Dynamical restoration of conformal invariance in σ-models

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Introduction and motivation

In a renormalizable field theory, the RG flow of the coupling λ is given by

$$\beta^{\lambda} = \frac{d\lambda}{d\ln\mu^2}\,, \qquad \mu \text{ is the energy scale}$$

- Usually they are determined perturbatively

$$\lambda\Phi^4: \quad \beta^{\lambda} = \frac{3\lambda^2}{16\pi^2} + \cdots\,, \quad \text{QED:} \quad \beta^{e} = \frac{e^3}{24\pi^2} + \cdots\,,$$
$$\text{QCD:} \quad \beta^{g} = -\left(11N_c - 2N_f\right)\frac{g^3}{96\pi^2} + \cdots$$

An important class are the two-dimensional (integrable) σ-models:

- Technically simpler, combination of CFT and gravitational techniques.
- Embedding into supergravity and connection with holography.

Focal Points

In this talk we will focus on the non-Abelian bosonic Thirring model:

1. Introduce the model
2. It's integrable all-loop effective action
3. Dynamically promote its parameter(s) – restoration of conformality
4. Conclusion and future directions

Plan of the talk

Non-Abelian Thirring model

The effective action

Dynamical restoration of conformality

NON-ABELIAN THIRRING MODEL

Let us consider the WZW model at level k in light-cone coordinates Witten '83

$$S_k(\mathfrak{g}) = -\frac{k}{2\pi}\int \mathrm{d}^2\sigma\, \mathrm{Tr}\left(\mathfrak{g}^{-1}\partial_+\mathfrak{g}\,\mathfrak{g}^{-1}\partial_-\mathfrak{g}\right) + \frac{k}{12\pi}\int_B \mathrm{Tr}\left(\mathfrak{g}^{-1}\mathrm{d}\mathfrak{g}\right)^3 , \quad \sigma^\pm = \tau \pm \sigma$$

where $\mathfrak{g} \in G$ and it is invariant under the current algebra symmetry.

It has two conserved (anti-)chiral currents

$$J_{a+} = i\sqrt{k}\ \mathrm{Tr}(t_a\partial_+\mathfrak{g}\mathfrak{g}^{-1}) , \quad J_{a-} = -i\sqrt{k}\ \mathrm{Tr}(t_a\mathfrak{g}^{-1}\partial_-\mathfrak{g})$$

satisfying two current algebras at level k

$$J_a(z_1)J_b(z_2) = \frac{\delta_{ab}}{z_{12}^2} + \frac{1}{\sqrt{k}}\frac{f_{abc}J_c(z_2)}{z_{12}} + \cdots , \quad z_{12} := z_1 - z_2$$

The (bosonized) non-Abelian Thirring model is defined as follows

$$S = S_k(\mathfrak{g}) - \frac{\lambda}{\pi}\int \mathrm{d}^2\sigma\, J_{a+}\, J_{a-}$$

Non-Abelian Thirring model

Symmetries of the non-abelian bosonized Thirring model:

$$S = S_k(\mathfrak{g}) - \frac{\lambda}{\pi}\int \mathrm{d}^2\sigma\, J_{a+}\, J_{a-}$$

- Using conformal perturbation theory we find Kutasov '89

$$\beta^\lambda = \frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu^2} = -\frac{c_G}{2k}\frac{\lambda^2}{(1+\lambda)^2} + \mathcal{O}\left(\frac{c_G^2}{k^2}\right),$$

$$f_{acd}f_{bcd} = -c_G\delta_{ab} \quad \text{e.g.} \quad c_G = 2N \quad \text{for} \quad \mathfrak{g} \in SU(N)$$

  where μ is the RG flow energy scale.

- The perturbing operator is marginally relevant, UV is at $\lambda = 0$ & IR as $\lambda \to 1^-$

- The effective action is expected to be invariant under Kutasov '89

$$\lambda \to \lambda^{-1}, \quad k \to -k - c_G$$

  as does the RG flow for $k \gg 1$.

Can we capture the λ dependence in an effective action?

Plan of the talk

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The effective action

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The effective action

We propose as an all-loop action Sfetsos '13

$$S_{k,\lambda}(\mathfrak{g}) = S_k(\mathfrak{g}) - \frac{1}{\pi}\int d^2\sigma \left\{\left(\mathbb{I} - \lambda D^T\right)^{-1}\lambda\right\}_{ab} J_{a+}J_{b-}\,, \quad 0 \leqslant \lambda < 1$$

and there is also a scalar $e^{-2\Phi} = \det(\mathbb{I} - \lambda D^T)$ and $D_{ab} = \mathrm{Tr}(t_a \mathfrak{g} t_b \mathfrak{g}^{-1})\,, DD^T = \mathbb{I}$.

Properties:

- Explicit weak-strong duality: $S_{-k,\lambda^{-1}}(\mathfrak{g}^{-1}) = S_{k,\lambda}(\mathfrak{g})$ Itsios, Sfetsos, KS '14
- For $\lambda \ll 1$ we obtain the non-Abelian Thirring model.
- The eom take the form of a Lax connection. Itsios, Sfetsos, KS, Torrieli '14
- It possesses well behaved zoom-in limits around $\lambda = \pm 1$ and $\mathfrak{g} = \mathbb{I}$.
- Using σ-model techniques we find the expression of the Thirring model Itsios, Sfetsos, KS '14

$$\frac{d\lambda}{d\ln\mu^2} = -\frac{c_G\,\lambda^2}{2k\,(1+\lambda)^2} + \mathcal{O}\left(\frac{c_G^2}{k^2}\right)\,, \quad 0 \leqslant \lambda < 1$$

Marginal deformation

Let us consider the $SU(2)$ case & the deformation matrix $\lambda_{ab} = \mathrm{diag}(0,0,\lambda_3)$ with

- This corresponds to the ${}^{SU(2)_k \times U(1)}/_{U(1)}$ gauged WZW
  Horne, Horowitz '92; Giveon, Kiritsis '94

$$ds^2 = 2k\left(d\omega^2 + \frac{(1-\lambda_3)\cos^2\omega d\theta^2 + (1+\lambda_3)\sin^2\omega d\phi^2}{1+\lambda_3\cos 2\omega}\right)$$

$$B = k\frac{\lambda_3 + \cos 2\omega}{1+\lambda_3\cos 2\omega}d\theta \wedge d\phi\,, \quad \Phi = -\frac{1}{2}\ln(1+\lambda_3\cos 2\omega)$$

- Obtained via an $O(2,2)$ transformation on the $SU(2)_k$ exact string background.
  Hassan, Sen '92

NON-MARGINAL DEFORMATION

The simplest example is the λ-def $SU(2)_k/U(1)$ coset CFT – $\lambda_{ab} = \mathrm{diag}(\lambda,\lambda,1)$
Sfetsos 13'

$$S = \frac{k}{\pi}\int \mathrm{d}^2\sigma\left(\frac{1-\lambda}{1+\lambda}\left(\partial_+\beta\partial_-\beta + \cot^2\beta\,\partial_+\alpha\partial_-\alpha\right) + \frac{4\lambda}{1-\lambda^2}\left(\cos\alpha\,\partial_+\beta + \sin\alpha\,\cot\beta\,\partial_+\alpha\right)\left(\cos\alpha\,\partial_-\beta + \sin\alpha\,\cot\beta\,\partial_-\alpha\right)\right)$$

and the scalar $\Phi = -\ln\sin\beta$

1. Classically integrable and the conserved charges are in involution
   Hollowood, Miramontes, Schmidtt 14', 15'

   It respects the weak-strong duality Itsios, Sfetsos, KS 14'

   $$\lambda \to \lambda^{-1}\,,\quad k \to -k\,,\quad k \gg 1$$

2. Renormalizable at one-loop in $1/k$ expansion Itsios, Sfetsos, KS 14'

   $$\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu^2} = -\frac{\lambda}{k} \quad\Longrightarrow\quad \lambda = \left(\frac{\mu_0}{\mu}\right)^{2/k}, \quad \mathrm{UV}_{\lambda=0} \quad\Longrightarrow\quad \mathrm{IR}_{\lambda\to 1}$$

   The driving operator is relevant $\Delta_{\mathcal{O}} = 2 - 2/k$ – parafermionic bilinear.

Dynamical promotion – preservation or restoration of conformality?

Plan of the talk

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PRESERVATION OF CONFORMALITY

Consider again the ${}^{SU(2)_k\times U(1)}/_{U(1)}$ gauged WZW Aliaj, Sfetsos, KS '22

$$ds^2 = 2k\left(d\omega^2 + \frac{(1-\lambda_3)\cos^2\omega d\theta^2 + (1+\lambda_3)\sin^2\omega d\phi^2}{1+\lambda_3\cos 2\omega}\right)$$
$$B = k\frac{\lambda_3 + \cos 2\omega}{1+\lambda_3\cos 2\omega} d\theta\wedge d\phi\,, \quad \Phi = -\frac{1}{2}\ln(1+\lambda_3\cos 2\omega)$$

Set-up:

1. Add the term ${}^k/_\pi\,\partial_+ t\partial_- t$ to the Lagrangian: $\mathcal{L} = \frac{1}{2\pi}(G_{\mu\nu}+B_{\mu\nu})\partial_+X^\mu\partial_-X^\nu$
2. Let the parameter λ_3 to be a function of t & add $\Phi_0(t)$ to the dilaton Φ.
   Greene, Shapere, Vafa, Yau 90'; Kiritsis, Kounnas '93; Tseytlin '94
3. Demand conformality at one-loop order $\mathcal{O}({}^1/_k)$

$$R_{MN} - \frac{1}{4}H_{MKL}H_N{}^{KL} + 2\nabla_M\partial_N\Phi = 0\,, \quad \nabla^P(e^{-2\Phi}H_{MNP}) = 0\,,$$
$$w = R - \frac{1}{12}H_{MNP}H^{MNP} + 4\nabla^2\Phi - 4\big(\partial\Phi\big)^2 = \text{const.}$$

with the central charge read through $W = 4 - 3w$. Tseytlin 87' & 06'

PRESERVATION OF CONFORMALITY

4. Yields the system

$$\ddot{\lambda}_3 = \dot{\lambda}_3 \left(2h - \frac{\lambda_3 \dot{\lambda}_3}{1-\lambda_3^2}\right), \quad \dot{h} = -\frac{\lambda_3 \dot{\lambda}_3 h}{1-\lambda_3^2}, \quad h = \dot{\Phi}_0 ,$$
$$w = -\frac{2h^2}{k} + \frac{2}{k} + \frac{\dot{\lambda}_3(\dot{\lambda}_3 - 4\lambda_3 h)}{2k(1-\lambda_3^2)} = \text{const.}$$

   which can be easily integrated.

5. Trivial solution $\lambda_3(t) = \text{const.}$ and $\Phi_0(t) = Q\,t$, corresponding to the $SU(2)_k \times U(1) / U(1) \times \mathbb{R}_Q$ exact string background.

6. In the Lorentzian version $t \to it$, it corresponds to the Nappi–Witten exact CFT $\frac{SU(2)_k \times SL(2,\mathbb{R})_{-k}}{U(1) \times U(1)}$ Tseytlin '94, Nappi–Witten '92

$$\lambda_3(t) = \frac{\sin\alpha + \cos 2t}{1 + \sin\alpha \cos 2t}, \quad \Phi_0(t) = -\frac{1}{2} \ln(1 + \sin\alpha \cos 2t)$$

   and

$$w = \frac{2h^2}{k} + \frac{2}{k} - \frac{\dot{\lambda}_3(\dot{\lambda}_3 - 4\lambda_3 h)}{2k(1-\lambda_3^2)} = 0$$

RESTORATION OF CONFORMALITY

Consider again the λ-def $SU(2)_k/U(1)$ coset CFT Aliaj, Sfetsos, KS '22

$$S = \frac{k}{\pi}\int d^2\sigma \left(\frac{1-\lambda}{1+\lambda}\left(\partial_+\beta\partial_-\beta + \cot^2\beta\,\partial_+\alpha\partial_-\alpha\right) \right.$$
$$\left. + \frac{4\lambda}{1-\lambda^2}\left(\cos\alpha\,\partial_+\beta + \sin\alpha\,\cot\beta\,\partial_+\alpha\right)\left(\cos\alpha\,\partial_-\beta + \sin\alpha\,\cot\beta\,\partial_-\alpha\right)\right)$$
$$\Phi = -\ln\sin\beta$$

Following the same strategy

1. Add the term ${}^k/_\pi\,\partial_+ t\partial_- t$ to the Lagrangian.
2. Let the parameter λ to depend on t & add $\Phi_0(t)$ to the scalar Φ.
3. Demand conformality at one-loop order $\mathcal{O}(1/k)$

$$\ddot{\lambda} = -4\lambda + 2\dot{\lambda}\left(h - \frac{\lambda\dot{\lambda}}{1-\lambda^2}\right), \quad \dot{h} = \frac{\dot{\lambda}^2}{(1-\lambda^2)^2}, \quad h = \dot{\Phi}_0$$
$$w = -\frac{1}{k}\left(2h^2 - \dot{h}\right) + \frac{2}{k}\frac{1+\lambda^2}{1-\lambda^2} = \text{const.}$$

Restoration of conformality

4. Trivial solution with $\lambda(t) = 0$ and $\Phi_0(t) = Q\,t$, corresponding to the $^{SU(2)_k}/_{U(1)} \times \mathbb{R}_Q$ CFT
$$\Delta_{\mathcal{O}} = 2 - \frac{2}{k}$$
5. In the dynamical case $\lambda(t)$ acquires dimension and at the linear level as $t \longrightarrow -\infty$ we find
$$\lambda(t) \simeq c\, \mathrm{e}^{h_i t} \sin\left[\sqrt{4 - h_i^2}\,(t - t_0)\right], \quad \Phi_0(t) \simeq h_i t$$
Here $0 < h_i < 2$ for reality, weak string coupling $\mathrm{e}^{\Phi(t)} \ll 1$, as $t \longrightarrow -\infty$.
6. The scaling dimension $(\Delta, \bar{\Delta})$ of $\lambda(t)$ is read through
$$T_{zz} = -\frac{s}{\alpha'}(\partial X)^2 + Q\,\partial^2 X\,, \quad V_{\Delta,\bar{\Delta}} =: \mathrm{e}^{aX} :\,, \quad \Delta = \bar{\Delta} = \frac{s a \alpha'}{2}\left(Q - \frac{a}{2}\right)$$
yielding $\Delta = \bar{\Delta} = {}^1/_k$ and $\lambda(t)\mathcal{O}$ is a marginal operator.
7. The central charge reads
$$W = 3 - 3w = 2 - \frac{6}{k} + 1 + \frac{6h_i^2}{k} = c_{\mathrm{2d}} + c_{\ell.\mathrm{d.}}$$

Restoration of conformality

8. Conformality beyond $\mathcal{O}(\lambda)$ is ensured from the consistency conditions

$$\ddot{\lambda} = -4\lambda + 2\dot{\lambda}\left(h - \frac{\lambda\dot{\lambda}}{1-\lambda^2}\right), \quad \dot{h} = \frac{\dot{\lambda}^2}{(1-\lambda^2)^2}$$

9. As time progresses the model approaches the strong coupling region

$$t \to 0^-, \quad \lambda = 1 - \alpha^2, \quad \alpha \simeq 2t^2, \quad e^{\Phi_0} \simeq -\frac{1}{t}$$

where the corresponding the constant $w = 0$ or equivalently $h_i = 1$.

Conclusion & Outlook

Dynamical restoration of conformal invariance in a class of integrable σ-models:

- The deformation parameters λ_{ab} become dynamical functions of time.
- ODE ensure conformal invariance at one-loop order.
- We revisited the ${}^{SU(2)_k \times U(1)}/_{U(1)}$ CFT – preservation of conformality.
- We studied the λ-def ${}^{SU(2)_k}/_{U(1)}$ – restoration of conformality.
- Extensions: Restoring conformality in exact CFT interpolating models
  Aliaj, Sfetsos, KS '22

$$\text{UV:} \quad G_{k_1} \times G_{k_2} \quad \Longrightarrow \quad \text{IR:} \quad G_{k_2-k_1} \times G_{k_1}$$

or

$$\text{UV:} \quad \frac{G_{k_1} \times G_{k_2}}{G_{k_1+k_2}} \quad \Longrightarrow \quad \text{IR:} \quad \frac{G_{k_2-k_1} \times G_{k_1}}{G_{k_2}}$$

- Extension in multi-parameter cases.
- Similarly, for the integrable Yang–Baxter deformed PCMs. Klimčík '02

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Fermionic model

Solvable QFT describing self-interacting massless Dirac fields in 1+1 dimensions.

Fermion in 1+1 dimension with $SU(N)$ symmetry Dashen, Frishman '73 & '75

$$\mathcal{L} = i\,\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \rho\, J_{a\mu}J_{a}^{\mu}\,, \quad J_{a\mu} = \bar{\psi}t_{a}\gamma_{\mu}\psi$$

where t_a are Hermitian matrices with vanishing trace and

$$[t_a, t_b] = f_{abc}t_c\,, \quad \mathrm{Tr}(t_a t_b) = \delta_{ab}\,, \quad a = 1, 2, \cdots, N^2 - 1\,.$$

Properties:

- Abelian case $N = 1$
  Thirring '58; Johnson '61; Hagen '67; Klaiber '68; Koperin '79
- Classically integrable Vega, Eichenherr, Maillet '83
- (UV) Conformal point at $\rho = 0$ with $\Delta_{\psi} = \frac{1}{2}$
- Two (anti-)chiral conserved currents satisfying current algebras (OPE) at level one

$$J_a(z_1)J_b(z_2) = \frac{\delta_{ab}}{z_{12}^2} + \frac{1}{\sqrt{1}}\frac{f_{abc}\,J_c(z_2)}{z_{12}} + \cdots$$

- (IR) Conformal point at

$$\rho_{\star} = \frac{4\pi}{N+1} \quad \text{with} \quad \Delta_{\psi} = \frac{1}{2} + \frac{N-1}{N}$$

INTERPOLATING BETWEEN EXACT CFTS

Consider the λ-deformed $SU(2)_{k_1} \times SU(2)_{k_2}$ Georgiou, Sfetsos (2017)

$$S = S_{k_1}(\mathfrak{g}_1) + S_{k_2}(\mathfrak{g}_2) - \frac{1}{\pi}\lambda \int \mathrm{d}^2\sigma\, J^{(\mathbf{1})}_{a+} J^{(\mathbf{2})}_{a-}$$

1. The model is not marginal

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu^2} = -\frac{c_G}{2k}\frac{\lambda^2(\lambda-\lambda_0)(\lambda-\lambda_0^{-1})}{(1-\lambda^2)^2}\,, \quad \lambda_0 = \sqrt{\frac{k_1}{k_2}} < 1$$

   and it flows between

$$\text{UV:} \quad G_{k_1} \times G_{k_2} \quad \Longrightarrow \quad \text{IR:} \quad G_{k_2-k_1} \times G_{k_1}$$

2. Dynamical extension leads to the system for $\lambda(t)$ and $h(t) = \dot{\Phi}_0(t)$

$$\ddot{\lambda} = 2h\dot{\lambda} - \frac{4\lambda^2(\lambda-\lambda_0)(\lambda-\lambda_0^{-1})}{(1-\lambda^2)^2} + \frac{\lambda\dot{\lambda}^2}{1-\lambda^2}$$

$$\dot{h} = \frac{6\lambda^3(\lambda-\lambda_0)(\lambda-\lambda_0^{-1})}{(1-\lambda^2)^3} - \frac{3\lambda^2\dot{\lambda}^2}{(1-\lambda^2)^2} - \frac{3\lambda\dot{\lambda}h}{1-\lambda^2}$$

3. It admits interpolations

$$t \to -\infty : SU(2)_{k_1} \times SU(2)_{k_2} \times \mathbb{R}_{h_i} \Longrightarrow t \to +\infty : SU(2)_{k_1} \times SU(2)_{k_2-k_1} \times \mathbb{R}_{h_f}$$

   with $\quad h_f^2 - h_i^2 = \dfrac{\lambda_0^3}{2(1-\lambda_0^2)} > 0$