The anapole moment of a charged lepton in softly-broken Supersymmetric QED

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29 June 2022

Motivating this talk

Much attention lately has been given to the anomalous magnetic moment of the muon (and the electron), which suggest the presence of new BSM physics. Also of interest are experimental limits on the electric dipole moment of the electron.

What about the anapole moment?

And why give this talk at a Supersymmetry meeting?

The following paper appeared in arXiv:1910.09545, but it was never published:

C. Aydin, *Anapole Moment of Leptons in the Minimal Supersymmetric Standard Model*

In my opinion, there were a number of deficiencies in the calculations presented in this paper.

- Some subtleties associated with the $\gamma - Z$ system of the electroweak sector must be addressed to obtain a gauge-invariant, physical result (Gongora-T. and Stuart, 1992). It is not clear whether these were correctly treated.

- Necessary pieces of the calculation appear to be absent.

Meanwhile, Herbi Dreiner, Steve Martin and I have (finally!) submitted our long overdue manuscript to Cambridge University Press, entitled *From Spinors to Supersymmetry* (to appear in Spring of 2023).

This book contains many explicit computations of Standard Model and supersymmetric processes (employing two-component spinor technology). After providing explicit details of the one-loop computation of the muon anomalous magnetic moment in softly-broken SUSY QED, we use the same methods to obtain the anapole moment of the muon.

Some details of this computation will be presented in this talk.

Outline

1. The electromagnetic vertex structure

2. Interpretation of the form factors

3. Isolating the form factors via the projection operator technique

4. Computation of the anapole moment of a charged lepton in SUSY-QED
   - The one loop vertex contribution
   - Wave function renormalization

5. Challenges of a more complete computation

6. Will the anapole moment of the electron (or muon) ever be measured?

The Electromagnetic Vertex Structure

Consider the scattering of a negatively charged muon (with electric charge $Q = -e$ and mass m) off an external static EM field, A^μ field. The first order S-matrix amplitude is given by,

$$\langle p', s'| \, S^{(1)} \, |p, s\rangle = ie\bar{u}(\vec{\boldsymbol{p}}\,', s')\Gamma^\mu(p, p')u(\vec{\boldsymbol{p}}, s)\widetilde{A}_\mu(q) \, ,$$

where $q \equiv p' - p$ is the momentum transfer, $p^2 = p'^{\,2} = m^2$, $\widetilde{A}_\mu(q)$ is the four-dimensional Fourier transform of $A_\mu(x)$, and the effective electromagnetic vertex function is,

$$\Gamma^\mu(p, p') = F_1(q^2)\gamma^\mu + \frac{i}{2m}\sigma^{\mu\nu}q_\nu F_2(q^2) + \frac{1}{2m}\gamma_5\sigma^{\mu\nu}q_\nu F_3(q^2)$$
$$+\frac{1}{4m^2}\big(q^\mu q^\nu - q^2 g^{\mu\nu}\big)\gamma_\nu\gamma_5 F_4(q^2) \, .$$

form factor	name	P	C	T	chirality flip
F_1	electric charge	$+$	$+$	$+$	no
F_2	anomalous magnetic dipole	$+$	$+$	$+$	yes
F_3	electric dipole	$-$	$+$	$-$	yes
F_4	anapole	$-$	$-$	$+$	no

P, C and T properties of the terms of the effective vertex, $\bar{u}(\vec{\boldsymbol{p}}')\,\Gamma^\mu(p,p')u(\vec{\boldsymbol{p}})A_\mu(q)$. Chirality flip refers to the nature of the interaction in the ultrarelativistic limit.

For a static EM field,

$$\widetilde{A}_\mu(q) = \int_{-\infty}^{\infty} dt\, e^{i(E'-E)t} \int d^3x\, A_\mu(\vec{\boldsymbol{x}})e^{-i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}}} = 2\pi\delta(E'-E)\widetilde{A}_\mu(\vec{\boldsymbol{q}})\,,$$

where $\widetilde{A}_\mu(\vec{\boldsymbol{q}})$ is the three-dimensional Fourier transform of $A_\mu(\vec{\boldsymbol{x}})$,

$$\widetilde{A}_\mu(\vec{\boldsymbol{q}}) \equiv \int d^3x\, A_\mu(\vec{\boldsymbol{x}})e^{-i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}}}\,.$$

Interpretation of the Form Factors

In nonrelativistic scattering theory, the S-matrix element is related to the interaction potential $V(\vec{\boldsymbol{x}})$ via

$$\langle \vec{\boldsymbol{p}}\,', s'|\, S\, |\vec{\boldsymbol{p}}, s\rangle = (2\pi)^3\, 2E\, \delta^3(\vec{\boldsymbol{p}}\,' - \vec{\boldsymbol{p}})\delta_{ss'} \\ -2\pi i \delta(E' - E)\, \langle \vec{\boldsymbol{p}}\,', s'|\, V(\vec{\boldsymbol{x}}) |\vec{\boldsymbol{p}}\,^{(+)}, s\rangle\,,$$

using covariant normalization of the one particle momentum states, where $|\vec{\boldsymbol{p}}\,^{(+)}\rangle$ is given by the Lippmann-Schwinger equation,

$$|\vec{\boldsymbol{p}}\,^{(+)}, s\rangle = |\vec{\boldsymbol{p}}, s\rangle + \lim_{\varepsilon \to 0^+} \frac{1}{E - H_0 + i\varepsilon} V |\vec{\boldsymbol{p}}\,^{(+)}, s\rangle\,,$$

and the Hamiltonian is given by $H = H_0 + V$. To leading order in V,

$$\langle \vec{\boldsymbol{p}}\,', s'|\, V(\vec{\boldsymbol{x}})\, |\vec{\boldsymbol{p}}, s\rangle = -e\bar{u}(\vec{\boldsymbol{p}}\,', s')\Gamma^\mu(p, p')u(\vec{\boldsymbol{p}}, s)\widetilde{A}_\mu(\vec{\boldsymbol{q}})\,.$$

The energy conserving delta function imposes $q^0 = E' - E = 0$.

Case 1: Static electric field with $A^\mu(x) = \big(\phi(x)\,;\,\vec{\mathbf{0}}\big)$

For covariantly normalized states, $\langle \vec{\boldsymbol{x}} | \vec{\boldsymbol{p}} \rangle = \sqrt{2E_{\vec{\boldsymbol{p}}}}\, e^{i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}}$. In the nonrelativistic limit, $\vec{\boldsymbol{p}}$, $\vec{\boldsymbol{p}}\,' \to \vec{\mathbf{0}}$, and we obtain

$$\langle \vec{\boldsymbol{p}}\,', s'|\, V(\vec{\boldsymbol{x}})\, |\vec{\boldsymbol{p}}, s\rangle = -eF_1(0)\, \langle \vec{\boldsymbol{p}}\,', s'|\, \phi(\vec{\boldsymbol{x}})\, |\vec{\boldsymbol{p}}, s\rangle \\ -\frac{eF_3(0)}{2m} \langle \vec{\boldsymbol{p}}\,', s'\big|\boldsymbol{\sigma}\boldsymbol{\cdot}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})\,\big|\vec{\boldsymbol{p}}, s\rangle,$$

after an integration by parts and using $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\phi$. That is,

$$V(\vec{\boldsymbol{x}}) = -e\phi(\vec{\boldsymbol{x}}) - \vec{\boldsymbol{d}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}})\,,$$

where

$$Q = -eF_1(0)\,, \qquad \vec{\boldsymbol{d}} = \frac{e}{m}F_3(0)\vec{\boldsymbol{S}}\,,$$

and $\vec{\boldsymbol{S}} = \frac{1}{2}\vec{\boldsymbol{\sigma}}$ is the nonrelativistic spin operator.

That is, $F_1(0) = 1$ and $F_3(0)$ yield the muon electric dipole moment.

Case 2: Static magnetic field with $A^\mu(x) = \big(0\,;\,\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}})\big)$.

We use the Fourier transform of the magnetic field, $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$,

$$\widetilde{\vec{\boldsymbol{B}}}(\vec{\boldsymbol{q}}) = \int (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}) e^{-i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}}}\, d^3x = i\vec{\boldsymbol{q}} \times \widetilde{\vec{\boldsymbol{A}}}\,,$$

after an integration by parts. The end result is

$$V(\vec{\boldsymbol{x}}) = \frac{e}{2m} F_1(0)\big(\vec{\boldsymbol{P}}\cdot\vec{\boldsymbol{A}} + \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{P}}\big) + \frac{e}{2m}\big[F_1(0) + F_2(0)\big]\boldsymbol{\sigma}\cdot\vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})$$
$$+\frac{e}{4m^2} F_4(0)\vec{\boldsymbol{\sigma}}\cdot\big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})\big).$$

The second term above corresponds to $V = -\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{B}}$, where

$$\vec{\boldsymbol{m}} = -\frac{e}{m}\big[1 + F_2(0)\big]\vec{\boldsymbol{S}} = -\frac{eg}{2m}\vec{\boldsymbol{S}}\,,$$

after putting $F_1(0) = 1$. That is, the anomalous magnetic moment is

$$F_2(0) = \tfrac{1}{2}(g - 2)\,.$$

Some details on the term proportional to $F_4(0)$

In the nonrelativistic limit, the term in the potential proportional to the form factor F_4 is given by,

$$\begin{aligned}\langle \vec{\boldsymbol{p}}\,', s'|\, V(\vec{\boldsymbol{x}})\, |\vec{\boldsymbol{p}}, s\rangle \supset & -\frac{e}{2m} F_4(0) \widetilde{A}^i(\vec{\boldsymbol{q}}) \big(q_i q_j - |\vec{\boldsymbol{q}}|^2 \delta_{ij}\big) \chi^\dagger_{s'} \sigma^j \chi_s \\ = & -\frac{e}{2m} F_4(0) \big[\widetilde{\vec{\boldsymbol{A}}}\cdot\vec{\boldsymbol{q}}\, \chi^\dagger_{s'} \vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{q}}\, \chi_s - |\vec{\boldsymbol{q}}|^2\, \chi^\dagger_{s'} \vec{\boldsymbol{\sigma}}\cdot\widetilde{\vec{\boldsymbol{A}}}\, \chi_s\big] \\ = & \frac{ie}{2m} F_4(0)\, \big[\chi^\dagger_{s'}\, \vec{\boldsymbol{\sigma}}\cdot(\vec{\boldsymbol{q}}\times\widetilde{\vec{\boldsymbol{B}}})\, \chi_s\big] \\ = & \frac{e}{4m^2} F_4(0)\, \langle \vec{\boldsymbol{p}}\,', s'|\, \vec{\boldsymbol{\sigma}}\cdot\big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})\big)\, |\vec{\boldsymbol{p}}, s\rangle\ ,\end{aligned}$$

where χ_s is the non-relativistic two-component spinor.

In analogy with the magnetic and electric dipole vectors, one can define the anapole vector $\vec{\boldsymbol{a}}$,

$$\vec{\boldsymbol{a}} = -\frac{e}{2m^2} F_4(0) \vec{\boldsymbol{S}}\,.$$

Thus, the P-violating interaction potential of a particle of charge $-e$ with anapole moment $\vec{\boldsymbol{a}}$ moving in a static magnetic field is given by,

$$V(\vec{\boldsymbol{x}}) = \frac{e}{4m^2} F_4(0) \vec{\boldsymbol{\sigma}} \cdot \big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})\big) = -\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) \,,$$

where $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}})$ is the external current that produces the static magnetic field.

In summary, the interaction energy of a particle with a magnetic, electric and anapole moment is given by

$$V = -\vec{\boldsymbol{m}} \cdot \vec{\boldsymbol{B}} - \vec{\boldsymbol{d}} \cdot \vec{\boldsymbol{E}} - \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{J}}.$$

The contribution of the anapole moment vanishes unless the source of the magnetic field $\vec{\boldsymbol{J}}$ is nonzero. That is, the coupling of the anapole moment to the external electromagnetic fields is of relevance only in matter [M. Nowakowski et al., Eur. J. Phys. **26**, 545 (2005)].

Isolating the Form Factors via projection

To isolate the form factors, the following identities are useful:

$$F_2(q^2) = \mathrm{Tr}\left\{ \left(g_1(q^2)\gamma_\mu + \frac{g_2(q^2)}{2m}(p+p')_\mu \right) (\not{p}' + m)\Gamma^\mu(p,p')(\not{p} + m) \right\},$$

$$F_3(q^2) = -\frac{i g_3(q^2)}{2m}(p+p')_\mu \,\mathrm{Tr}\{\gamma_5(\not{p}' + m)\Gamma^\mu(p,p')(\not{p} + m)\},$$

$$F_4(q^2) = g_4(q^2)\,\mathrm{Tr}\{\gamma_\mu\gamma_5(\not{p}' + m)\Gamma^\mu(p,p')(\not{p} + m)\},$$

where,

$$g_1(q^2) = \frac{m^2}{\left(\frac{1}{2}d - 1\right)q^2(4m^2 - q^2)}, \quad g_2(q^2) = -\frac{2m^2\left[2m^2 + \left(\frac{1}{2}d - 1\right)q^2\right]}{\left(\frac{1}{2}d - 1\right)q^2(4m^2 - q^2)^2},$$

$$g_3(q^2) = \frac{2m^2}{q^2(4m^2 - q^2)}, \quad g_4(q^2) = \frac{m^2}{\left(\frac{1}{2}d - 1\right)q^2(4m^2 - q^2)}.$$

Here, we perform the Dirac algebra in $d = 4 - 2\epsilon$ dimensions.

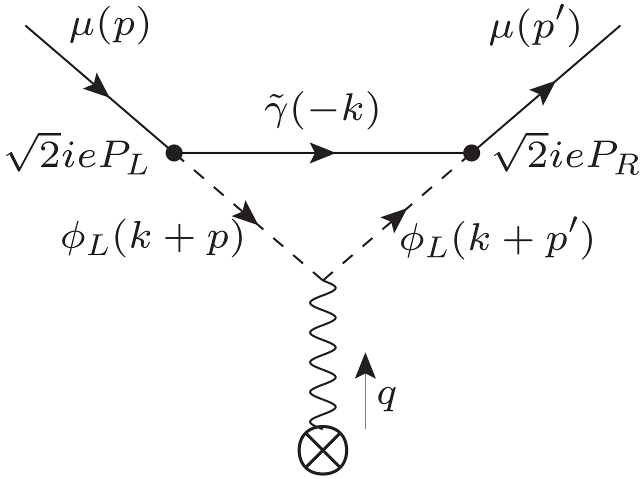
We shall evaluate Γ^μ at one-loop order using dimensional regularization. Two important checks of our calculations are:

- The singularities of the g_i at $q^2 = 0$ and $q^2 = 4m^2$ must cancel in the final result.
- The final result must be UV finite in which case one can set $\epsilon = 0$. Moreover, the intermediate steps of the calculation will *not* produce a finite term that can potentially result from $\epsilon \cdot \epsilon^{-1}$.

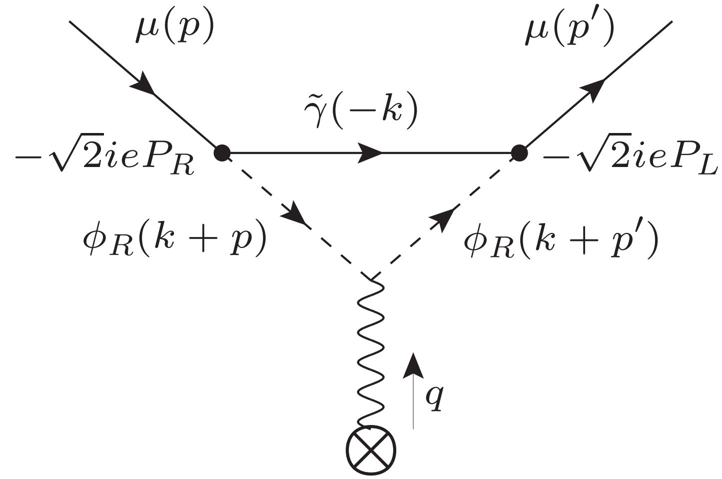
Remark: For a (neutral) Majorana fermion, the form factors $F_1(q^2) = F_2(q^2) = F_3(q^2) = 0$. Thus, the only nonzero form factor is $F_4(q^2)$. This is relevant in the study of Majorana fermion dark matter.

See Merlin Reichard's talk in Monday's parallel sessions entitled "Anapole Moment of Majorana Fermions and Implications for Direct Detection of Neutralino Dark Matter."

One-loop vertex contributions to $F_4(q^2)$



(a)



(b)

$$ieF_4(q^2)_{\rm vertex} = \frac{2m^2e^3}{q^2(4m^2-q^2)} \int \frac{d^dk}{(2\pi)^d} \frac{(p+p'+2k)^\mu}{(k^2-M^2)[(k+p)^2-m_L^2][(k+p')^2-m_L^2]}$$

$$\times\, {\rm Tr}\big[(\not{p}+m)\gamma_\mu\gamma_5(\not{p}'+m)\not{k}P_L\big] + (L \leftrightarrow R)\,,$$

where M is the photino mass, m_L and m_R are the masses of $\tilde{\mu}_L$ and $\tilde{\mu}_R$, respectively (smuon mixing is neglected), and $P_{R,L} \equiv \frac{1}{2}(1+\gamma_5)$.

After evaluating the trace and replacing $e \to e\mu^\epsilon$, where μ is the scale of DimReg,

$$F_4(q^2)_{\rm vertex} = \frac{-4im^2e^2\mu^{2\epsilon}}{q^2(4m^2-q^2)} \int \frac{d^dk}{(2\pi)^d} \left[k^2(4m^2{-}q^2){-}4p{\cdot}k\, p'{\cdot}k\right] \left(\frac{1}{D_L} - \frac{1}{D_R}\right),$$

where $D_{L,R} \equiv (k^2 - M^2)[(k+p)^2 - m^2_{L,R}][(k+p')^2 - m^2_{L,R}]$.

In terms of the Passarino-Veltman C functions,

$$F_4(q^2)_{\rm vertex} = \frac{\alpha m^2}{\pi q^2(4m^2-q^2)} \Big\{ m^2q^2\big[C_{21} + 4C_{22} - 4C_{23}\big] + \big[4m^2(d-1) - q^2(d-2)\big]C_{24} - (L \to R)\Big\},$$

where $\alpha \equiv e^2/(4\pi)$. After setting $p^2 = p'^{\,2} = m^2$, the arguments of the C-functions above are $(m^2, q^2, m^2; M^2, m_L^2, m_L^2)$. In the "$(L \to R)$" terms, one simply makes the replacement $m_L^2 \to m_R^2$.

The Passarino-Veltman loop functions

The loop functions are evaluated in $d = 4 - 2\epsilon$ dimensions.

$$B_1(p^2; m_a^2, m_b^2)p^\mu = -16\pi^2 i\mu^{2\epsilon} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu}{(q^2 - m_a^2 + i\varepsilon)[(q+p)^2 - m_b^2 + i\varepsilon]},$$

$$C_{21}p_1^\mu p_1^\nu + C_{22}p_2^\mu p_2^\nu + C_{23}(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) + C_{24}g^{\mu\nu} = -16\pi^2 i\mu^{2\epsilon} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{D_C},$$

where $C_{ij} \equiv C_{ij}(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2)$ and

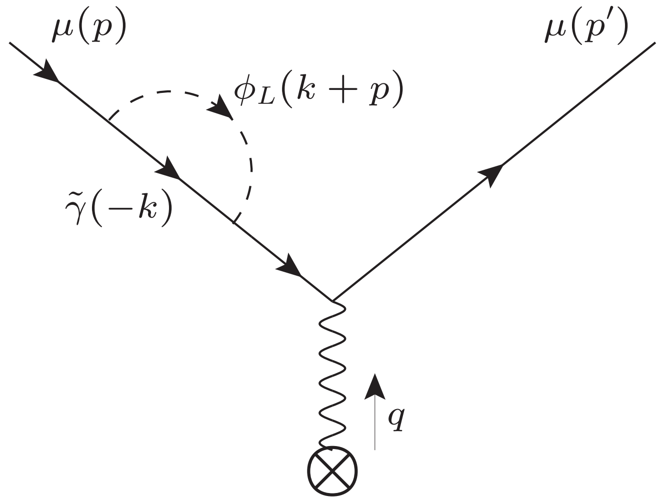
$$D_C \equiv (q^2 - m_a^2 + i\varepsilon)[(q+p_1)^2 - m_b^2 + i\varepsilon][(q+p_1+p_2)^2 - m_c^2 + i\varepsilon]\,.$$

C_{21}, C_{22} and C_{23} are finite as $\epsilon \to 0$, whereas B_1 and C_{24} are divergent,

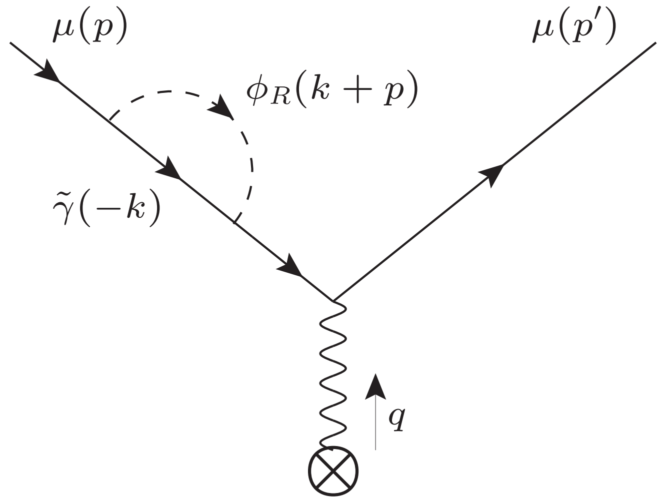
$$[B_1]_{\rm div} = -\frac{1}{2\epsilon}\,,$$

$$[C_{24}]_{\rm div} = \frac{1}{4\epsilon}\,.$$

Wave function renormalization contributions to $F_4(q^2)$



(a)



(b)

plus two diagrams where the self-energy insertions appear on the outgoing muon.

Self-energy corrections to external on-shell fermion lines are implemented by making the following replacements:

- $u(\vec{\boldsymbol{p}}, s) \to Z^{1/2} u(\vec{\boldsymbol{p}}, s)$ for an incoming fermion line.
- $\bar{u}(\vec{\boldsymbol{p}}, s) \to \bar{u}(\vec{\boldsymbol{p}}, s)\overline{Z}^{1/2}$ for an outgoing fermion line.

Here, Z is the wave function renormalization constant of the fermion and $\overline{Z} \equiv \gamma^0 Z^* \gamma^0$. In general, Z and $\overline{Z}$ can be decomposed into left-handed and right-handed contributions,

$$Z = P_L Z_L + P_R Z_R \,, \qquad \overline{Z} = P_R Z_L^* + P_L Z_R^* \,.$$

Note that $Z_{L,R} = 1 + \delta Z_{L,R}$, where $\delta Z_{L,R}$ represent the loop corrections. Hence, to one loop accuracy,

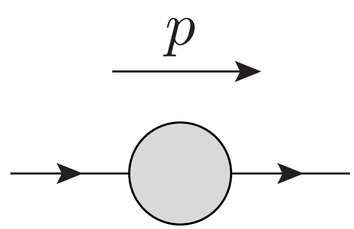
$$\bar{u}(\vec{\boldsymbol{p}}, s)\gamma_\mu u(\vec{\boldsymbol{p}}, s) \to \bar{u}(\vec{\boldsymbol{p}}, s)\gamma_\mu u(\vec{\boldsymbol{p}}, s) + \tfrac{1}{2}\bar{u}(\vec{\boldsymbol{p}}, s)\left[\delta\overline{Z}\gamma_\mu + \gamma_\mu \delta Z\right]u(\vec{\boldsymbol{p}}, s).$$

That is, the one-loop self-energy diagrams yield

$$\tfrac{1}{2}\bar{u}(\vec{\boldsymbol{p}}, s)\left[(\delta Z_L + \delta Z_L^*)\gamma_\mu P_L + (\delta Z_R + \delta Z_R^*)\gamma_\mu P_R\right]u(\vec{\boldsymbol{p}}, s).$$

One can compute δZ_L and δZ_R by evaluating the 1PI self-energy function of the fermion in the on-shell renormalization scheme.

Here, we shall (mostly) follow the analysis of B.A. Kniehl and A. Pilaftsis, Nucl. Phys. B **474**, 286 (1996).



$$-i\Sigma(p)$$

$$\Sigma(p) = \not{p}\left[P_L\Sigma_L(p^2) + P_R\Sigma_R(p^2)\right] + P_L\Sigma_D(p^2) + P_R\overline{\Sigma}_D(p^2)$$

One can then show that

$$\tfrac{1}{2}(\delta Z_L + \delta Z_L^*) = \Sigma_L(m^2) + \mathcal{D}\,,$$

$$\tfrac{1}{2}(\delta Z_R + \delta Z_R^*) = \Sigma_R(m^2) + \mathcal{D}\,,$$

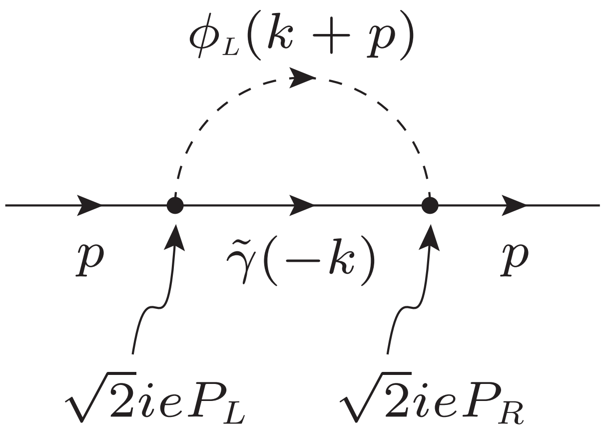
where

$$\mathcal{D} \equiv m^2\left[\Sigma'_L(m^2) + \Sigma'_R(m^2)\right] + m\left[\Sigma'_D(m^2) + \overline{\Sigma}'_D(m^2)\right].$$

and $\Sigma'(m^2) \equiv (d\Sigma(p^2)/dp^2)_{p^2=m^2}$.

Contribution of the photino–smuon loop to the muon self-energy

$-i\Sigma(p) =$



$+ \quad (L \to R \text{ and } e \to -e)$

$$= -2e^2\mu^{2\epsilon}\gamma_\mu P_L \int \frac{d^d k}{(2\pi)^d} \, \frac{k^\mu}{(k^2 - M^2)[(k+p)^2 - m_L]} \quad + \quad (L \to R)$$

In light of the decomposition of $\Sigma(p)$ given previously,

$$\Sigma_L(p^2) = \frac{\alpha}{2\pi} B_1(p^2; M^2, m_L^2)$$

$$\Sigma_R(p^2) = \frac{\alpha}{2\pi} B_1(p^2; M^2, m_R^2)$$

$$\Sigma_D(p^2) = \overline{\Sigma}_D(p^2) = 0 \, .$$

It follows that

$$F_4(q^2)_{\text{self-energy}} = \frac{m^2}{q^2(4m^2 - q^2)} \times \mathrm{Tr}\big[\gamma_\mu\gamma_5(\not{p}' + m)(c_L\gamma^\mu P_L + c_R\gamma^\mu P_R)(\not{p} + m)\big]\,,$$

where $c_L \equiv \frac{1}{2}(\delta Z_L + \delta Z_L^*)$ and $c_R \equiv \frac{1}{2}(\delta Z_R + \delta Z_R^*)$.

Plugging in the expressions for c_L and c_R obtained previously and evaluating the trace, the end result is proportional to $c_L - c_R$. Explicitly,

$$F_4(q^2)_{\text{self-energy}} = \frac{m^2}{q^2(4m^2 - q^2)}\big[4m^2(d-1) - q^2(d-2)\big]\big(\Sigma_L(m^2) - \Sigma_R(m^2)\big)\,.$$

We thus obtain,

$$F_4(q^2)_{\text{self-energy}} = \frac{\alpha m^2}{2\pi q^2(4m^2 - q^2)}\big[4m^2(d-1) - q^2(d-2)\big] \times\big[B_1(m^2; M^2, m_L^2) - B_1(m^2; M^2, m_R^2)\big]\,.$$

Full one-loop SUSY QED contribution to $F_4(q^2)$

Adding up the contributions from the vertex and self-energies yields,

$$F_4(q^2) = \frac{\alpha m^2}{\pi q^2(4m^2 - q^2)}\Big\{m^2 q^2\big[C_{21} + 4C_{22} - 4C_{23}\big] + \big[4m^2(d-1) - q^2(d-2)\big]\big[C_{24} + \tfrac{1}{2}B_1(m^2; M^2, m_L^2)\big] - (L \to R)\Big\},$$

Note that $C_{24} + \frac{1}{2}B_1$ is UV finite!

A nicer expression emerges by using the following two identities:

$$C_{21}(m^2, q^2, m^2; M^2, m_L^2, m_L^2) = 2C_{23}(m^2, q^2, m^2; M^2, m_L^2, m_L^2)$$

$$2C_{24}(m^2, q^2, m^2; M^2, m_L^2, m_L^2) + B_1(m^2; M^2, m_L^2) = q^2\big[C_{23}(m^2, q^2, m^2; M^2, m_L^2, m_L^2) - 2C_{22}(m^2, q^2, m^2; M^2, m_L^2, m_L^2)\big].$$

Thus, we arrive at our final result (setting $d = 4$),

$$F_4(q^2) = \frac{\alpha m^2}{\pi} \Big\{ C_{23}(m^2, q^2, m^2; M^2, m_L^2, m_L^2) \\ -2C_{22}(m^2, q^2, m^2; M^2, m_L^2, m_L^2) - (L \to R) \Big\} .$$

Notice that the singularities at $q^2 = 0$ and $q^2 = 4m^2$ have indeed canceled, and the loop corrections induced by $\widetilde{\mu}_L$ and $\widetilde{\mu}_R$ are separately UV finite.

Moreover, the anapole form factor vanishes exactly when $m_L = m_R$, as expected, since in this limit SUSY QED is parity invariant.

The static anapole moment of the muon is obtained by setting $q^2 = 0$. Using the integral representations of the C-functions and assuming that $m \ll m_{L,R}$, it follows that

$$F_4(0) = \frac{\alpha m^2}{6\pi} \int_0^1 \left(\frac{1}{(m_L^2 - M^2)x + M^2} - \frac{1}{(m_R^2 - M^2)x + M^2} \right) x^3 \, dx \, .$$

As an example, in the limit of $M = 0$,

$$F_4(0) = \frac{\alpha m^2}{18\pi} \left(\frac{1}{m_L^2} - \frac{1}{m_R^2} \right) .$$

In compressed SUSY, $m_L^2 = M^2 + \delta m_L^2$ and $m_R^2 = M^2 + \delta m_R^2$ where δm_L^2, $\delta m_R^2 \ll M^2$. In this limit,

$$F_4(0) = \frac{\alpha m^2}{30\pi} \left(\frac{\delta m_R^2 - \delta m_L^2}{M^4} \right) .$$

Challenges of a more complete calculation

A more complete calculation should incorporate the full electroweak sector, which will contribute to the anapole moment due to the parity-violating couplings of W and Z to the fermions. But, the literature is quite confusing on this matter.

"The gauge dependence of the static $(q^2 = 0)$ characteristics of charged leptons in the framework of the GSW (Glashow-Salam-Weinberg) model was studied. It was found that the anapole moments of leptons are gauge dependent and hence cannot be considered as observables."

H. Czyk et al., *Is the anapole moment a physical observable?*, Can. J. Phys. **66** (1988) 132.

"In contrast with the charge, magnetic [dipole] moment, and EDM, the anapole moment is not an intrinsic and well-defined property of an elementary particle."

M.J. Musolf and B.R. Holstein, *Observability of the anapole moment and neutrino charge radius*, Phys. Rev. D **43**, 2956 (1991).

"We derive an expression for the charge radius and anapole moment of a free fermion induced at one loop in the standard Glashow-Salam-Weinberg model of electroweak interactions. The result, despite earlier claims to the contrary, is demonstrably gauge-invariant and observable in principle."

A. Góngora-T. and R.G. Stuart, Z. Phys. C **55**, 101 (1992).

"The neutrino anapole moment and neutrino charge radius have finite gauge-invariant expressions in the SM; they define the axial vector (anapole) and the vector (NCR) contact interactions of any fermion with an external electromagnetic current, respectively. The situation with the anapole was also unclear because of the wrong statements of various authors about the electromagnetic interaction induced by the anapole moment of the particle."

V.M. Dubovik and V.E. Kuznetsov, *The Toroid Dipole Moment of the Neutrino*, Int. J. Mod. Phys. A **13**, 5257 (1998).

"Using the pinch technique we construct at one-loop order a neutrino charge radius, which is finite, depends neither on the gauge-fixing parameter nor on the gauge-fixing scheme employed, and is process independent."

J. Bernabéu et al., *Charge radius of the neutrino*, Phys. Rev. D **62**, 113012 (2000).

Will the anapole moment of the electron (or muon) ever be measured?

- First, one must clarify the definition of a physical gauge invariant anapole moment of a point particle. This will dictate the type of experiment needed to measure it. For example, in J. Erler, A. Kurylov and M.J. Ramsey-Musolf, Phys. Rev. D **68**, 016006 (2003), a formula is displayed for the weak charge of the proton that depends on one term called Δ'_e.

  "The latter, which corresponds to the anapole moment of the electron, depends on the choice of EW gauge and is not by itself a physical observable."

- The nuclear anapole moment of $^{133}\mathrm{Cs}$ has been extracted successfully from a measurement of the hyperfine dependence of the atomic parity violation (expected from the hadronic weak interaction). For more details, see W.C. Haxton and C.E. Wieman, Ann. Rev. Nucl. Part. Sci. **51**, 261 (2001).

Conclusions

1. It is desirable to clarify the meaning of a physical (gauge invariant) anapole moment of a (pointlike) charged lepton. Assuming such a process-independent quantity exists, it would be a universal property of the lepton as important as its charge and electric and magnetic dipole moments.

2. The anapole moment of the muon (or electron) in softly-broken SUSY QED, computed at one-loop order, is a finite gauge invariant quantity, which is physically meaningful.

3. Extension to the full MSSM requires addressing point 1 above.