Flat asymptotics, charges and dual charges

What the Cotton can do

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Why asymptotic symmetries and charges? [Komar '59; ADM '60; BMS '62]

- Universal features of solutions to Einstein's equations
- Hints to holography and in particular flat holography

Irrespective of holographic correspondence...

...a solution is captured by a set fields defined on a conformal boundary and obeying conformal boundary dynamics [Penrose '63]

Can we compute the charges from a boundary perspective?

Yes as a synthesis of bry. symmetry and bry. dynamics

Why Carrollian dynamics?

Asymptotically flat spacetimes $\rightarrow$ null boundary $\rightarrow$ Carrollian geometry

What is the Cotton? [Émile Cotton, 1899]

- Covariant derivative of the Einstein tensor in Riemannian geometries – remarkable in 3 dimensions (no Weyl)
- Admits Carrollian relatives on Carrollian geometries

The main message for 4-dim Ricci-flat spactimes

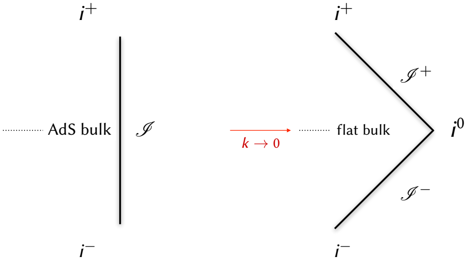
Boundary energy and momentum & Carrollian Cotton

- generate infinite *dual towers of charges* determined from a boundary account (e.g. mass vs. nut)
- carry part of the *infinite Chthonian information* required for reconstructing the bulk

A new asymptotic structure

From AdS$_n$ to flat$_n$ asymptotics

$$\Lambda = -\frac{(n-1)(n-2)}{2} k^2 \rightarrow 0$$



$k \equiv$ boundary velocity of light $\leftrightarrow$ $k \rightarrow 0$ Carrollian limit

The null bry. $\mathscr{I}^{\pm}$ is a Carrollian geometry in $n-1$ dimensions

Carrollian geometry [Lévy-Leblond '65; Duval et al. '14; Bekaert et al. '16]

Basic ingredients in $d+1$ dimensions (coordinates $t, \mathbf{x}$)

- degenerate metric: $\mathrm{d}s^2 = 0 \times \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j$
- field of observers (kernel): $\frac{1}{\Omega}\partial_t$ (t should be spelled u)
- clock form: $\boldsymbol{e} = \Omega \mathrm{d}t - b_i \mathrm{d}x^i$ (Ehresmann connection)

General covariance

Carrollian diffeomorphisms: $t' = t'(t, \mathbf{x}) \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})$

Consequences of the bry. Carrollian structure

I – Ricci-flat spacetime reconstruction in n dimensions

should be *Weyl invariant and Carrollian covariant* wrt the $n-1$ -dim conformal bry. – gauges as Bondi, Newmann–Unti are not

II – Flat holography - *if it exists*

calls for a *Carrollian conformal field theory* on an $n-1$-dim bry. e.g. Ricci-flat$_4$ dual to CCFT$_3$

Dynamics

General-covariant action and energy–momentum tensor

Pseudo-Riemannian spacetimes in $d+1$ dimensions

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

- Weyl invariance $\rightarrow T^{\mu}{}_{\mu} = 0$
- general covariance ($\xi = \xi^{\mu}(t, \mathbf{x})\partial_{\mu}$ diffeos) $\rightarrow \nabla_{\mu} T^{\mu\nu} = 0$
- ξ conformal Killing $\rightarrow I^{\mu} = \xi_{\nu} T^{\mu\nu}$ $\quad Q_{\xi} = \int_{\Sigma_d} * I$ conserved

Carrollian-covariant action, energy and momentum

$$\begin{cases} \Pi^{ij} = \frac{2}{\sqrt{a}\Omega} \frac{\delta S}{\delta a_{ij}} & \text{energy–stress tensor} \\ \Pi^{i} = \frac{1}{\sqrt{a}\Omega} \frac{\delta S}{\delta b_{i}} & \text{energy flux} \\ \Pi = -\frac{1}{\sqrt{a}} \left(\frac{\delta S}{\delta \Omega} + \frac{b_i}{\Omega} \frac{\delta S}{\delta b_i} \right) & \text{energy density} \end{cases}$$

In Carrollian spacetimes

- Weyl covariance $\to \Pi^i{}_i = \Pi$
- Carrollian covariance ($\xi = \xi^t(t, \mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i$ diffeos)

$$\to \begin{cases} \frac{1}{\Omega}\hat{\mathscr{D}}_t\Pi + \hat{\mathscr{D}}_i\Pi^i + \Pi^{ij}\xi_{ij} = 0 & \text{time} \\ \hat{\mathscr{D}}_i\Pi^i{}_j + 2\Pi^i\varpi_{ij} = -\left(\frac{1}{\Omega}\hat{\mathscr{D}}_t\delta^i_j + \xi^i{}_j\right)P_i & \text{space} \end{cases}$$

$\to$ momentum P_i

Currents and charges

- Carrollian current: Carrollian scalar κ and vector K^i
- Carrollian divergence: $\mathcal{K} = \frac{1}{\Omega}\hat{\mathscr{D}}_t\kappa + \hat{\mathscr{D}}_j K^j$
- Charge: $Q_K = \int_{\Sigma_d} \mathrm{d}^d x \sqrt{a}\left(\kappa + b_i K^i\right)$ conserved if $\mathcal{K} = 0$

Carrollian conformal isometries

Conformal Killings via $\mathscr{L}_\xi a_{ij}$ and $\mathscr{L}_\xi \frac{1}{\Omega}\partial_t$ *but not* $\mathscr{L}_\xi \boldsymbol{e}$

- $\kappa = \xi^i P_i - \xi^{\hat t}\Pi$ and $K^i = \xi^j \Pi_j{}^i - \xi^{\hat t}\Pi^i \quad \xi^{\hat t} = \xi^t - \xi^i \frac{b_i}{\Omega}$
- not conserved: $\mathcal{K} = -\Pi^i \mathscr{L}_\xi e_i$ [Petkou, Petropoulos, Rivera-Betancour, Siampos '22]

Remarkable property [Ciambelli, Leigh, Marteau, Petropoulos '19]

$\frac{1}{2\Omega} a^{ik}\left(\partial_t a_{kj} - a_{kj}\partial_t \ln\sqrt{a}\right) = \xi^i{}_j = 0 \Leftrightarrow a_{ij}(t,\mathbf{x}) = \mathrm{e}^{\sigma(t,\mathbf{x})}\bar{a}_{ij}(\mathbf{x}) \Leftrightarrow$
conformal Carroll isometries $\equiv$ conf$[\bar{a}_{ij}(\mathbf{x})] \ltimes$ supertranslations

$$\boxed{d+1=3 \to \mathfrak{so}(3,1) \ltimes \text{supertranslations} \equiv \mathrm{BMS}_4}$$

In summary

Mid-term main messages

- Null boundaries in asymptotically flat spacetimes are *Carrollian geometries* – zero speed of light
- Carrollian geometries with $\xi^{ij} = 0$ have an *infinite tower of conformal Killings* – in 3 dim *always* BMS_4
- Conformal-Killing charges exist but are *not always conserved*

Pure gravity – asymptotically flat or AdS

Basic field in pure gravity: metric G_{AB} in $n = d + 2$ dim

$\left\{r, t, x^i\right\}$, $i = 1, \ldots, d$ plus *gauge fixing* ($n = d + 2$ conditions)
$\rightarrow$ find the solutions as $\mathrm{O}\left(1/r^n\right)$ with coefficients $f(t, \mathbf{x})$

Gauge choice: twofold goal

- reach *manifest Weyl invariance and general covariance* for the dynamics on the $n - 1$-dim conformal boundary
- define charges from a *purely boundary perspective*

Einstein spacetimes covariantly reconstructed

Solution space with incomplete Newman–Unti gauge and mild boundary conditions [Ciambelli, Marteau, Petropoulos, Ruzziconi '20]

- $\frac{n(n-1)+2}{2}$ Einstein's equations $\rightarrow n^2 - 3$ functions of $(t, \mathbf{x})$ $\rightarrow$ boundary data $\mu, \nu, \ldots \in \{0, 1, \ldots, n-2 = d\}$
  - $g_{\mu\nu}$ symmetric $\leftarrow \frac{n(n-1)}{2}$
    boundary metric
  - $T_{\mu\nu}$ symmetric and traceless $\leftarrow \frac{n(n-1)}{2} - 1$
    conformal boundary energy–momentum tensor
  - $u^\mu \leftarrow n - 2$ (due to the gauge incompleteness $G_{ri} \neq 0$)
    boundary normalized vector field
- remaining $n - 1$ Einstein's equations $\boxed{\nabla_\mu T^{\mu\nu} = 0}$

On arbitrary (boundary) geometry $g_{\mu\nu}$ of dim $d+1$

$$\boxed{T^{\mu\nu} = \varepsilon \frac{u^\mu u^\nu}{k^2} + \frac{\varepsilon}{d} h^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\mu}{k^2} + \frac{u^\nu q^\mu}{k^2}}$$

- $\|\mathrm{u}\|^2 = -k^2 \quad h^{\mu\nu} = g^{\mu\nu} + \frac{u^\mu u^\nu}{k^2}$
- q^μ, $\tau^{\mu\nu}$ transverse plus $\tau^\mu{}_\mu = 0$

$\rightarrow$ "relativistic Weyl-covariant fluid"

In 4-dim bulk (3-dim boundary – $d = 2$): the Cotton tensor

$C_{\mu\nu} = \eta_\mu{}^{\rho\sigma} \nabla_\rho \left(R_{\nu\sigma} - \frac{R}{4} g_{\nu\sigma} \right)$ symmetric, traceless, $\nabla_\mu C^{\mu\nu} = 0$

decomposed along u^μ: c^μ, $c^{\mu\nu}$ transverse plus $c^\mu{}_\mu = 0$

$$\boxed{C_{\mu\nu} = c \frac{u^\mu u^\nu}{k^2} + \frac{c}{2} h^{\mu\nu} + c^{\mu\nu} + \frac{u^\mu c^\mu}{k^2} + \frac{u^\nu c^\mu}{k^2}}$$

What the Cotton can do in AdS$_4$ asymptotics

$C_{\mu\nu} \neq 0 \Leftrightarrow$ non-conformally flat bry. $\leftrightarrow$ asymptotically *locally* AdS bulk

ξ bry. conformal Killing $\rightarrow I^{\mu} = \xi_{\nu} T^{\mu\nu}$ and $I^{\mu}_{\mathrm{Cot}} = \xi_{\nu} C^{\mu\nu}$

$$Q_{\xi} = \int_{\Sigma_2} *\mathrm{I} \quad \text{and} \quad Q_{\mathrm{Cot}\,\xi} = \int_{\Sigma_2} *\mathrm{I}_{\mathrm{Cot}}$$

electric and *magnetic* dual conserved charges (bulk mass vs. nut)

- Remark: $Q_{\mathrm{Cot}\,\xi} \sim$ magnetic Komar charges
- "Self-duality": $\mathbf{q} - \frac{1}{8\pi G} * \mathbf{c} = 0$ and $\boldsymbol{\tau} + \frac{1}{8\pi G k^2} * \boldsymbol{c} = 0 \rightarrow$ resummed bulk metric $\rightarrow$ Petrov algebraically special
- Why? $T_{\mu\nu}$ and $C_{\mu\nu}$ enter asymptotically the bulk Weyl
- Limitation in AdS: at most 10 conformal Killings $(d + 1 = 3)$

Extendable in Ricci-flat spacetimes – more interesting

Ricci-flat in incomplete Newman–Unti gauge

Full solution space in $n = 4$ [Brussels & Paris groups]

$ds^2_{\text{Ricci-flat}}$ described in terms of $2+1$ *Carrollian boundary data*

- Carrollian geometry (6)
  - degenerate metric (3)
  - Ehresmann connection (3)
- "Carrollian conformal fluid" (5)
  - energy (1)
  - momenta – heat current (2) and stress tensor (2)
- Carrollian-fluid "velocity" (2) – hydro-frame freedom
- Carrollian dynamical shear (2) $\mathscr{C}_{ij}$
- *infinite* number of further Carrollian data obeying Carrollian dynamics – at every $\mathrm{O}\left(1/r^n\right)$: *Chthonian*

Salient features

- Weyl invariance & Carrollian covariance wrt boundary
- shear $\rightarrow$ news $\leftrightarrow$ bulk gravitational radiation
- boundary Carrollian fluid with Π, Π^i, Π^{ij}, P^i *under external force* – free if zero shear
- always $\xi_{ij} = 0 \Leftrightarrow \infty$ conformal Carrollian group $\mathrm{BMS}_4 \equiv$ asymptotic bulk symmetries
- boundary Carrollian Cotton in the form
  - Π_{Cot}, Π^i_{Cot}, Π^{ij}_{Cot}, P^i_{Cot}
  - Π'_{Cot}, Π'^i_{Cot}, Π'^{ij}_{Cot}, P'^i_{Cot}

  obeying Carrollian dynamics

Spin off: towers of Carrollian charges

From the boundary Carrollian fluid: electric tower

Q_ξ conserved in the absence of shear and for $\mathscr{L}_\xi \boldsymbol{e} = 0$

From the boundary Carrollian Cotton: magnetic tower

$Q_{\mathrm{Cot}\,\xi}$ conserved for $\mathscr{L}_\xi \boldsymbol{e} = 0$

Self-dual tower

$Q_{\mathrm{Cot}'\,\xi}$ conserved $\forall \xi$

Other towers

From Chthonian Carrollian data associated with the *subleading* $\mathrm{O}\left({}^{1}/_{r^n}\right)$ terms in the bulk action – under construction: Penrose charges and comparison with bulk approaches [Godazgar², Pope '18–21]

Final summary and prospects

Quotable facts

- n-dim Ricci-flat bulk $\leftrightarrow$ $n-1$-dim Carrollian boundary
- bulk reconstruction $\leftrightarrow$ infinite Chthonian Carrollian dofs
- towers of charges $\leftrightarrow$ Carrollian isometries & dynamics
- $n=4$ $\leftrightarrow$ prominent role of the Cotton and BMS_4 – *electric* vs. *magnetic*

Hints for flat "holography"

- Expected duality $\text{flat}_4/\text{CCFT}_3$
  - local (Chthonian)?
  - Carrollian CFTs (quantum)? [Le Bellac, Lévy- Leblond '67 & '73; Souriau '85; Duval et al. '14; Bagchi et al. '20; Henneaux, Salgado–Rebolledo '79 & '21; Rivera-B., Vilatte '22]
- What about $\text{flat}_4/\text{CFT}_2$ *celestial* holography? [Harvard school]
  - developed mostly for *radiation S*-matrix
  - based on "$SL(2,\mathbb{C})$" invariance – vs. BMS_4

Starring

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6 Appendix

EXAMPLE: $n = 3$ $(\Lambda = -k^2)$

NEWMAN–UNTI GAUGE WITH DIRICHLET [BAÑADOS '99]

- Asymptotic symmetry: $SO(2,2) \to$ Virasoro $\oplus$ Virasoro
- $\mathrm{d}s^2_{\text{locally AdS}_3} = -\frac{1}{k}\left(\mathrm{d}x^+ - \mathrm{d}x^-\right)\mathrm{d}r + r^2\mathrm{d}x^+\mathrm{d}x^- + \frac{1}{k^2}\left[L_+(x^+)\mathrm{d}x^+ - L_-(x^-)\mathrm{d}x^-\right]\left(\mathrm{d}x^+ - \mathrm{d}x^-\right)$ $(x^\pm = x \pm kt)$
- Bañados zero-modes include BTZ solutions [BTZ '92; BHTZ '93]
  $L_+ + L_- = M$ & $L_+ - L_- = kJ$

CHARGES AND ALGEBRA [BROWN, HENNEAUX '86]

$$L_m^\pm = \frac{1}{8\pi kG}\int_0^{2\pi}\mathrm{d}x\, \mathrm{e}^{\mathrm{i}mx^\pm}\left(L_\pm + \frac{1}{4}\right)$$

$$\mathrm{i}\left\{L_m^\pm, L_n^\pm\right\} = (m-n)L_{m+n}^\pm + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0}$$

$c = {}^3\!/_{2kG}$

In $n = 4$ dimensions $\Lambda = -3k^2$

General solution: $6 + 5 + 2$ arbitrary boundary data

- $\mathrm{d}s^2 = -k^2 \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j \to \{c, c^\mu, c^{\mu\nu}\}$
- $T_{\mu\nu} \to \{\varepsilon = 2p, q^\mu, \tau^{\mu\nu}\}$
- $\mathbf{u} = u_\mu \mathrm{d}x^\mu \to \left\{\sigma^{\mu\nu}, \omega^{\mu\nu}, \mathbf{A} = \frac{1}{k^2}\left(\mathbf{a} - \frac{\Theta}{2}\mathbf{u}\right), \mathscr{D}_\mu\right\}$

Incomplete Newman–Unti gauge $\to$ covariant wrt bry.

$$
\begin{aligned}
\mathrm{d}s^2_{\text{Einstein}} \quad = \quad & 2\frac{\mathbf{u}}{k^2}(\mathrm{d}r + r\mathbf{A}) + r^2 \mathrm{d}s^2 - 2\frac{r}{k^2}\sigma_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{\mathrm{S}}{k^4} \\
& + \frac{8\pi G}{k^4 r}\left[\varepsilon \mathbf{u}^2 + \frac{4\mathbf{u}}{3}\left(\mathbf{q} - \frac{1}{8\pi G} * \mathbf{c}\right)\right. \\
& \left. + \frac{2k^2}{3}\left(\boldsymbol{\tau} + \frac{1}{8\pi G k^2} * \boldsymbol{c}\right)\right] + \frac{1}{r^2}\left(c\gamma \frac{\mathbf{u}^2}{k^4} + \cdots\right) \\
& + \mathrm{O}\left({}^1\!/\!_{r^3}\right) \qquad \gamma^2 = \frac{1}{2k^4}\omega_{\alpha\beta}\omega^{\alpha\beta}
\end{aligned}
$$

$$
S_{\mu\nu} = 2u_{(\mu}\mathscr{D}_\lambda\left(\sigma_{\nu)}{}^\lambda + \omega_{\nu)}{}^\lambda\right) - \frac{\mathscr{R}}{2}u_\mu u_\nu + 2\omega_{(\mu}{}^\lambda \sigma_{\nu)\lambda} + (\sigma^2 + k^4\gamma^2)\,h_{\mu\nu}
$$

RICCI-FLAT SPACETIMES UP TO $O\left(1/r^3\right)$ – BRY. CARROLLIAN & WEYL COVARIANT

$$
\begin{aligned}
\mathrm{d}s^2_{\text{Ricci-flat}} \;=\;& 2\boldsymbol{\mu}\left(\mathrm{d}r + r\varphi_a \boldsymbol{\mu}^a - r\frac{\theta}{2}\boldsymbol{\mu} + *\boldsymbol{\mu}^b \hat{\mathscr{D}}_b * \varpi - \frac{1}{2}\boldsymbol{\mu}^a \hat{\mathscr{D}}_b \mathscr{C}^b{}_a\right) \\
& + \left(\rho^2 + \frac{\mathscr{C}_{cd}\mathscr{C}^{cd}}{8}\right)\mathrm{d}\ell^2 + \mathscr{C}_{ab}\left(r\boldsymbol{\mu}^a\boldsymbol{\mu}^b - *\varpi * \boldsymbol{\mu}^a\boldsymbol{\mu}^b\right) \\
& + \frac{1}{r}\left[\left(8\pi G\varepsilon - \hat{\mathscr{K}}\right)\boldsymbol{\mu}^2 + \frac{32\pi G}{3}\left(\pi_a - \frac{1}{8\pi G} * \psi_a\right)\boldsymbol{\mu}\boldsymbol{\mu}^a \right. \\
& \left. - \frac{16\pi G}{3}E_{ab}\boldsymbol{\mu}^a\boldsymbol{\mu}^b\right] + \frac{1}{r^2}\left[*\varpi c \boldsymbol{\mu}^2 + \cdots\right] + O\left(1/r^3\right)
\end{aligned}
$$