^4 - 44a^4 m\, q^2\alpha^2 r^3$$

$$+ \left(-33a^6\alpha^2 m^2 + 27a^4 m^2\right) r^2 - 18a^4 m\, q^2 r + a^4 q^4\Big]\cos(\theta)^4$$

$$+80r^2\left(\frac{11m\, q^2 r^2}{20} + \left(a^2 m^2 - \frac{7q^4}{20}\right) r - \frac{11a^2 m\, q^2}{20}\right) a^2\alpha\cos(\theta)^3$$

$$+ \left(q^4\alpha^2 r^6 + 18a^2\alpha^2 m\, q^2 r^5 + \left(27a^4\alpha^2 m^2 - 33a^2 m^2\right) r^4 + 44a^2 m\, q^2 r^3 - 14a^2 q^4 r^2\right)\cos(\theta)^2$$

$$-24r^4\left(\frac{m\, q^2 r^2}{12} + \left(a^2 m^2 - \frac{q^4}{12}\right) r - \frac{3a^2 m\, q^2}{4}\right)\alpha\cos(\theta)$$

$$+ \left(-3a^2\alpha^2 m^2 + m^2\right) r^6 - 2m\, q^2 r^5 + q^4 r^4 \Big)$$

$$\times \left(a^2\alpha\left(a^2 m + q^2 r\right)\cos(\theta)^3 + \left(-3a^2 mr + a^2 q^2\right)\cos(\theta)^2 + \left(-3a^2\alpha m\, r^2 - \alpha q^2 r^3\right)\cos(\theta) + m\, r^3 - q^2 r^2\right) \tag{62}$$

$$I_4 = \frac{864\,(\alpha r\cos(\theta)-1)^9\, a}{\left(r^2+a^2\cos(\theta)^2\right)^9}\left(\frac{\cos(\theta)^3\, a^2 m}{3} + r\alpha\left(a^2 m + \frac{2q^2 r}{3}\right)\cos(\theta)^2 + \left(-m\,r^2 + \frac{2}{3}q^2 r\right)\cos(\theta) - \frac{\alpha r^3 m}{3}\right)$$
$$\times\Bigg[a^4\left(\alpha^2 q^4 r^2 + 2a^2\alpha^2 m\,q^2 r + a^2 m^2\left(a^2\alpha^2 - \frac{1}{3}\right)\right)\cos(\theta)^6 - 8a^4\left(\frac{11m\,q^2 r^2}{12} + \left(a^2 m^2 - \frac{q^4}{4}\right) r\right.$$
$$\left. - \frac{a^2 m\,q^2}{4}\right)\alpha\cos(\theta)^5 + \left(-\frac{10a^2\alpha^2 q^4 r^4}{3} - 12a^4\alpha^2 m\,q^2 r^3 + \left(-9a^6\alpha^2 m^2 + 11a^4 m^2\right) r^2\right.$$
$$\left. - \frac{22a^4 m\,q^2 r}{3} + a^4 q^4\right)\cos(\theta)^4 + \frac{80r^2 a^2\left(\frac{9m\,q^2 r^2}{20} + \left(a^2 m^2 - \frac{q^4}{4}\right) r - \frac{9a^2 m\,q^2}{20}\right)\alpha\cos(\theta)^3}{3} + \left(\alpha^2 q^4 r^6\right.$$
$$\left. + \frac{22a^2\alpha^2 m\,q^2 r^5}{3} + \left(11a^4\alpha^2 m^2 - 9a^2 m^2\right) r^4 + 12a^2 m\,q^2 r^3 - \frac{10a^2 q^4 r^2}{3}\right)\cos(\theta)^2 - 8r^4\alpha\left(\frac{m\,q^2 r^2}{4}\right.$$
$$\left. + \left(a^2 m^2 - \frac{q^4}{4}\right) r - \frac{11a^2 m\,q^2}{12}\right)\cos(\theta) - \frac{r^4\left(m^2\left(a^2\alpha^2-3\right) r^2 + 6m\,q^2 r - 3q^4\right)}{3}\Bigg], \tag{63}$$

$$I_5 = 4\Lambda, \tag{64}$$

$$I_6 = \frac{4}{\left(r^2+a^2\cos(\theta)^2\right)^4}\Big(\cos(\theta)^8\,\alpha^8 q^4 r^8 - 8\cos(\theta)^7\,\alpha^7 q^4 r^7 + 28\cos(\theta)^6\,\alpha^6 q^4 r^6$$
$$- 56\cos(\theta)^5\,\alpha^5 q^4 r^5 + \Lambda^2\cos(\theta)^8\, a^8 + 4\Lambda^2\cos(\theta)^6\, a^6 r^2 + 70\cos(\theta)^4\,\alpha^4 q^4 r^4 + 6\Lambda^2\cos(\theta)^4\, a^4 r^4$$
$$- 56\cos(\theta)^3\,\alpha^3 q^4 r^3 + 4\Lambda^2\cos(\theta)^2\, a^2 r^6 + \Lambda^2 r^8 + 28\cos(\theta)^2\,\alpha^2 q^4 r^2 - 8\cos(\theta)\,\alpha q^4 r + q^4\Big), \tag{65}$$

$$I_7 = \frac{4}{\left(r^2 + a^2 \cos(\theta)^2\right)^4} \left(3 \cos(\theta)^8 \alpha^8 q^4 r^8 - 24 \cos(\theta)^7 \alpha^7 q^4 r^7 + 84 \cos(\theta)^6 \alpha^6 q^4 r^6 - 168 \cos(\theta)^5 \alpha^5 q^4 r^5\right.$$
$$+ \Lambda^2 \cos(\theta)^8 a^8 + 4\Lambda^2 \cos(\theta)^6 a^6 r^2 + 210 \cos(\theta)^4 \alpha^4 q^4 r^4 + 6\Lambda^2 \cos(\theta)^4 a^4 r^4$$
$$\left. - 168 \cos(\theta)^3 \alpha^3 q^4 r^3 + 4\Lambda^2 \cos(\theta)^2 a^2 r^6 + \Lambda^2 r^8 + 84 \cos(\theta)^2 \alpha^2 q^4 r^2 - 24 \cos(\theta) \alpha q^4 r + 3q^4 \right) \Lambda, \tag{66}$$

$$I_9 = \frac{16q^4}{\left(r^2 + a^2 \cos(\theta)^2\right)^7} (\alpha r \cos(\theta) - 1)^{11} \left(\cos(\theta)^3 a^4 \alpha m + \cos(\theta)^3 a^2 \alpha q^2 r\right.$$
$$\left. - 3 \cos(\theta) a^2 \alpha m r^2 - \cos(\theta) \alpha q^2 r^3 - 3a^2 \cos(\theta)^2 m r + a^2 \cos(\theta)^2 q^2 + m r^3 - q^2 r^2 \right), \tag{67}$$

$$
\begin{aligned}
I_8 = {} & \frac{1}{\left(r^2 + a^2 \cos(\theta)^2\right)^8} \\
& \times \Bigg(\left(4\alpha^{16} q^8 r^{16} + 24\Lambda^2 a^8 \alpha^8 q^4 r^8 + 4\Lambda^4 a^{16}\right) \cos(\theta)^{16} + \left(-64\alpha^{15} q^8 r^{15} - 192\Lambda^2 a^8 \alpha^7 q^4 r^7\right) \cos(\theta)^{15} \\
& + 32 r^2 \left(15\alpha^{14} q^8 r^{12} + 3\Lambda^2 a^6 \alpha^8 q^4 r^8 + 21\Lambda^2 a^8 \alpha^6 q^4 r^4 + \Lambda^4 a^{14}\right) \cos(\theta)^{14} - 1344 r^5 \alpha^5 q^4 \Bigg[\frac{5}{3} q^4 r^8 \alpha^8 + \frac{4}{7} a^6 r^4 \alpha^2 \Lambda^2 \\
& + a^8 \Lambda^2\Bigg] \cos(\theta)^{13} + 112 r^4 \left(\left(65 q^8 \alpha^{12} + \frac{9}{7} a^4 q^4 \alpha^8 \Lambda^2\right) r^8 + 24 a^6 q^4 r^4 \alpha^6 \Lambda^2 + a^8 \Lambda^2 \left(15 q^4 \alpha^4 + a^4 \Lambda^2\right)\right) \cos(\theta)^{12} \\
& - 1344 r^3 \alpha^3 \left(\left(13 q^4 \alpha^8 + \frac{6}{7} a^4 \alpha^4 \Lambda^2\right) r^8 + 4 a^6 r^4 \alpha^2 \Lambda^2 + a^8 \Lambda^2\right) q^4 \cos(\theta)^{11} + 224 r^2 \Bigg[\frac{3\Lambda^2 a^2 \alpha^8 q^4 r^{12}}{7} \\
& + \left(143 q^8 \alpha^{10} + 18\Lambda^2 a^4 \alpha^6 q^4\right) r^8 + a^6 \Lambda^2 \left(30 q^4 \alpha^4 + a^4 \Lambda^2\right) r^4 + 3 a^8 q^4 \alpha^2 \Lambda^2\Bigg] \cos(\theta)^{10} - 192 r \alpha q^4 \Bigg[4 a^2 r^{12} \alpha^6 \Lambda^2 \\
& + \left(\frac{715}{3} q^4 \alpha^8 + 42 a^4 \alpha^4 \Lambda^2\right) r^8 + 28 a^6 r^4 \alpha^2 \Lambda^2 + a^8 \Lambda^2\Bigg] \cos(\theta)^9 + \Bigg[24\Lambda^2 \alpha^8 q^4 r^{16} + 2688\Lambda^2 a^2 \alpha^6 q^4 r^{12} \\
& + \left(51480 q^8 \alpha^8 + 10080\Lambda^2 a^4 \alpha^4 q^4 + 280\Lambda^4 a^8\right) r^8 + 2688\Lambda^2 a^6 \alpha^2 q^4 r^4 + 24\Lambda^2 a^8 q^4\Bigg] \cos(\theta)^8 \\
& - 768 r^3 \alpha q^4 \left(\frac{r^{12} \alpha^6 \Lambda^2}{4} + 7 a^2 r^8 \alpha^4 \Lambda^2 + \left(\frac{715}{12} q^4 \alpha^6 + \frac{21}{2} a^4 \alpha^2 \Lambda^2\right) r^4 + a^6 \Lambda^2\right) \cos(\theta)^7 + 96 r^2 \Bigg[7 q^4 r^{12} \alpha^6 \Lambda^2 \\
& + \left(70 a^2 q^4 \alpha^4 \Lambda^2 + \frac{7}{3} a^6 \Lambda^4\right) r^8 + \left(\frac{1001}{3} q^8 \alpha^6 + 42 a^4 q^4 \alpha^2 \Lambda^2\right) r^4 + a^6 q^4 \Lambda^2\Bigg] \cos(\theta)^6 - 1152 r^5 \alpha \Bigg[\frac{7}{6} r^8 \alpha^4 \Lambda^2 \\
& + \frac{14}{3} a^2 r^4 \alpha^2 \Lambda^2 + \frac{91}{6} q^4 \alpha^4 + a^4 \Lambda^2\Bigg] q^4 \cos(\theta)^5 + \Bigg(1680\Lambda^2 \alpha^4 q^4 r^{12} + 112\Lambda^4 a^4 r^{12} + 2688\Lambda^2 a^2 \alpha^2 q^4 r^8 \\
& + 7280\alpha^4 q^8 r^4 + 144\Lambda^2 a^4 q^4 r^4\Bigg) \cos(\theta)^4 - 768 r^3 \left(\frac{7}{4} r^8 \alpha^2 \Lambda^2 + a^2 r^4 \Lambda^2 + \frac{35}{12} q^4 \alpha^2\right) \alpha q^4 \cos(\theta)^3 \\
& + 96 r^2 \left(\frac{1}{3} \Lambda^4 a^2 r^{12} + 7 q^4 r^8 \alpha^2 \Lambda^2 + \Lambda^2 a^2 q^4 r^4 + 5 q^8 \alpha^2\right) \cos(\theta)^2 - 64 q^4 r \alpha \left(3\Lambda^2 r^8 + q^4\right) \cos(\theta) + 4\Lambda^4 r^{16} \\
& + 24\Lambda^2 q^4 r^8 + 4 q^8\Bigg), \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
I_{10} = {} & -\frac{16 a q^4 \left(\alpha r \cos(\theta) - 1\right)^{11}}{\left(r^2 + a^2 \cos(\theta)^2\right)^7} \Bigg(3 \cos(\theta)^2 a^2 \alpha m r + 2 \cos(\theta)^2 \alpha q^2 r^2 + \cos(\theta)^3 a^2 m \\
& - \alpha r^3 m - 3 \cos(\theta)\, m r^2 + 2 \cos(\theta)\, q^2 r\Bigg), \qquad (69)
\end{aligned}
$$

$$I_{11}=\frac{64q^4\left(\alpha r\cos(\theta)-1\right)^{14}}{\left(r^2+a^2\cos(\theta)^2\right)^{10}}\Big[\left(q^2r\alpha+am\left(a\alpha+1\right)\right)a^2\cos(\theta)^3+\left(2a\,q^2\alpha r^2+3a^2m\left(a\alpha-1\right)r+a^2q^2\right)\cos(\theta)^2$$

$$+\left(-\alpha q^2r^3-3am\left(a\alpha+1\right)r^2+2a\,q^2r\right)\cos(\theta)-r^2\left(m\left(a\alpha-1\right)r+q^2\right)\Big]$$

$$\times\Big(a^2\left(q^2r\alpha+am\left(a\alpha-1\right)\right)\cos(\theta)^3+\left(-2a\,q^2\alpha r^2-3a^2m\left(a\alpha+1\right)r+a^2q^2\right)\cos(\theta)^2$$

$$+\left(-\alpha q^2r^3-3am\left(a\alpha-1\right)r^2-2a\,q^2r\right)\cos(\theta)+\left(m\left(a\alpha+1\right)r-q^2\right)r^2\Big),\tag{70}$$

$$I_{12}=-\frac{128aq^4\left(\alpha r\cos(\theta)-1\right)^{14}}{\left(r^2+a^2\cos(\theta)^2\right)^{10}}\Big(3\cos(\theta)^2a^2\alpha mr+2\cos(\theta)^2\alpha q^2r^2+\cos(\theta)^3a^2m$$

$$-\alpha r^3m-3\cos(\theta)\,m\,r^2+2\cos(\theta)\,q^2r\Big)$$

$$\times\Big[\cos(\theta)^3a^4\alpha m+\cos(\theta)^3a^2\alpha q^2r-3\cos(\theta)\,a^2\alpha m\,r^2-\cos(\theta)\,\alpha q^2r^3-3a^2\cos(\theta)^2\,mr$$

$$+a^2\cos(\theta)^2q^2+m\,r^3-q^2r^2\Big],\tag{71}$$

$$I_{15}=\frac{4q^4}{\left(r^2+a^2\cos(\theta)^2\right)^8}\Big(\left(\alpha^2q^4r^2+2a^2\alpha^2m\,q^2r+a^2m^2\left(a^2\alpha^2+1\right)\right)\cos(\theta)^2$$

$$+2q^2\alpha\left(a^2m-m\,r^2+q^2r\right)\cos(\theta)+m^2\left(a^2\alpha^2+1\right)r^2-2m\,q^2r+q^4\Big)\left(\alpha r\cos(\theta)-1\right)^{14},\tag{72}$$

$$I_{16}=-\frac{8\left(\alpha r\cos(\theta)-1\right)^{17}q^4}{\left(r^2+a^2\cos(\theta)^2\right)^{11}}\Big[\left(\alpha^2q^4r^2+2a^2\alpha^2m\,q^2r+a^2m^2\left(a^2\alpha^2+1\right)\right)\cos(\theta)^2$$

$$+2q^2\alpha\left(a^2m-m\,r^2+q^2r\right)\cos(\theta)+m^2\left(a^2\alpha^2+1\right)r^2-2mr\,q^2+q^4\Big]$$

$$\times\Big(a^2\alpha\left(a^2m+q^2r\right)\cos(\theta)^3+\left(-3a^2mr+a^2q^2\right)\cos(\theta)^2$$

$$+\left(-3a^2\alpha m\,r^2-\alpha q^2r^3\right)\cos(\theta)+m\,r^3-q^2r^2\Big),\tag{73}$$

$$I_{17} = \frac{24aq^4\left(\alpha r\cos\left(\theta\right)-1\right)^{17}}{\left(r^2+a^2\cos(\theta)^2\right)^{11}}\left(\frac{\cos\left(\theta\right)^3 a^2 m}{3} + r\alpha\left(a^2 m + \frac{2q^2 r}{3}\right)\cos(\theta)^2\right.$$
$$+\left.\left(-m\,r^2+\frac{2}{3}q^2 r\right)\cos(\theta) - \frac{\alpha r^3 m}{3}\right)\left(\left(\alpha^2 q^4 r^2 + 2a^2\alpha^2 m\,q^2 r + a^2 m^2\left(a^2\alpha^2+1\right)\right)\cos(\theta)^2\right.$$
$$\left.+2q^2\alpha\left(a^2 m - m\,r^2 + q^2 r\right)\cos(\theta) + m^2\left(a^2\alpha^2+1\right)r^2 - 2m\,q^2 r + q^4\right). \tag{74}$$

We summarise our results as follows:

Theorem

The exact algebraic expressions for the curvature invariants calculated for the accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime are given in Equations (61)-(74) and (60).

Remark

For vanishing acceleration of the black hole, i.e. $\alpha = 0$, *we recover the results of Theorem 1.*

We note that we have also checked our results for the curvature invariants for accelerating Kerr-Newman-(anti-)de Sitter black holes within the NP formalism , as we did for the case of non-accelerating Kerr-Newman-(anti-)de Sitter black holes. For instance, the only non-zero curvature scalars in the NP-formalism for the metric (56) relative to a natural null tetrad are the Ricci scalars:

$$\boxed{\Phi_{11} = \frac{1}{2} q^2 \frac{(1-\alpha r\cos(\theta))^4}{(r^2+a^2\cos(\theta)^2)^2}, \quad \text{and} \quad \Lambda,} \tag{75}$$

and the Weyl scalar:

$$\boxed{\Psi_2 = -\frac{(1-\alpha r\cos(\theta))^3 \left(m\,(ia\alpha+1)\,(r+ia\cos(\theta)) - q^2\,(1+\alpha r\cos(\theta))\right)}{(r-ia\cos(\theta))^3\,(r+ia\cos(\theta))}} \tag{76}$$

As a result, we obtain for the curvature invariant I_6 the explicit compact form:

$$I_6 = \frac{4q^4 \left(1 - \alpha r \cos(\theta)\right)^8}{\left(r^2 + a^2 \cos(\theta)^2\right)^4} + 4\Lambda^2, \tag{77}$$

which is in total agreement with eqn.(65) obtained with tensorial computation using a Maple code. Likewise within the NP formalism we derive the following explicit algebraic expression for the curvature invariants I_7, I_8:

$$I_7 = \frac{12\Lambda q^4 \left(1 - \alpha r \cos(\theta)\right)^8}{\left(r^2 + a^2 \cos(\theta)^2\right)^4} + 4\Lambda^3, \tag{78}$$

$$\begin{aligned} I_8 = {} & \frac{4q^8 \left(1 - \alpha r \cos(\theta)\right)^{16}}{\left(r^2 + a^2 \cos(\theta)^2\right)^8} + 4\left(\frac{12\Lambda q^4 \left(1 - \alpha r \cos(\theta)\right)^8}{\left(r^2 + a^2 \cos(\theta)^2\right)^4} + 4\Lambda^3\right)\Lambda \\ & - 6\left(\frac{4q^4 \left(1 - \alpha r \cos(\theta)\right)^8}{\left(r^2 + a^2 \cos(\theta)^2\right)^4} + 4\Lambda^2\right)\Lambda^2 + 12\Lambda^4. \end{aligned} \tag{79}$$

a result that fully agrees with eqns.(66) and (68) respectively.

The sign of the invariant I_1

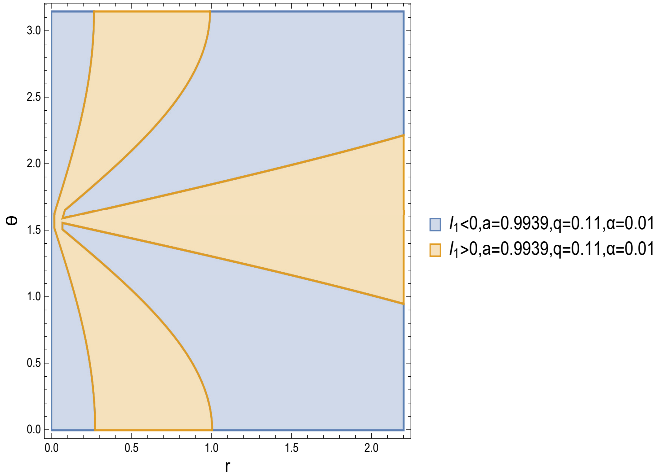


Figure: Regions of negative and positive I_1, eqn.(61) for an accelerating, charged and rotating black hole for the choice of values for the black hole parameters: $a = 0.9939, q = 0.11, \alpha = 0.01, m = 1$.

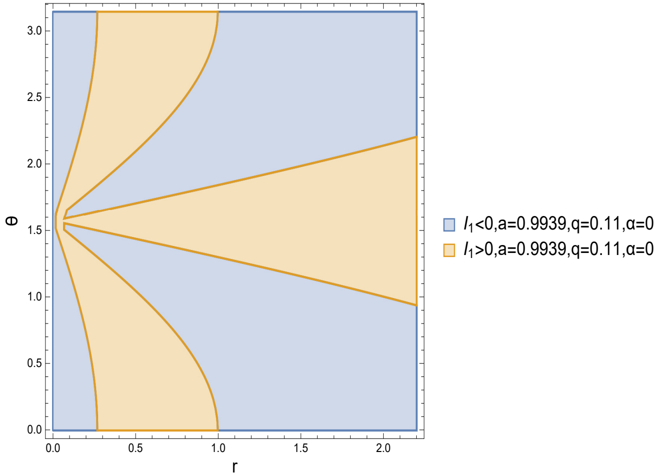


Figure: Regions of negative and positive I_1, eqn.(31) for a non-accelerating, charged and rotating black hole for the choice of values for the black hole parameters: $a = 0.9939, q = 0.11, \alpha = 0, m = 1$.

The sign of the Chern-Pontryagin invariant K_2

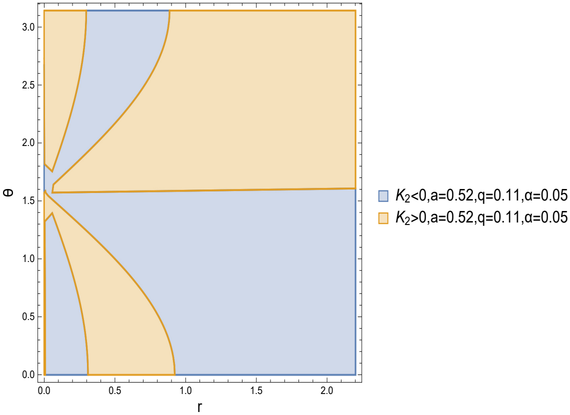


Figure: Regions of negative and positive Chern-Pontryagin invariant K_2, eqn.(60) for an accelerating Kerr-Newman black hole in (anti-)de Sitter spacetime for the choice of values for the black hole parameters: $a = 0.52, q = 0.11, \alpha = 0.05, m = 1$.

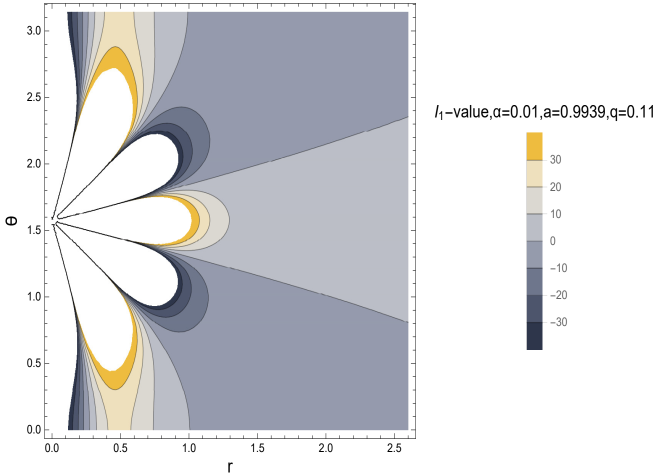


Figure: Contour plot of level curves of constant I_1 (eqn.61) for the choice of parameter values: $a = 0.9939, q = 0.11, \alpha = 0.01, m = 1$, in the $r - \theta$ plane.

The meaning/importance of regions with positive and negative values of the invariants I_1, K_2

We now briefly comment the meaning/importance of regions with positive and negative values of the invariants I_1, K_2 and the zero-value boundaries between the regions in the graphs. The regions of spacetime where the invariants I_1, K_2 vanish can be determined analytically. For reasons of simplicity of the presentation we focus the discussion in the case of zero acceleration, i.e. $\alpha = 0$. Solving $I_1 = 0$ for $\cos(\theta)$, applying the method of Tartaglia and Cardano, and assuming $\alpha = 0$ we obtain:

$$\cos(\theta) = \pm\frac{a^2q^2 - 3a^2mr}{3a^3m}$$

$$\pm\left(-a^4q^4 + 12a^4mq^2r - 18a^4m^2r^2\right) / \Bigg(3a^3m\Bigg(-a^6q^6 + 18a^6mq^4r - 54a^6m^2q^2r^2 + 54a^6m^3r^3$$

$$+ 3\sqrt{6}\sqrt{-a^{12}m^2q^8r^2 + 18a^{12}m^3q^6r^3 - 72a^{12}m^4q^4r^4 + 108a^{12}m^5q^2r^5 - 54a^{12}m^6r^6}\Bigg)^{1/3}\Bigg)$$

$$\mp\frac{1}{3a^3m}\Bigg(-a^6q^6 + 18a^6mq^4r - 54a^6m^2q^2r^2 + 54a^6m^3r^3$$

$$+ 3\sqrt{6}\sqrt{-a^{12}m^2q^8r^2 + 18a^{12}m^3q^6r^3 - 72a^{12}m^4q^4r^4 + 108a^{12}m^5q^2r^5 - 54a^{12}m^6r^6}\Bigg)^{1/3}, \tag{80}$$

$$\cos(\theta) = \pm\frac{a^2q^2 - 3a^2mr}{3a^3m}$$

$$\mp\left(\left(1 + i\sqrt{3}\right)\left(-a^4q^4 + 12a^4mq^2r - 18a^4m^2r^2\right)\right) / \Bigg(6a^3m\Bigg(-a^6q^6 + 18a^6mq^4r - 54a^6m^2q^2r^2 + 54a^6m^3r^3$$

$$+ 3\sqrt{6}\sqrt{-a^{12}m^2q^8r^2 + 18a^{12}m^3q^6r^3 - 72a^{12}m^4q^4r^4 + 108a^{12}m^5q^2r^5 - 54a^{12}m^6r^6}\Bigg)^{1/3}\Bigg)$$

$$\pm\frac{1}{6a^3m}\left(1 - i\sqrt{3}\right)\Bigg(-a^6q^6 + 18a^6mq^4r - 54a^6m^2q^2r^2 + 54a^6m^3r^3$$

$$+ 3\sqrt{6}\sqrt{-a^{12}m^2q^8r^2 + 18a^{12}m^3q^6r^3 - 72a^{12}m^4q^4r^4 + 108a^{12}m^5q^2r^5 - 54a^{12}m^6r^6}\Bigg)^{1/3}, \tag{81}$$

$$
\begin{aligned}
\cos(\theta) &= \pm\frac{a^2q^2-3a^2mr}{3a^3m} \\
&\mp\left(\left(1-i\sqrt{3}\right)\left(-a^4q^4+12a^4mq^2r-18a^4m^2r^2\right)\right)\Big/\left(6a^3m\left(-a^6q^6+18a^6mq^4r-54a^6m^2q^2r^2+54a^6m^3r^3\right.\right. \\
&\left.\left.+3\sqrt{6}\sqrt{-a^{12}m^2q^8r^2+18a^{12}m^3q^6r^3-72a^{12}m^4q^4r^4+108a^{12}m^5q^2r^5-54a^{12}m^6r^6}\right)^{1/3}\right) \\
&\pm\frac{1}{6a^3m}\left(1+i\sqrt{3}\right)\left(-a^6q^6+18a^6mq^4r-54a^6m^2q^2r^2+54a^6m^3r^3\right. \\
&\left.+3\sqrt{6}\sqrt{-a^{12}m^2q^8r^2+18a^{12}m^3q^6r^3-72a^{12}m^4q^4r^4+108a^{12}m^5q^2r^5-54a^{12}m^6r^6}\right)^{1/3}, \qquad (82)
\end{aligned}
$$

while solving $K_2 = 0$ (again for zero acceleration $\alpha = 0$) yields:

$$\cos(\theta) = 0, \qquad (83)$$

$$\cos(\theta) = \pm\sqrt{\frac{3mr^2-2q^2r}{m}}\frac{1}{a}, \qquad (84)$$

$$\cos(\theta) = \pm\frac{1}{a}\sqrt{\frac{q^2r^2-mr^3}{q^2-3mr}}. \qquad (85)$$

Thus the zero boundary expressed by eqns(80)-(82) can be interpreted as separating regions of electric dominance of the Weyl tensor ($I_1 > 0$) from regions of Weyl magnetic dominance ($I_1 < 0$) (Kraniotis Class.Quantum Grav. 39 (2022) 145002). We mention at this point that an observer with a timelike velocity vector field u^α is said to measure the *electric* and *magnetic* components, $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$, respectively, of the Weyl tensor by

$$\mathcal{E}_{\alpha\beta} \equiv C_{\alpha\gamma\beta\delta}u^\gamma u^\delta, \quad \mathcal{H}_{\alpha\beta} \equiv C^*_{\alpha\gamma\beta\delta}u^\gamma u^\delta. \tag{86}$$

The curvature invariant I_1 is related to the electric and magnetic components of the Weyl tensor as follows (Filipe L *et al*)(2021)):

$$\frac{I_1}{8} = \mathcal{E}^{\alpha\beta}\mathcal{E}_{\alpha\beta} - \mathcal{H}^{\alpha\beta}\mathcal{H}_{\alpha\beta}, \tag{87}$$

while the Chern-Pontryagin invariant K_2 is expressed as follows :

$$\frac{1}{16}K_2 = \mathcal{E}^{\alpha\beta}\mathcal{H}_{\alpha\beta}. \tag{88}$$

Equation (87) clarifies the introduction of the region of Weyl electric dominance: $\mathcal{E}^{\alpha\beta}\mathcal{E}_{\alpha\beta} > \mathcal{H}^{\alpha\beta}\mathcal{H}_{\alpha\beta}$, i.e. $I_1 > 0$, and regions of Weyl magnetic dominance: $\mathcal{E}^{\alpha\beta}\mathcal{E}_{\alpha\beta} < \mathcal{H}^{\alpha\beta}\mathcal{H}_{\alpha\beta}$, i.e. $I_1 < 0$.

The zeros of the Hirzebruch invariant K_2 in eqns.(83)-(85) define purely electric/magnetic Weyl surfaces. For $\theta = \pi/2$ and $\cos(\theta) = \pm\frac{\sqrt{mr(3mr-2q^2)}}{ma}$ Weyl tensor is purely electric while for the zeros in (85) the Weyl tensor is purely magnetic. A Weyl tensor is called purely electric (purely magnetic) when $\mathcal{H}_{\alpha\beta} = 0$ ($\mathcal{E}_{\alpha\beta} = 0$) R. Arianhod CQG 1994.

Chiral photon anomaly for a gravitational background with a non-trivial Chern-Pontryagin invariant K_2

A non-trivial Hirzebruch signature density invariant K_2 also appears to play a role in the electromagnetic duality anomaly in curved spacetimes. As is known the source-free Maxwell action is invariant under electric-magnetic duality rotations in arbitrary spacetimes (Deser and Teitelboim 1976):

$$F_{\mu\nu} \to F_{\mu\nu}\cos(\theta) + F^{*}_{\mu\nu}\sin(\theta). \tag{89}$$

This leads to a conserved classical Noether charge. In the work by I Agullo *et al* PRL (2017),inspired by earlier work of A.D. Dolgov *et al* NPB 1989,[1], it was shown that this conservation law was broken at the quantum level in the presence of a background field with a non-trivial Chern-Pontryagin invariant.

[1] The result of Dolgov, was further explored in (Reuter 1988) where it was shown that for antisymmetric gauge fields of rank $2n-1$ coupled to gravity in $4n$ dimensions, the symmetry under duality rotations is broken by quantum effects.

In particular quantum effects may induce violation of helicity conservation for photons propagating in curved spacetimes. Observable effects of this photon chiral anomaly are directly related to the variation of electromagnetic helicity $\mathcal{H}_{\rm em}$ (Galaverni and Gabriele GERG 2021):

$$\Delta\langle\mathcal{H}_{\rm em}\rangle \propto \int_{t_1}^{t_2}\int_{\Sigma^3} R_{\alpha\beta\mu\nu}R^{*\alpha\beta\mu\nu}\sqrt{-g}\,\mathrm{d}^4x \tag{90}$$

If the integral on the right term is different from zero, then $\mathcal{H}_{\rm em}$ is not conserved. The difference between the numbers of right circularly polarised photons and left circularly polarised photons changes: the degree of circular polarisation is not conserved.

Indeed, at large distances from a Kerr-Newman-(anti-)de Sitter black hole the Hirzebruch density, eqn.(28), has the expansion:

$$
\begin{aligned}
K_2 &= {}^* \mathbf{R} \cdot \mathbf{R} = \frac{96a}{r^{12}} \Big(-3m^2 r^5 \cos(\theta) + 5\cos(\theta) m q^2 r^4 - 2q^4 r^3 \cos(\theta) \\
&\quad + 10 m^2 a^2 r^3 \cos^3(\theta) \Big) \left(1 - \frac{6a^2}{r^2} \cos^2(\theta) + \cdots \right) \\
&= -288 \frac{m^2 a}{r^7} \cos(\theta) + 480 \frac{a\cos(\theta) m q^2}{r^8} \\
&\quad - 192 \frac{a\cos(\theta)}{r^9} \left(q^4 - 14 m^2 a^2 \cos^2(\theta) \right) + O\left(\frac{1}{r^9} \right)
\end{aligned} \tag{91}
$$

Then integration yields the result (Kraniotis Class.Quantum Grav. 39 (2022) 145002):

$$
\int R_{\alpha\beta\mu\nu} R^{*\alpha\beta\mu\nu} \sqrt{-g}\, \mathrm{d}^4 x \propto \int_0^\pi \cos(\theta) \sin^2(\theta) (r^2 + a^2 \cos^2(\theta)) \mathrm{d}\theta = 0. \tag{92}
$$

Thus despite the fact that, for a non-accelerating KN(a)dS black hole, the Hirzebruch invariant is non-trivial its integral over all space is zero-in this case there are no observable effects related to the quantum anomaly. This is in agreement with the recent calculation for the Kerr metric in (Galaverni and Gabriele GERG 2021).

Let us investigate now the case of accelerating Kerr-Newman black holes in (anti-)de Sitter spacetime. The Chern-Pontryagin invariant K_2 in equation (60) for large radii takes the form:

$$K_2 =^* \mathbf{R}\cdot\mathbf{R} = \frac{96a\,(\alpha r\cos(\theta)-1)^6}{r^{12}}\Big(\cos(\theta)^3\,a^4\alpha m+\cos(\theta)^3\,a^2\alpha q^2 r$$

$$-3\cos(\theta)\,a^2\alpha m\,r^2-\cos(\theta)\,\alpha q^2 r^3-3a^2\cos(\theta)^2\,mr+a^2\cos(\theta)^2\,q^2+m\,r^3-q^2r^2\Big)$$

$$\times\Big[3\cos(\theta)^2\,a^2\alpha mr+2\cos(\theta)^2\,\alpha q^2r^2+\cos(\theta)^3\,a^2m-\alpha r^3 m-3\cos(\theta)\,m\,r^2$$

$$+2\cos(\theta)\,q^2r\Big]\left(1-6\frac{a^2}{r^2}\cos^2(\theta)+\cdots\right)$$

$$=\frac{96a\,(\alpha r\cos(\theta)-1)^6}{r^6}(mq^2\alpha^2\cos(\theta)-m^2\alpha)+\cdots \tag{93}$$

Interestingly, the following polar angular integration of the leading term in (93), which is a part of the spacetime integral $\iiiint R_{\alpha\beta\mu\nu}R^{*\alpha\beta\mu\nu}\sqrt{-g}\mathrm{d}^4x$, gives a non-zero result:

$$\int_0^{\pi}\frac{96a}{r^6}(mq^2\alpha^2\cos(\theta)-m^2\alpha)\sin^2(\theta)(r^2+a^2\cos^2(\theta))(\alpha r\cos(\theta)-1)^2\mathrm{d}\theta\neq 0. \tag{94}$$

Thus, it appears that accelerating Kerr-Newman black holes in (anti-)de Sitter spacetime yield a non-zero effect for the quantum photon chiral anomaly since a nonzero Chern-Pontryagin integrated term is present.

Conclusions

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- We mentioned the quantum photon chiral anomaly in connection to the Chern-Pontryagin invariant and the difference between the case of non-accelerated and accelerated black hole is highlighted.