Wavefunction of the universe: Diffeomeorphism invariance and field redefinitions

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H.P., N. Toumbas, B. de Vaulchier, Nucl. Phys. B 973 (2021), 115600
A. Kehagias, H.P., N. Toumbas JHEP 12 (2021), 165

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Introduction

■ In Quantum Mechanics, wavefunctions $\Longrightarrow$ probabilities

- Schrödinger equation
- Path integral

■ In Quantum Gravity, wavefunction of the universe

$\Longrightarrow$ probabilities favoring realistic aspects of the Universe?

- Wheeler-DeWitt equation ['62]: Ambiguity in its form
- Hartle-Hawking proposal for spatially closed universes with cosmological constant $\Lambda > 0$ ['83]

Euclidean path integral

$$\Psi[h_{ij}] = \int \frac{\mathcal{D}g}{\mathrm{Vol(Diff)}}\; e^{-\frac{1}{\hbar}S_{\mathrm{E}}[g]}$$

where the sum is over all **compact four-manifolds of metric g and having a single 3D boundary of metric h_{ij}.**

This is the **"No-boundary proposal"**

■ Homogeneous and isotropic: Space is a 3-sphere

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\,\mathrm{d}\Omega_3^2$$

$$\Psi(a_0) = \int_{\substack{a_{\mathrm{i}}=0\\ a_{\mathrm{f}}=a_0}} \frac{\mathcal{D}N\,\mathcal{D}a}{\mathrm{Vol(Diff)}}\; e^{-\frac{1}{\hbar}S_{\mathrm{E}}[N,a]}$$

where Diff = Euclidean-time reparametrizations

$\Psi(a_0)$ is the probability amplitude for creating a Universe of scale factor a_0 from "nothing" [Vilenkin, '82]

Plan

■ **Fix consistently the gauge of Euclidean-time reparametrization.**

■ **Field redefinitions of the scale factor** are symmetries of the classical action but

$$a = A(q) \quad \Longrightarrow \quad \mathcal{D}a \neq \mathcal{D}q$$

We obtain **different results for the wavefunction.**

■ For each prescription we **lift the ambiguity in the Wheeler-DeWitt equation,** at the semi-classical level.

■ However, all prescriptions are equivalent: **Same quantum predictions,** at least at the semi-classical level.

■ They **reproduce classical cosmology** in the $\hbar \to 0$ limit.

Gauge fixing of Euclidean-time reparametrizations

■ The Euclidean action is

$$S_{\mathrm{E}} = 6\pi \int_{x^0_{\mathrm{Ei}}}^{x^0_{\mathrm{Ef}}} \mathrm{d}x^0_{\mathrm{E}} \sqrt{g_{00}} \left[a\, g^{00} \left(\frac{\mathrm{d}a}{\mathrm{d}x^0_{\mathrm{E}}} \right)^2 + a - \frac{\Lambda}{3}\, a^3 \right]$$

The potential has contributions from the scalar curvature and the cosmological constant.

■ It describes a non-linear σ-model:

- There is a **line segment $[x^0_{\mathrm{Ei}}, x^0_{\mathrm{Ef}}]$** of metric $g_{00} \equiv N^2$

- **The target space is $\mathbb{R}_+$ parametrized by the scale factor a,** with metric $G_{aa} = a$.

■ Let us concentrate on the line-segment of Euclidean time:

All metrics g_{00} are not equivalent up to a change of coordinate, since the proper length ℓ of a line segment is invariant under a change of coordinate.

$\Longrightarrow$ **To fix a gauge, we choose a metric $\hat{g}_{00}[\ell]$ in each equivalence class ℓ, which is a modulus**

$$\int \frac{\mathcal{D}N}{\mathrm{Vol(Diff)}} = \int_0^{+\infty} \mathrm{d}\ell \; \cancel{\int_{\mathrm{Diff}} \frac{\mathcal{D}\xi}{\mathrm{Vol(Diff)}}} \; \Delta_{\mathrm{FP}}[\hat{g}_{00}[\ell]]$$

■ **Fadeev-Popov determinant**

$$1 = \Delta_{\mathrm{FP}}[\hat{g}_{00}[\ell]] \int_0^{+\infty} \mathrm{d}\ell' \int_{\mathrm{Diff}} \mathcal{D}\xi \; \delta\big[\hat{g}_{00}[\ell] - \hat{g}_{00}^{\xi}[\ell']\big]$$

• Introducing **anticommuting ghosts b^{00}, c_0**

$$\Delta_{\rm FP}[\hat{g}_{00}[\ell]] = \int_{\substack{c^0(\hat{x}^0_{\rm Ei})=0 \\ c^0(\hat{x}^0_{\rm Ef})=0}} \mathcal{D}c \int \mathcal{D}b \left(b, \frac{\hat{g}[\ell]}{\ell}\right) \exp\left\{4i\pi \left(b, \hat{\nabla}c\right)\right\}$$

where $(f, h) \equiv \int_{\hat{x}^0_{\rm Ei}}^{\hat{x}^0_{\rm Ef}} \mathrm{d}\hat{x}^0_{\rm E} \sqrt{\hat{g}_{00}[\ell]}\, f^{0\cdots 0}\, h_{0\cdots 0}$

• By expanding in Fourrier modes on $[\hat{x}^0_{\rm Ei}, \hat{x}^0_{\rm Ei}]$ and **using gauge-invariant measures,**

$$\Delta_{\rm FP}[\hat{g}_{00}[\ell]] = 1$$

■ The result is not always trivial: Replacing the line-segment by a circle, the result is $1/\ell$

Path integral over the scale factor

■ Gauge $\hat{g}_{00}[\ell] = \ell^2$ defined on $[\hat{x}_{\mathrm{Ei}}, \hat{x}_{\mathrm{Ef}}] = [0, 1]$

$$\Psi(a_0) = \int_0^{+\infty} \mathrm{d}\ell \int_{\substack{a(0)=0 \\ a(1)=a_0}} \mathcal{D}a \; e^{-\frac{1}{\hbar} S_{\mathrm{E}}[\ell, a]}$$

where the action

$$S_{\mathrm{E}}[\ell, a] = 6\pi \int_0^1 \mathrm{d}\tau \left[\frac{a}{\ell} \left(\frac{\mathrm{d}a}{\mathrm{d}\tau} \right)^2 + \ell \, V(a) \right] , \qquad V(a) = a - \frac{\Lambda}{3} \, a^3$$

■ It is not quadratic $\Longrightarrow$ **semi-classical approximation**

■ In the literature, one often uses a gauge that depends on a

$$g_{00} = \frac{\ell^2}{a^2} \quad \Longrightarrow \quad S_{\mathrm{E}} = \int_0^{\int_0^1 \mathrm{d}\tau\, a(\tau)} \mathrm{d}\zeta\,(\cdots) \;\; \text{replaced by} \int_0^{1} \mathrm{d}\zeta\,(\cdots)$$

Problem : ℓ is no more the invariant length parametrizing correctly the classes of equivalence of the metrics.

They use this gauge because $q = a^2$ renders the action Gaussian and thus the path integral exactly computable.

■ **Steepest-descent method** on

$$\Psi(a_0) = \int_0^{+\infty} \mathrm{d}\ell \int_{\substack{a(0)=0 \\ a(1)=a_0}} \mathcal{D}a \; e^{-\frac{1}{\hbar} S_{\mathrm{E}}[\ell,a]}$$

- Find instanton solutions: Two solutions $(\bar{a}_\epsilon, \bar{\ell}_\epsilon)$, $\quad \epsilon = \pm 1$
- Expand at quadratic order and integrate over fluctuations

■ $$S_{\mathrm{E}}[\ell,a] = \bar{S}_{\mathrm{E}}^{\epsilon} + 6\pi^2 \int_0^1 \mathrm{d}\tau \bar{\ell}_\epsilon \left[\delta a \, \mathcal{Q}_\epsilon \, \delta a + 2 \, \delta a \, \frac{V_a(\bar{a})}{\bar{\ell}_\epsilon} \, \delta\ell + \delta\ell \, \frac{V(\bar{a})}{\bar{\ell}_\epsilon^2} \, \delta\ell \right] + \cdots$$

$$\text{where} \quad \mathcal{Q}_\epsilon = -\frac{\bar{a}_\epsilon}{\bar{\ell}_\epsilon^2} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - \frac{1}{\bar{\ell}_\epsilon^2} \frac{\mathrm{d}\bar{a}_\epsilon}{\mathrm{d}\tau} \frac{\mathrm{d}}{\mathrm{d}\tau} - \frac{2}{3} \Lambda \, \bar{a}_\epsilon$$

It is a self-adjoint operator acting in the Hilbert space of functions vanishing at 0 and 1.

- Diagonalizing,

$$\Psi(a_0) = \sum_{\epsilon=\pm 1} e^{-\frac{1}{\hbar}\bar{S}_{\mathrm{E}}^{\epsilon}} \int_{\substack{\delta a(0)=0\\ \delta a(1)=0}} \mathcal{D}\delta a \, \exp\left\{-\frac{6\pi^2}{\hbar}\,(\delta a, \mathcal{Q}_\epsilon\, \delta a)\right\} \int \mathrm{d}\delta\ell \, \exp\left\{-\frac{\mathcal{K}_\epsilon}{\hbar}\,\delta\ell^2\right\}\ (1+\mathcal{O}(\hbar))$$

$$= \sum_{\epsilon=\pm 1} \frac{e^{-\frac{1}{\hbar}\bar{S}_{\mathrm{E}}^{\epsilon}}}{\sqrt{\det \mathcal{Q}_\epsilon}\ \sqrt{\mathcal{K}_\epsilon}}\ (1+\mathcal{O}(\hbar))$$

- To compute $\det \mathcal{Q}_\epsilon \equiv \prod_{k\geq 1} \nu_{\epsilon,k}$ we improve the method of Affleck and Coleman ['77]

$$\Psi(a_0) = \sum_{\epsilon=\pm 1} \frac{1}{\sqrt{\epsilon}} \frac{\exp\left[\epsilon\,\frac{12\pi^2}{\hbar\,\Lambda}\left(1-\frac{\Lambda}{3}a_0^2\right)^{\frac{3}{2}}\right]}{a_0^{\frac{1}{8}}\left(1-\frac{\Lambda}{3}a_0^2\right)^{\frac{1}{4}}}\,(1+\mathcal{O}(\hbar))$$

Field redefinitions

■ Classically, the action is invariant under redefinitions $\boldsymbol{a = A(q)}$.
At the quantum level $\boldsymbol{\mathcal{D}a \neq \mathcal{D}q}$ **due to a Jacobian**

$$\widetilde{\Psi}(q_0) = \int_0^{+\infty} \mathrm{d}\ell \int_{\substack{q(0)=q_\mathrm{i} \\ q(1)=q_0}} \mathcal{D}q \; e^{-\frac{1}{\hbar} S_\mathrm{E}[\ell^2, q]} \qquad \text{where} \qquad a_0 = A(q_0)\,, \quad 0 = A(q_\mathrm{i})$$

$$= \sum_{\epsilon=\pm 1} \frac{1}{\sqrt{\epsilon}} \frac{\exp\left[\epsilon \, \frac{12\pi^2}{\hbar\,\Lambda} \left(1 - \frac{\Lambda}{3} a_0^2\right)^{\frac{3}{2}}\right]}{|A'(q_0)|^{-\frac{1}{4}} \, a_0^{\frac{1}{8}} \left(1 - \frac{\Lambda}{3} a_0^2\right)^{\frac{1}{4}}} \left(1 + \mathcal{O}(\hbar)\right)$$

There are **infinitely many different choices of wavefunctions!**

Wheeler-DeWitt equation

■ For each prescription $\mathcal{D}q$, **all possible states/wavefunctions satisfy an equation** similar to Schrödinger in quantum mechanics.

To derive it,

$$0 = \int \frac{\mathcal{D}N\,\mathcal{D}q}{\mathrm{Vol(Diff)}}\,\frac{\delta}{\delta N}\,e^{iS[N,q]} = -i\int \frac{\mathcal{D}N\,\mathcal{D}q}{\mathrm{Vol(Diff)}}\,\frac{\mathcal{H}}{N}\,e^{iS[N,q]} \tag{1}$$

where the **classical Hamiltonian is**

$$\frac{\mathcal{H}}{N} = -\frac{1}{24\pi}\,\frac{\pi_q^2}{AA'^2} - 6\pi V \qquad \text{where} \qquad \pi_q = -12\pi^2\,\frac{AA'^2}{N}\,\dot{q}$$

$\Longrightarrow$ **The quantum Hamiltonian vanishes on all states of the Hilbert space $\Leftrightarrow$ Wheeler-DeWitt equation**

■ Classically, we have for arbitrary functions $\rho_1(q), \rho_2(q)$

$$\pi_q^2 = \frac{1}{\rho_1\,\rho_2}\,\pi_q\,\rho_1\,\pi_q\,\rho_2$$

- **canonical quantization**

$$q \longrightarrow q_0 \,, \qquad \pi_q \longrightarrow -i\hbar \frac{\mathrm{d}}{\mathrm{d}q_0}$$

yields an **ambiguity**

$$\frac{\hbar^2}{24\pi}\,\frac{1}{AA'^2}\,\frac{1}{\rho}\,\frac{\mathrm{d}}{\mathrm{d}q_0}\Big(\rho\,\frac{\mathrm{d}\Phi}{\mathrm{d}q_0}\Big) + \Big(\hbar^2\,\omega - 6\pi V\Big)\Phi = 0$$

where Φ is a generic wavefunction of the Hilbert space.

■ We can find ρ by solving this equation at the semi-classical level using the **WKB method**

$$\Phi(q_0) = \sum_{\epsilon=\pm 1} N_\epsilon \, \frac{\exp\Big[\epsilon s\,\frac{12\pi^2}{\hbar\,\Lambda}\,\big(1-\frac{\Lambda}{3}a_0^2\big)^{\frac{3}{2}}\Big]}{\sqrt{a_0\,\rho(q_0)|A'(q_0)|}\,\big(1-\frac{\Lambda}{3}a_0^2\big)^{\frac{1}{4}}}\,(1+\mathcal{O}(\hbar))$$

Comparing with a particular wavefunction, the “no-boundary state”

$$\Longrightarrow \quad \rho(q_0) = a_0^{-\frac{3}{4}}\,|A'(q_0)|^{-\frac{3}{2}}$$

Universality at the semi-classical

■ **Different wavefunction prescriptions $\mathcal{D}q$ and Wheeler-DeWitt equations**
$\Longrightarrow$ **different quantum gravities with same classical limits?**

- To discuss probabilities, we **define an inner product in each Hilbert space.**

Denoting $\Phi(q_0) \equiv \Phi_A(a_0)$, $\quad \big(\text{where } a_0 = A(q_0)\big)$

$$\langle \Phi_{A1}, \Phi_{A2} \rangle = \int_0^{+\infty} \mathrm{d}a_0 \, \mu(a_0) \, \Phi_{A1}(a_0)^* \, \Phi_{A2}(a_0)$$

- Imposing **Hermiticity of the Hamiltnonian**

$$\Big\langle \Phi_{A1}, \frac{\mathcal{H}}{N} \Phi_{A2} \Big\rangle = \Big\langle \frac{\mathcal{H}}{N} \Phi_{A1}, \Phi_{A2} \Big\rangle$$

$\Longrightarrow$ Differential equation $\Longrightarrow$ $\mu = a_0 \, \rho \, |A'|$

$$\implies \quad \sqrt{\mu(a_0)}\,\Phi_A(a_0) = \sum_{\epsilon=\pm1} N_\epsilon \, \frac{\exp\left[\epsilon\,\frac{12\pi^2}{\hbar\,\Lambda}\,\left(1-\frac{\Lambda}{3}a_0^2\right)^{\frac{3}{2}}\right]}{\left(1-\frac{\Lambda}{3}a_0^2\right)^{\frac{1}{4}}}\,(1+\mathcal{O}(\hbar))$$

is **independent of $\boldsymbol{\rho}$ and $\boldsymbol{A}$ *i.e.* is independent of the choice of field redefinition, at the semi-classical level.**

So is the inner product $\quad \langle \Phi_{A1}, \Phi_{A2}\rangle = \int_0^{+\infty} \mathrm{d}a_0\,\mu\,\Phi_{A1}^*\,\Phi_{A2}$

$\Longrightarrow$ **All probabilities are independent of the choice of measure $\boldsymbol{\mathcal{D}q}$, at least at the semi-classical level.**

■ **If we assume this statement is exact in $\hbar$,** we can lift completely the remaining ambiguity in the Wheeler-DeWitt equation

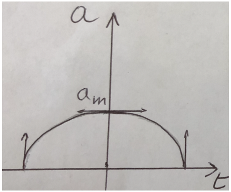
$$\implies \quad \omega = -\frac{1}{24\pi^2 a_0}\left[\frac{5}{16}\,\frac{1}{a_0^2} + \frac{1}{4}\Big(\frac{\rho'}{\rho}\Big)^2 - \frac{\rho''}{\rho}\right]$$

Recovering the classical cosmology

■ Generalize to the case where the **universe is filled with a perfect fluid** of state equation [A.Kehagias, H.P., N. Toumbas, '21]

$$p_m = w\rho_m \quad \text{where} \quad -1 \leq w \leq 1 \qquad (\text{not only } w = -1)$$

■ Classical cosmology for $\boldsymbol{w > -\frac{1}{3}}$:



To compare with Quantum Gravity which provides probabilities, we define

classical probability = duration the scale factor lies in the range $[a, a + \mathrm{d}a]$, divided by the total duration of the cosmological evolution

$$P_{cl}(a)\,\mathrm{d}a = \frac{1}{\sqrt{\pi}}\,\frac{\left|\Gamma\left(\frac{1}{3w+1}\right)\right|}{\Gamma\left(\frac{1}{2}+\frac{1}{3w+1}\right)}\,\frac{a^{\frac{3w+1}{2}}\,\mathrm{d}a}{a_m\sqrt{a_m^{3w+1}-a^{3w+1}}}$$

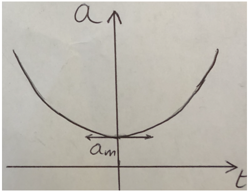
■ For the choice of field a the Hilbert space measure is $\mu = a\rho$
(In the literature, one often takes incorrectly ρ alone.)

We solve the Wheeler-DeWitt equation at the semiclassical level using the WKW method

For $a < a_m$: $P(a) \equiv \mu|\Phi|^2 \longrightarrow P_{cl}(a)$ when $\hbar \to 0$

For $a > a_m$ (classically forbiden): $P(a) \equiv \mu|\Phi|^2 \longrightarrow 0$ when $\hbar \to 0$

■ For $\boldsymbol{w < -\frac{1}{3}}$:



The classical trajectory has an infinite duration $\Longrightarrow$ The "classical probability" cannot be normalized, but **ratios of it (at different values of the scale factor) can be understood as relative probabilities.**
The quantum wavefunction is non-normalizable. Relative probabilities are well defined and reproduce the classical ones when $\hbar \to 0$.

Conclusion

■ We have considered the **Hartle-Hawking wavefunction for spatially closed universes, homogeneous and isotropic.**

■ The system can be seen as a non-linear σ-model with a line segment of Euclidean time and a target space parametrized by the scale factor.

■ The **gauge fixing of time reparametrization** is done by:

- Integrating over the proper length of the line-segment.
- The Faddeev-Popov determinant is trivial.
- Using gauge invariant measures.

■ The **field redefinitions of the scale factor yield different path-integral measures and wavefunctions, but the Hilbert spaces are equivalent at least semi-classically.**

■ The **quantum probabilities reproduce in the $\hbar \to 0$ limit the classical cosmological evolution.**