α_s from soft parton-to-hadron fragmentation in jets

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Precision measurements of the strong coupling constant
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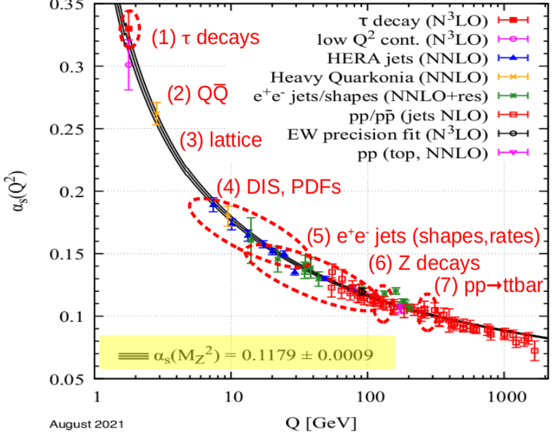
Outline

- **Jets in pQCD: parton cascade**
- **Resummation of infrared, collinear singularities via DLA/MLLA/NMLLA approach. . .**
- **Combined NMLLA+DGLAP evolution of Fragmentation Functions (FFs). NLO splitting functions and NNLO running coupling**
- **Parametrization of the FFs via distorted Gaussian. Energy evolution of its moments (mean multiplicity, mean peak position, width, skewness and kurtosis)**
- **Data–Theory comparison for jets from e^+e^--annihilation and DIS jets data sets in the range $2 - 200$ GeV**
- **Determination of $\alpha_s(M^2_{Z^0})$ from the energy evolution of FF moments**

World α_s determination (PDG 2021)

- Determined today by comparing 7 experimental observables to pQCD NNLO,N^{3}LO predictions, plus global average at the Z pole scale:

[Huston/Rabbertz/Zanderighi]



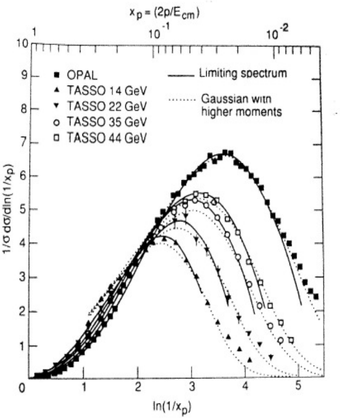
Motivation

The QCD coupling constant α_s

1. Less precisely determined among the coupling constants of the SM of particle physics
2. Importance: many fundamental SM observables at the LHC and future FCC-ee depend on this key parameter described by QCD
3. Current uncertainty of the strong coupling world-average value: $\alpha_s(m_Z) = 0.1179 \pm 0.0009$ is about 0.9%
4. Motivation: reduce the uncertainty by combining current α_s extractions with novel high-precision observables
5. Novel NMLLA+NNLO* $\alpha_s(m_Z)$ determination from the energy-evolution of the FF moments

Motivation

Can we do better than this MLLA results in the PDG?



19. Fragmentation functions in e^+e^-, ep and pp collisions

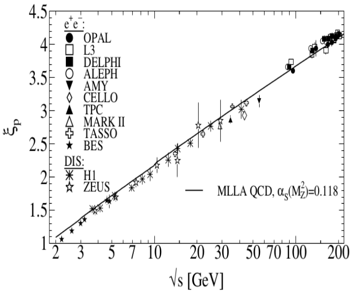
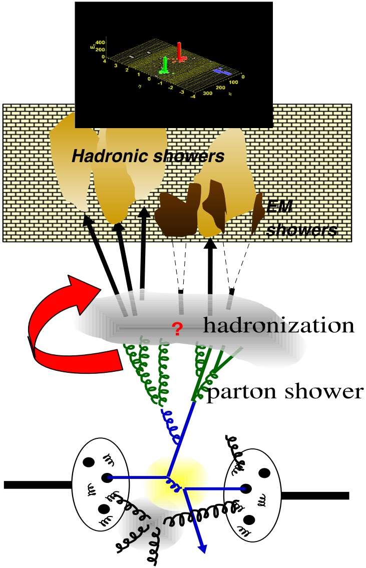


Figure 19.5: Evolution of the peak position, ξ_p, of the ξ distribution with the CM energy $\sqrt{s}$. The MLLA QCD prediction using $\alpha_S(s = M_Z^2) = 0.118$ is superimposed to the data of Refs. [26,28,29,32–34,36,41,55,56,73,74,77–85].

Part I

Jet fragmentation in pQCD

Jets in $e^+e^- \to q\bar{q}$, DIS, p-p, p-pbar

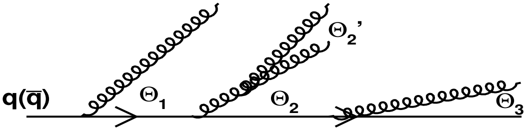


- Parton shower evolution in the Leading-Log Approx. (and extensions: MLLA, NMLLA,...$D \sim \delta\left(1 - \frac{x}{z}\right)$ for $Q_0 \to \Lambda_{\mathrm{QCD}}$: $1 \leftrightarrow 1$)

- Hadronization in the Local Parton Hadron Duality Hypothesis (LPHD):

  - Parton FFs $\simeq$ hadron distributions modulo overall constant factor $\mathcal{K}^{ch}$. [Dokshitzer, Khoze, Mueller]

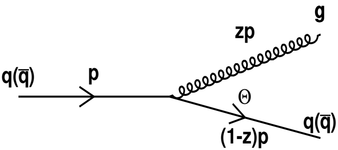
k_t-ordering (DGLAP) vs. Angular Ordering (MLLA)



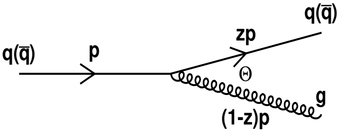
AO: $\Theta_1 \geq \Theta_2 (\geq \Theta_2') \geq \Theta_3$

Successive parton decays (soft/collinear and/or hard/collinear) ruled by:

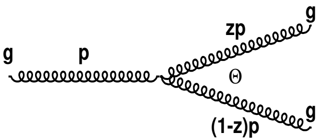
- $k_\perp$-ordering $\rightarrow$ DGLAP LLA evolution equations at large $x \sim 1$: ev. time variable "$t = \ln k_\perp$"
  - Hard FF $x > 0.1$: hard-hadrons in jets
- QCD coherence $\rightarrow$ Angular Ordering (AO) $\rightarrow$ MLLA evolution equations for FFs at small $x \ll 1$: ev. time variable $t = \ln \Theta$
  - Soft FF $x < 0.1$: bulk of hadron production in jets



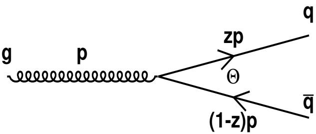
$$P_q^{gq}(z)=C_F\frac{1+(1-z)^2}{z}$$



$$P_q^{qg}(z)=C_F\frac{1+z^2}{1-z}$$



$$P_g^{gg}(z)=2C_A\left(\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right)$$



$$P_g^{q\bar{q}}(z)=T_R\left[z^2+(1-z)^2\right]$$

Soft/collinear res.: DLA, MLLA, next-to-MLLA,...

DLA: $\alpha_s \log(1/x) \log\Theta \sim 1$

1. resummation of soft and collinear gluons
2. main ingredient to the estimation of inclusive observables in jets
3. neglects recoil effects (i.e. energy conservation)
4. Anomalous dimension: $\gamma \sim \sqrt{\alpha_s}$

$$d\sigma_q^g = C_F \frac{\alpha_s}{\pi} \frac{dz}{z} \frac{dk_\perp^2}{k_\perp^2}, \qquad \gamma^{\mathrm{DLA}} = \frac{1}{2}\left(-\omega + \sqrt{\omega^2 + 8N_c \frac{\alpha_s}{\pi}}\right)$$

Soft/collinear res.: DLA, MLLA, next-to-MLLA,...

MLLA: $\alpha_s \log(1/x) \log\Theta + \alpha_s \log\Theta \sim 1 + \sqrt{\alpha_s}$

1. $\mathcal{O}(\alpha_s)$ collinear splittings (i.e. LLA FFs, PDFs at large $x \sim 1$)
2. partially "restores" recoil effects
3. includes α_s running coupling effects ($\propto \beta_0, \beta_1$)
4. Anomalous dimension: $\gamma \sim \sqrt{\alpha_s} + \alpha_s$

$$d\sigma_q^g = C_F \frac{\alpha_s(k_\perp^2)}{\pi} P_{qg}(z) dz \frac{dk_\perp^2}{k_\perp^2}, \qquad \gamma^{\mathrm{MLLA}} = \gamma^{\mathrm{DLA}} + \gamma^{\mathrm{SL}}$$

Soft/collinear res.: DLA, MLLA, next-to-MLLA,...

Next-to-...-MLLA anomalous dimension

After diagonalisation, the $D^h_{q,\bar{q},g}$ FFs can be determined throughout:

$$\gamma_{++}^{\mathrm{NMLLA}} \sim \sqrt{\alpha_s} + \alpha_s + \alpha_s^{3/2} + \dots$$

1. Further improve recoil effects
2. Includes higher order running coupling effects $\propto (\beta_0, \beta_1)$
3. Anomalous dimension: Note expansion in half-powers of α_s

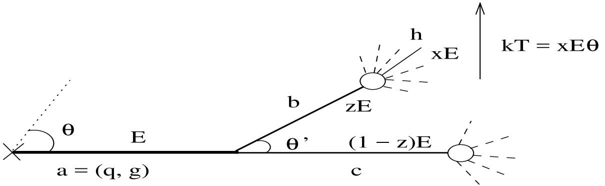
DGLAP versus MLLA evolution equations:

Renormalized QCD evolution equations for $a[1] \to b[z]c[1-z]$:

$$\frac{d}{d\ln\theta}\left[xD_a^b(x,\ln E\theta)\right]=\sum_c\int_0^1 dz\,P_{ac}(z)\frac{\alpha_s(\ln zE\theta)}{\pi}\left[\frac{x}{z}D_c^b\left(\frac{x}{z},\ln zE\theta\right)\right]$$

- z: energy fraction of intermediate parton; x: energy fraction of the hadron
- Identical but for one detail: for hard partons the shift in $\ln z$ in the argument of D and α_s is negligible:
- for soft/collinear splittings $z \ll 1$: $|\ln z| \gg 1$, $\Theta \ll 1$: DLA
  - for hard/collinear splittings $z \sim 1$: $\ln z \sim 0$, $\Theta \ll 1$: LLA
  - for soft/collinear + hard/collinear corrections: Modified-LLA (MLLA)

Solving the evolution equations at small x (or small ω)



Mellin transform:

$$\mathcal{D}(\omega, Y) = \int_0^\infty d\xi e^{-\omega\xi} D(\xi, Y), \quad \hat{\xi} = \ln\frac{1}{z}, \quad Y = \ln\frac{E\theta}{Q_0}$$

$$\Rightarrow \frac{\partial}{\partial Y}\mathcal{D}(\omega, Y) = \int_0^\infty d\hat{\xi}\left[e^{-\omega\hat{\xi}}\right] P(\hat{\xi})\frac{\alpha_s(Y-\hat{\xi})}{2\pi}\mathcal{D}(\omega, Y-\hat{\xi})$$

with $\overline{\mathrm{MS}}$ NLO:

$$\alpha_s(Y) = \frac{2\pi}{\beta_0(Y+\lambda)}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln 2(Y+\lambda)}{Y+\lambda}\right]$$

Solving the evolution equations at small x (or ω)

- Diagonalization of the matrix for $P(\Omega) \to 2$ eigenvalues: $P_{\pm\pm}(\Omega)$
- Express $\mathcal{D}_q$ and $\mathcal{D}_g$ as the linear combination of the corresponding eigenvectors: $\mathcal{D}^{\pm}$:

$$\frac{\partial}{\partial Y}\mathcal{D}^{\pm}(\omega, Y, \lambda) = P_{\pm\pm}(\Omega)\frac{\alpha_s(Y)}{2\pi}\mathcal{D}^{\pm}(\omega, Y, \lambda), \quad P_{++}(\Omega) = \frac{4N_c}{\Omega} - a_1 + 4N_c a_2 \Omega$$

- NMLLA evolution equation for $\mathcal{D}^{+}$:

$$\left(\omega + \frac{\partial}{\partial Y}\right)\frac{\partial}{\partial Y}\mathcal{D}^{+} = \left[1 - \frac{a_1}{4N_c}\left(\omega + \frac{\partial}{\partial Y}\right) + a_2\left(\omega + \frac{\partial}{\partial Y}\right)^2\right]4N_c\frac{\alpha_s}{2\pi}\mathcal{D}^{+}$$

- $\overline{\mathrm{MS}}$ NLO: $\alpha_s(Y) = \frac{2\pi}{\beta_0(Y+\lambda)}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln 2(Y+\lambda)}{Y+\lambda}\right]$
- DLA term: $\propto \mathcal{O}(1)$.
- Hard single logs: $\propto a_1 \sim \mathcal{O}(\sqrt{\alpha_s})$ (MLLA) & $\propto a_2 \sim \mathcal{O}(\alpha_s)$ (NMLLA)

Solution throughout anomalous dimension γ

- Replace the ansatz:

$$\mathcal{D}^{+}(\omega, Y, \lambda) = \exp\left[\int_0^Y dy\, \gamma(\omega, \alpha_s(y+\lambda))\right] \mathcal{D}^{+}(\omega, \lambda)$$

- $\Rightarrow$ quadratic equation for the anomalous dimension γ_{++}:

$$(\omega+\gamma_{++})\gamma_{++} - \frac{2N_c\alpha_s}{\pi} = -\beta(\alpha_s)\frac{d\gamma_{++}}{d\alpha_s} - a_1(\omega+\gamma_{++})\frac{\alpha_s}{2\pi} - \frac{a_1}{2\pi}\beta(\alpha_s) + a_2(\omega^2+2\omega\gamma_{++}+\gamma_{++}^2)\frac{\alpha_s}{2\pi},$$

• Approximate NLO (NLO*): Splitting functions at LO, $\overline{\rm MS}$ NLO α_s. Solved iteratively:

$$\beta(\alpha_s) = -\beta_0\frac{\alpha_s^2}{2\pi} - \beta_1\frac{\alpha_s^3}{4\pi^2} + \mathcal{O}(\alpha_s^4)$$

NMLLA anomalous dimension

$$\gamma_{++}^{\rm NMLLA} = \gamma^{\rm DLA}\left[\mathcal{O}(\sqrt{\alpha_s})\right] + \delta\gamma^{\rm MLLA}\left[\mathcal{O}(\alpha_s)\right] + \delta\gamma^{\rm NMLLA}\left[\mathcal{O}(\alpha_s^{3/2})\right]$$

Anomalous dimension (first attempt)

NMLLA anomalous dimension

Average multiplicity rate in QCD jets (for $\omega = 0$):

$$\gamma_{++}^{\mathrm{NMLLA}} = P_{++}^{(0)}(\omega) + P_{++}^{(1)}(\omega) + P_{++}^{*(2)}(\omega)$$

DLA:

$$P_{++}^{(0)} = \frac{1}{2}\omega(s-1) = \mathcal{O}(\sqrt{\alpha_s})$$

with

$$s = \sqrt{1 + \frac{4\gamma_0^2}{\omega^2}}, \text{ where } \gamma_0^2 = \frac{4N_c\alpha_s}{2\pi}$$

Anomalous dimension (first attempt)

NMLLA anomalous dimension

Average multiplicity rate in QCD jets (for $\omega = 0$):

$$\gamma_{++}^{\mathrm{NMLLA}} = P_{++}^{(0)}(\omega) + P_{++}^{(1)}(\omega) + P_{++}^{*(2)}(\omega)$$

MLLA:

$$P_{++}^{(1)} = \frac{\alpha_s}{2\pi}\left[-\frac{1}{2}a_1(1+s^{-1}) + \frac{\beta_0}{4}(1-s^{-2})\right] = \mathcal{O}(\alpha_s)$$

with

$$s = \sqrt{1+\frac{4\gamma_0^2}{\omega^2}},\ \text{where } \gamma_0^2 = \frac{4N_c\alpha_s}{2\pi}$$

Anomalous dimension (first attempt)

NMLLA anomalous dimension

Average multiplicity rate in QCD jets (for $\omega = 0$):

$$\gamma_{++}^{\mathrm{NMLLA}} = P_{++}^{(0)}(\omega) + P_{++}^{(1)}(\omega) + P_{++}^{*(2)}(\omega)$$

NMLLA:

$$P_{++}^{*(2)} = \frac{\alpha_s^2(\omega s)^{-1}}{104\pi^2}\left[4a_1^2(1-s^{-2}) + 8a_1\beta_0(1-s^{-3}) + \beta_0^2(1-s^{-2})(3+5s^{-2}) - 64N_c\frac{\beta_1}{\beta_0}\ln(Y+\lambda)\right] + \frac{N_c\alpha_s}{2\pi}a_2(\omega s)\left[\left(1+s^{-1}\right)^2\right] = \mathcal{O}\left(\alpha_s^{3/2}\right)$$

with

$$s = \sqrt{1+\frac{4\gamma_0^2}{\omega^2}}, \text{ where } \gamma_0^2 = \frac{4N_c\alpha_s}{2\pi}$$

Anomalous dimension (first attempt)

Extension using $\overline{\mathrm{MS}}$ NLO+resummed (small-x) splittings functions (based on C.-H. Kom, A. Vogt, K. Yeats JHEP 1210 (2012) 033)

$$
\begin{aligned}
P^T_{qq}(N) &= \frac{4}{3}\frac{C_F n_f}{C_A} a_s \left\{ \frac{1}{2\xi}(S-1)(\mathcal{L}+1)+1 \right\} \\
&+ \frac{1}{18}\frac{C_F n_f}{C_A^3} a_s \bar{N} \Big\{ (-11 C_A^2 + 6 C_A n_f - 20 C_F n_f)\frac{1}{2\xi}(S-1+2\xi) + 10 C_A^2 \frac{1}{\xi}(S-1)\mathcal{L} \\
&- (51 C_A^2 - 6 C_A n_f + 12 C_F n_f)\frac{1}{2}(S-1) + (11C_A^2+2C_A n_f - 4 C_F n_f) S^{-1}\mathcal{L} \\
&+ (5C_A^2 - 2C_A n_f + 6 C_F n_f)\frac{1}{\xi}(S-1)\mathcal{L}^2 + (51 C_A^2 - 14 C_A n_f + 36 C_F n_f)\mathcal{L} \Big\}, \qquad (3.2)
\end{aligned}
$$

$$
P^T_{qg}(N) = \frac{C_A}{C_F} P^T_{qq}(N) - \frac{2}{9}\frac{n_f}{C_A^2} a_s \bar{N} (C_A^2 + C_A n_f - 2 C_F n_f)\left\{\frac{1}{2\xi}(S-1)(\mathcal{L}+1)+1\right\}, \qquad (3.3)
$$

$$
\begin{aligned}
P^T_{gg}(N) &= \frac{1}{4} \bar{N}(S-1) - \frac{1}{6C_A} a_s (11C_A^2+2C_An_f-4C_Fn_f)(S^{-1}-1) - P^T_{qq}(N) \\
&+ \frac{1}{576 C_A^3} a_s \bar{N} \Big\{ \big([1193-576\zeta_2]C_A^4 - 140 C_A^3 n_f + 4 C_A^2 n_f^2 - 56 C_A^2 C_F n_f + 16 C_A C_F n_f^2 \\
&- 48 C_F^2 n_f^2\big)(S-1) + \big([830-576\zeta_2]C_A^4 + 96 C_A^3 n_f - 8 C_A^2 n_f^2 - 208 C_A^2 C_F n_f \\
&+ 64 C_A C_F n_f^2 - 96 C_F^2 n_f^2\big)(S^{-1}-1) + (11C_A^2+2C_An_f-4C_Fn_f)^2(S^{-3}-1)\Big\}, \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
P^T_{gq}(N) &= \frac{C_F}{C_A} P^T_{gg}(N) - \frac{1}{3} \frac{C_F}{C_A^2} a_s (C_A^2 + C_A n_f - 2C_F n_f)\frac{1}{\xi} (S-1+2\xi) \\
&+ \frac{1}{36}\frac{C_F}{C_A^4} a_s \bar{N} \Big\{ \big(11 C_A^4 + 13 C_A^2 n_f (C_A - 2C_F) + 2 C_A^2 n_f^2 - 8(C_A - C_F) C_F n_f^2\big)(1-S^{-1}) \\
&- \big(48 C_A^4 - 45 C_A^3 C_F - 72\zeta_2 C_A^3 (C_A - C_F) - 33 C_A^3 n_f + 2 C_A^2 n_f^2 + 48 C_A^2 C_F n_f \\
&- 8 C_F^2 n_f^2\big)\frac{1}{\xi}(S-1+2\xi) + \big(-54 C_A^4 + 45 C_A^3 C_F + 72\zeta_2 C_A^3 (C_A-C_F) + 23 C_A^3 n_f \\
&- 28 C_A^2 n_f C_F - 8(C_A - 2C_F)C_F n_f^2\big)\frac{1}{\xi} (S-1)\mathcal{L}\Big\} \qquad (3.5)
\end{aligned}
$$

Anomalous dimension (first attempt)

NNLL+NLO* anomalous dimension

Average multiplicity rate in QCD jets (for $\omega = 0$):

$$\gamma_{++}^{\mathrm{NNLL+NLO}^*} = P_{++}^{(0)}(\omega) + P_{++}^{(1)}(\omega) + P_{++}^{(2)}(\omega)$$

NNLL+NLO*:

$$P_{++}^{(2)} = \frac{\alpha_s^2}{104\pi^2}(\omega s)^{-1}\Big[4a_1^2(1-s^{-2}) + 8a_1\beta_0(1-s^{-3}) + \beta_0^2(1-s^{-2}) \times (3+5s^{-2}) - 64N_c\frac{\beta_1}{\beta_0}\ln(Y+\lambda)\Big] + \frac{N_c\alpha_s}{2\pi}a_2(\omega s)\Big[\left(1+s^{-1}\right)^2 + a_3\,(s-1) - a_4\left(1-s^{-1}\right) - a_6\Big] = \mathcal{O}\left(\alpha_s^{3/2}\right)$$

with

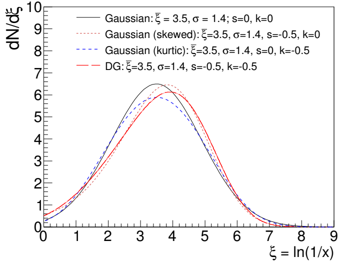
$$s = \sqrt{1+\frac{4\gamma_0^2}{\omega^2}}, \text{ where } \gamma_0^2 = \frac{4N_c\alpha_s}{2\pi}$$

Part II

Phenomenology: ansatz for FF

Single inclusive distribution: Distorted Gaussian

- $D^{+}(\xi,Y,\lambda)=\frac{\mathcal{N}}{\sigma\sqrt{2\pi}}\exp\left[\frac{1}{8}k-\frac{1}{2}s\delta-\frac{1}{4}(2+k)\delta^2+\frac{1}{6}s\delta^3+\frac{1}{24}k\delta^4\right]$



- $\delta=\frac{(\xi-\bar{\xi})}{\sigma}$
- Mean multiplicity: $\mathcal{N}=\mathcal{D}^{+}(\omega=0,Y,\lambda)$
- Mean peak position: $\bar{\xi}$
- Dispersion (width): σ
- Skewness: s, kurtosis: k

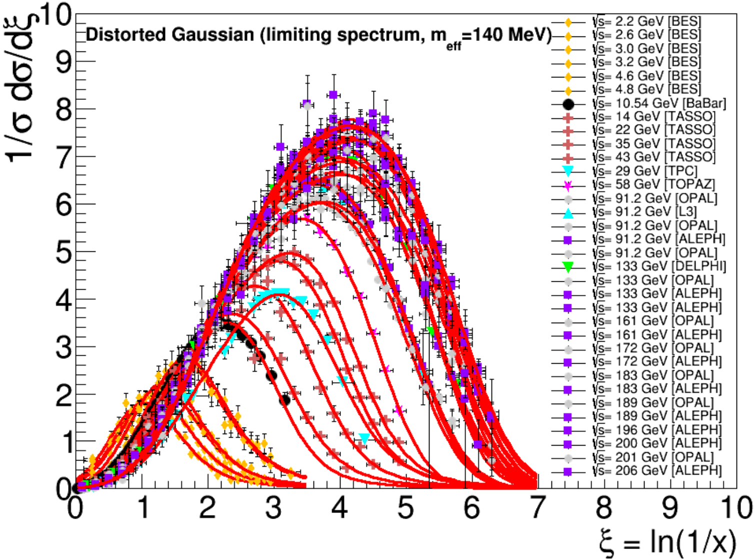
- Moments of the Distorted Gaussian (from anomalous dimension):

$$\mathcal{N}=K_0,\quad \bar{\xi}=K_1,\quad \sigma=\sqrt{K_2},\quad s=\frac{K_3}{\sigma^3},\quad k=\frac{K_4}{\sigma^4}$$

$$K_{n\geq 0}=\int_0^Y dy\left(-\frac{\partial}{\partial\omega}\right)^n\gamma_{++}^{\mathrm{NNLL+NLO^*}}(\alpha_s(y+\lambda))\Big|_{\omega=0},\ Y=\ln\frac{E\theta}{Q_0}$$

- Skewness and kurtosis (new ingredient) affect tails $\neq$ Gaussian shape!

Example: Distorted Gaussian fits to e^+e^- FFs



Evolution of the NNLL+NLO* moments of the DG FFs

Final expressions as a function of $Y = \ln(E\theta/Q_0)$ and $\lambda = \ln(Q_0/\Lambda_{QCD})$:
(N_f=5) initial jet energy — shower energy cutoff

- **Multiplicity:**

$$\mathcal{N}(Y) = \mathcal{K}^{\rm ch}\exp\Big[2.50217\left(\sqrt{Y+\lambda}-\sqrt{\lambda}\right) - 0.491546\ln\frac{Y+\lambda}{\lambda} + (0.0153206 + 0.41151\ln(Y+\lambda))\frac{1}{\sqrt{Y+\lambda}} - (0.0153206+0.41151\ln\lambda)\frac{1}{\sqrt{\lambda}}\Big]. \quad (71)$$

- **Average:**

$$\bar{\xi}(Y) = 0.5Y + 0.592722\left(\sqrt{Y+\lambda}-\sqrt{\lambda}\right) + 0.0763404\ln\frac{Y+\lambda}{\lambda}. \quad (73)$$

- **Peak position:**

$$\xi_{\max}(Y) = 0.5Y + 0.592722\left(\sqrt{Y+\lambda}-\sqrt{\lambda}\right) + 0.0763404\ln\frac{Y+\lambda}{\lambda} - 0.355325. \quad (74)$$

- **Width:**

$$\sigma(Y,\lambda) = \left(\frac{\beta_0}{144N_c}\right)^{1/4}\sqrt{(Y+\lambda)^{3/2}-\lambda^{3/2}}\Bigg\{1 - \frac{\beta_0}{64}f_1(Y,\lambda)\sqrt{\frac{16N_c}{\beta_0(Y+\lambda)}} + \Bigg[\frac{3}{16}(3a_2+a_3+2a_4)f_2(Y,\lambda) - \frac{3}{64}\left(\frac{3a_1^2}{16N_c^2}f_2(Y,\lambda) + \frac{a_1\beta_0}{8N_c^2}f_2(Y,\lambda) - \frac{\beta_0^2}{64N_c^2}f_2(Y,\lambda) + \frac{3\beta_0^2}{128N_c^2}f_1^2(Y,\lambda)\right) + \frac{\beta_1}{64\beta_0}(\ln 2(Y+\lambda)-2)f_3(Y,\lambda)\Bigg]\frac{16N_c}{\beta_0(Y+\lambda)}\Bigg\}, \quad (75)$$

$$\sigma(Y) = 0.36499\sqrt{(Y+\lambda)^{3/2}-\lambda^{3/2}}\Big\{1 - 0.299739f_1(Y,\lambda)\frac{1}{\sqrt{Y+\lambda}} - [1.61321f_2(Y,\lambda) + 0.0449219f_1^2(Y,\lambda) + (0.32239 - 0.246692\ln(Y+\lambda))f_3(Y,\lambda)]\frac{1}{Y+\lambda}\Big\}. \quad (76)$$

- **Skewness:**

$$s(Y) = -\frac{1.94704}{\sqrt{(Y+\lambda)^{3/2}-\lambda^{3/2}}}\left[1 - 0.299739f_1(Y,\lambda)\frac{1}{\sqrt{Y+\lambda}}\right]. \quad (78)$$

- **Kurtosis:**

$$k(Y) = -\frac{2.15812}{\sqrt{Y+\lambda}}\frac{1-\left(\frac{\lambda}{Y+\lambda}\right)^{5/2}}{\left[1-\left(\frac{\lambda}{Y+\lambda}\right)^{3/2}\right]^2}\Big\{1 + [1.19896f_1(Y,\lambda) - 1.99826f_4(Y,\lambda)]\frac{1}{\sqrt{Y+\lambda}} + [1.07813f_1^2(Y,\lambda) + 6.45283f_2(Y,\lambda) + 1.28956f_3(Y,\lambda) - 2.39583f_1(Y,\lambda)f_4(Y,\lambda) - 7.13372f_5(Y,\lambda) + 0.0217751f_6(Y,\lambda) - (0.986767f_3(Y,\lambda) - 0.822306f_6(Y,\lambda))\ln(Y+\lambda)]\frac{1}{Y+\lambda}\Big\}. \quad (80)$$

Evolution of the NNLL+NLO* moments of the (DG) FFs

Expressions evolved down to $\Lambda_{\rm QCD}$: $Q_0 \sim \Lambda_{\rm QCD}$:

$$
\begin{aligned}
*\text{Multiplicity}:\quad \mathcal{N}(Y) &= \mathcal{K}^{\rm ch}\exp\left[2.50217\sqrt{Y} - 0.491546\ln Y - (0.06889 - 0.41151\ln Y)\frac{1}{\sqrt{Y}}\right.\\
&+ \left.(0.00068 - 0.161658\ln Y)\frac{1}{Y}\right]
\end{aligned}
$$

$$*\text{Peak position}:\quad \xi_{\rm max}(Y) = 0.5Y + 0.592722\sqrt{Y} - 0.351319 + 0.002$$

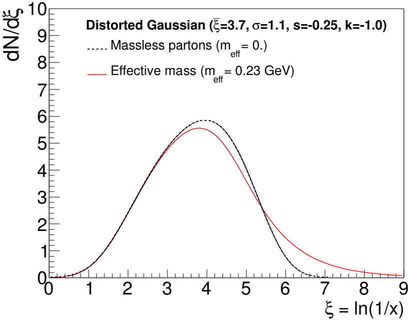
$$*\text{Width}:\quad \sigma(Y) = 0.36499Y^{3/4}\left[1 - 0.299739\frac{1}{\sqrt{Y}} - (1.4921 - 0.246692\ln Y)\frac{1}{Y} + \frac{1.98667}{Y^{3/2}}\right]$$

$$*\text{Skewness}:\quad s(Y) = -\frac{1.89445}{Y^{3/4}}\left[1 - 0.312499\frac{1}{\sqrt{Y}} - \frac{1.64009}{Y}\right]$$

$$*\text{Kurtosis}:\quad k(Y) = -\frac{2.15812}{\sqrt{Y}}\left[1 - 0.799305\frac{1}{\sqrt{Y}} + (0.730466 - 0.164461\ln Y)\frac{1}{Y} - \frac{8.05771}{Y^{3/2}}\right] \qquad (1)$$

$*$ Evolution of all moments depend on 1 single free parameter $\Lambda_{\rm QCD}$, which can be extracted from fits of exp. e^+e^- and $e^-p \to$ jets(hadrons) data

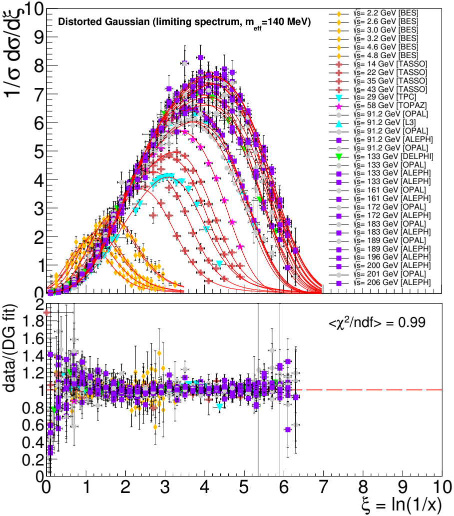
Hadron mass effects



Including hadron mass m_h effects (mixture of pions (65%), kaons (35%) and protons (5%)):

$$\frac{1}{\sigma_{\mathrm{tot}}}\frac{d\sigma^{\mathrm{h}}}{d\xi_p} \propto \frac{p_h}{E_h} D^+(\xi, Y) \qquad \xi = \ln(1/x) = \ln\left(\frac{\sqrt{s}/2}{\sqrt{(s/4)e^{-2\xi_p} + m_{\mathrm{eff}}^2}}\right)$$

$$m_h \sim \mathcal{O}(\Lambda_{\mathrm{QCD}}), \quad E_h = \sqrt{p_h^2 + m_{\mathrm{eff}}^2}, \quad p_h = (\sqrt{s}/2)\exp(-\xi_p)$$



- Best agreement reached for $m_h = 0.14$ GeV: consistent with a dominant pion composition of the inclusive charged hadron spectra.

Part III

Extraction of $\alpha_s(M^2_{Z_0})$ from fits

Fitting procedure

- Experimental distribution will be fitted to the DG parametrization as a function of ξ in the energy range $[0, Y]$

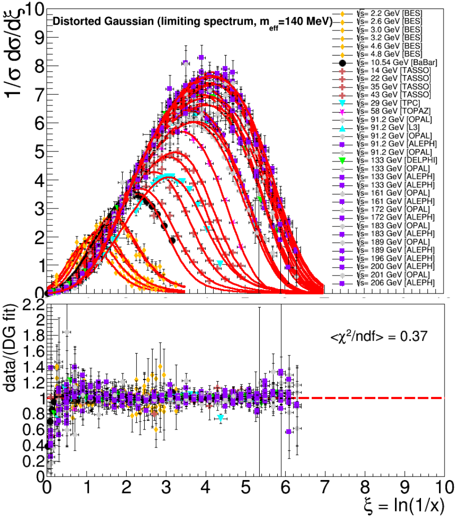
$$\frac{1}{\sigma_{\text{tot}}}\frac{d\sigma^{\text{h}}}{d\xi} = \mathcal{K}^{\text{ch}}\frac{2C_F}{N_c}D^{+}(\xi, Y), \quad Y = \ln\left(\frac{\sqrt{s}}{2\Lambda_{\text{QCD}}}\right)$$

- Each fit of the DG has five free parameters: maximum peak position, total multiplicity, width, skewness and kurtosis

- Each parameter or component of the DG is derived from the fit for each data set

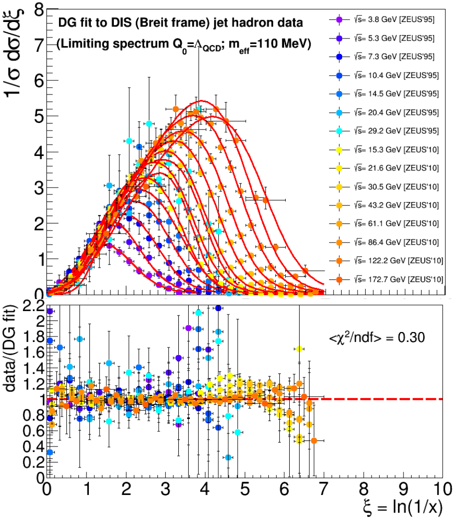
Distorted Gaussian fits to e^+e^- FFs

- 34 e^+e^- data-sets at $\sqrt{s} = 2.2 - 206$ GeV ~ 1200 data points
- For increasing energy: peak shifts to right, width increases, moderate non-Gaussian tails
- Excellent fit at all energies, with 5 free DG parameters: $\mathcal{N}_{ch}$, ξ_{max}, σ, s and k

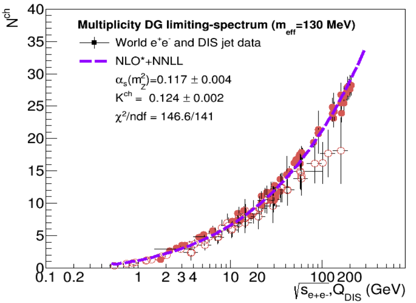


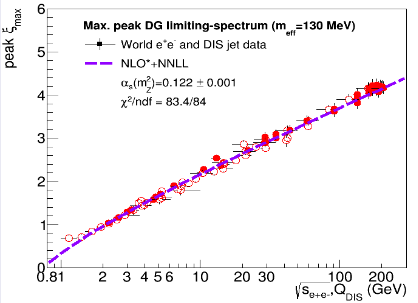
Distorted Gaussian fits to *ep* (DIS) FFs

- Brick wall frame: incoming quarks scatters off photons & returns along the same axis
- 15 ZEUS data-sets at $\sqrt{s} = 3.8 - 173$ GeV ~ 250 data points (other measured H1, ZEUS moments added to global fit)
- Excellent fits to DG but larger uncertainties than e^+e^- measurements



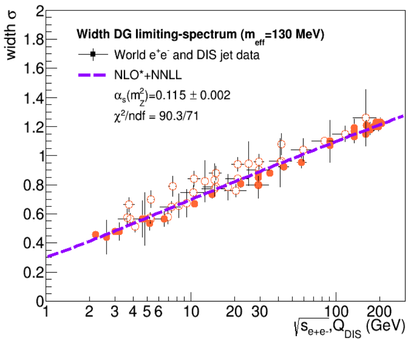
Fit of energy evolution of 1st & 2nd FF moments with **single free** parameter $\Lambda_{\rm QCD}$

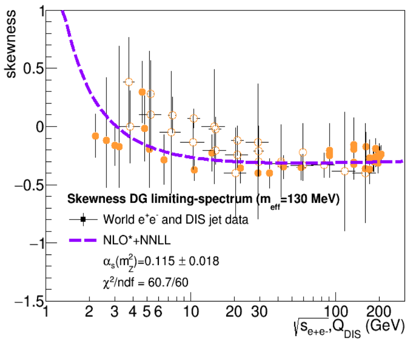




- Theoretical $\mathcal{N}_{ch}$ absolutely normalized to match data (LHPD). QCD coupling drives evolution: care about shape, NOT absolute Nch
- Very good agreement between e^+e^-, DIS and theory for the energy evolution of the FF Nch and peak position
- DIS multiplicity lower than e^+e^- but with larger uncertainties

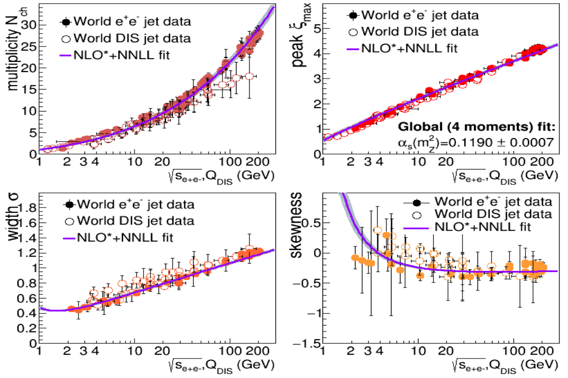
Fit of energy evolution of 3rd & 4th FF moments with **single free** parameter Λ_{QCD}





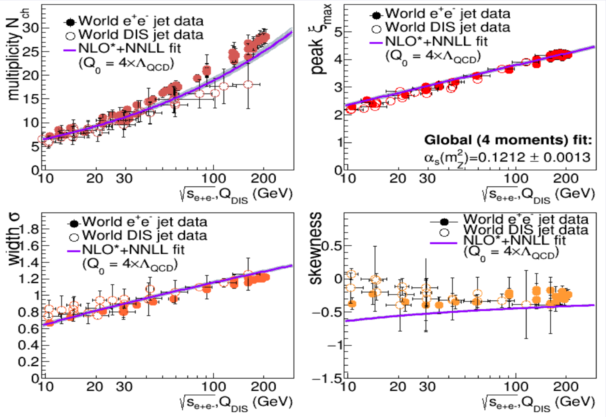
- Good data-theory agreement for the energy evolution of the FF width (skewness has large exp. uncertainties)
- Consistent e^+e^-& e-p moments (but larger DIS uncertainties)

Combined fit of all FF moments energy evolution with **single free** parameter Λ_{QCD}



- χ^2 averaging: increased uncertainty for few points fits to reach $\chi^2/\mathrm{ndf} \sim 1$
- Final α_s uncertainty of $\sim 1.2\%$ includes m_{eff}, exp. fits and corr.

α_s at NNLL+NLO*: scale uncertainty



- Extra uncertainty $\alpha_s(Q_0 = \Lambda_{\rm QCD}) - \alpha_s(Q_0 = 1{\rm GeV}) = 2\%$ due to scale variation (stopping the evolution at Q=1 GeV rather than going down to $\Lambda_{\rm QCD}$

$\alpha_s(M_{Z_0}^2)$ at NNLL+NLO* from low-z FFs evolution

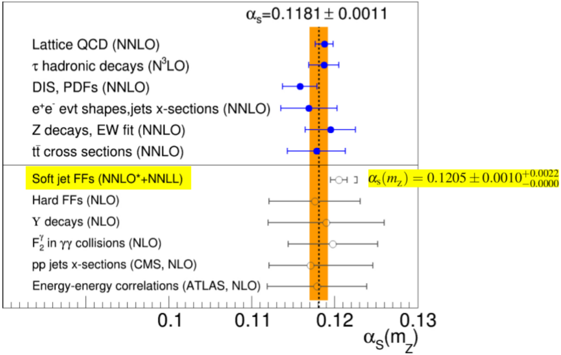


Figure 4: Summary of α_s determinations using different methods. The top points show $N^{2,3}LO$ extractions currently included in the PDG [1], the bottom ones shown those obtained with other approaches at lower degree of accuracy today [13], including the result of our work. The dashed line and shaded (orange) band indicate the current PDG world-average and its uncertainty.

Conclusion

Novel precision measurement of α_s at NNLL+NLO* accuracy

Part IV

Outlook

Outlook

$\mathrm{N^4LL + NLO}$ anomalous dimension

Average multiplicity rate in QCD jets (for $\omega = 0$):

$$\gamma_{++}^{\overline{\mathrm{MS}}} = P_{++}^{(0)}(\omega) + P_{++}^{(1)}(\omega) + P_{++}^{(2)}(\omega) + P_{++}^{(3)}(\omega) + P_{++}^{(4)}(\omega)$$

1. LL+LO: $P_{++}^{(0)}$ is available ($\overline{\mathrm{MS}}$ $\equiv$DLA)
2. NLL+LO: $P_{++}^{(1)}$ & NNLL+NLO: $P_{++}^{(2)}$ can be obtained from $\overline{\mathrm{MS}}$ (i.e. Vogt's resummation: JHEP 10 (2011) 025) and exact diagonalisation by Kotikov & Teryaev (Phys. Rev. D 103 (2021) 034002)
3. Repeat the steps of 2nd item to cast $P_{++}^{(3)}(\omega)$ & $P_{++}^{(4)}(\omega)$

Outlook

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1. LL+LO: $P_{++}^{(0)}$ is available ($\overline{\mathrm{MS}}$ $\equiv$DLA)
2. $$\gamma_{++}^{\mathrm{MLLA}} - \left[\gamma_{++}^{\overline{\mathrm{MS}}} + \beta(Q^2)\frac{d}{d\alpha_s}\ln\frac{\mathcal{R}^T(\alpha_s(Q^2))}{\mathcal{R}^T(\alpha_s(\Lambda^2))}\right] = -\frac{11}{12}(1+\ldots)$$
   Ref: Duff Neill, JHEP 03 (2021) 081
3. Repeat the steps of 2nd item to cast $P_{++}^{(3)}(\omega)$ & $P_{++}^{(4)}(\omega)$

Outlook

Order	LL (DLA)	NLL	NNLL	$\mathrm{N^3LL}$	$\mathrm{N^4LL}$
LO $P_{ac}^{(0)}$	$\mathcal{O}(\sqrt{\alpha_s})$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^{2})$	$\mathcal{O}(\alpha_s^{5/2})$
NLO $P_{ac}^{(1)}$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^{2})$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO $P_{ac}^{(2)}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$

NLL+LO: $\gamma_{++}^{\overline{\mathrm{MS}}} = \sqrt{\alpha_s} + \alpha_s$

Outlook

Order	LL (DLA)	NLL	NNLL	$\mathrm{N^3LL}$	$\mathrm{N^4LL}$
LO $P_{ac}^{(0)}$	$\mathcal{O}(\sqrt{\alpha_s})$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO $P_{ac}^{(1)}$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO $P_{ac}^{(2)}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$

NNLL+NLO: $\gamma_{++}^{\overline{\mathrm{MS}}} = \sqrt{\alpha_s} + \alpha_s + \alpha_s^{3/2}$

Outlook

Order	LL (DLA)	NLL	NNLL	$\mathrm{N^3LL}$	$\mathrm{N^4LL}$
LO $P_{ac}^{(0)}$	$\mathcal{O}(\sqrt{\alpha_s})$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^{2})$	$\mathcal{O}(\alpha_s^{5/2})$
NLO $P_{ac}^{(1)}$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^{2})$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO $P_{ac}^{(2)}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$

$\mathrm{N^3LL}$+NLO: $\gamma_{++}^{\overline{\mathrm{MS}}} = \sqrt{\alpha_s} + \alpha_s + \alpha_s^{3/2} + \alpha_s^2$

Outlook

Order	LL (DLA)	NLL	NNLL	$\mathrm{N^3LL}$	$\mathrm{N^4LL}$
LO $P_{ac}^{(0)}$	$\mathcal{O}(\sqrt{\alpha_s})$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO $P_{ac}^{(1)}$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO $P_{ac}^{(2)}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$

N^{4}LL+NNLO: $\gamma_{++}^{\overline{\mathrm{MS}}} = \sqrt{\alpha_s} + \alpha_s + \alpha_s^{3/2} + \alpha_s^2 + \alpha_s^{5/2}$

$\mathrm{N^3LL} = \mathcal{O}(\alpha_s^2)$

Extension using $\overline{\mathrm{MS}}$ NLO+resummed (small-x) splittings functions (based on C.-H. Kom, A. Vogt, K. Yeats JHEP 1210 (2012) 033)

$$
\begin{aligned}
P^T_{qg,\mathrm{N^3LL}}(N) = {} & \frac{C_A}{C_F} P^T_{qq,\mathrm{N^3LL}}(N) + \frac{1}{54}\frac{n_f}{C_A^3} a_s^2 \Big\{ 2([81 - 144\zeta_2]C_A^4 - 90C_A^3 C_F + 144\zeta_2 C_A^3 C_F \\
& -79C_A^3 n_f + 106C_A^2 C_F n_f + 6C_A^2 n_f^2 - 24C_A C_F n_f^2 + 24C_F^2 n_f^2)\frac{1}{\xi^2}(S - 1 + 2\xi + 2\xi^2) \\
& -2([483 - 576\zeta_2]C_A^4 - [360 - 576\zeta_2]C_A^3 C_F - 139C_A^3 n_f + 102C_A^2 C_F n_f + 56C_A C_F n_f^2 \\
& -6C_A^2 n_f^2 - 88C_F^2 n_f^2)\frac{1}{\xi}(S - 1 + 2\xi) + ([429 - 576\zeta_2]C_A^4 - [360 - 576\zeta_2]C_A^3 C_F - 213C_A^3 n_f \\
& +250C_A^2 C_F n_f - 2C_A^2 n_f^2 + 40C_A C_F n_f^2 - 72C_F^2 n_f^2)\frac{1}{\xi^2}(S - 1 + 2\xi)\mathcal{L} + 8([137 - 144\zeta_2]C_A^4 \\
& -[90 - 144\zeta_2]C_A^3 C_F - 17C_A^3 n_f - 6C_A^2 C_F n_f - 6C_A^2 n_f^2 + 36C_A C_F n_f^2 - 48C_F^2 n_f^2)(1 + \frac{1}{\xi}\mathcal{L}) \\
& +(11C_A^4 + 13C_A^3 n_f - 26C_A^2 C_F n_f + 2C_A^2 n_f^2 - 8C_A C_F n_f^2 + 8C_F^2 n_f^2)\frac{1}{\xi^2}(S^{-1} - 1 - 2\xi)\mathcal{L} \\
& +4([59 - 72\zeta_2]C_A^4 - [45 - 72\zeta_2]C_A^3 C_F - 20C_A^3 n_f + 20C_A^2 C_F n_f - 2C_A^2 n_f^2 + 14C_A C_F n_f^2 \\
& -20C_F^2 n_f^2)\frac{1}{\xi^2}(S - 1)\mathcal{L}^2 \Big\} . \qquad \text{(B.2)}
\end{aligned}
$$

N^{4}LL$= \mathcal{O}(\alpha_s^{5/2})$

Extension using $\overline{\mathrm{MS}}$ NLO+resummed (small-x) splittings functions (based on C.-H. Kom, A. Vogt, K. Yeats JHEP 1210 (2012) 033)

$$
\begin{aligned}
P_{\mathrm{gg}}^{T}(N)\Big|_{\mathrm{N^4LL}}^{C_F=0} = {} & \frac{1}{13271040}\,\frac{1}{C_A^2}\,a_s^2\,\bar{N}\Big\{16([15688235-19918080\zeta_2+7983360\zeta_3 \\
& +5059584\zeta_2^2]C_A^4+[914360+875520\zeta_2-2142720\zeta_3]C_A^3n_f-[134200+46080\zeta_2]C_A^2n_f^2 \\
& +5600n_f^3C_A-80n_f^4)(S-1)-32([7822505-2826000\zeta_2+1330560\zeta_3-134784\zeta_2^2]C_A^4 \\
& +[514490+83520\zeta_2-276480\zeta_3]C_A^3n_f+[16880+20160\zeta_2]C_A^2n_f^2-2840n_f^3C_A+80n_f^4) \\
& \cdot\frac{1}{\xi}(S-1+2\xi)+2([12686895+12997440\zeta_2-2471040\zeta_3-10907136\zeta_2^2]C_A^4 \\
& -[3309880+564480\zeta_2-1624320\zeta_3]C_A^3n_f+[37960-172800\zeta_2]C_A^2n_f^2+21280n_f^3C_A \\
& -1040n_f^4)\frac{1}{\xi^2}(S-1+2\xi+2\xi^2)-4([3135445+6822720\zeta_2+190080\zeta_3-5868288\zeta_2^2]C_A^4 \\
& -[1973120+587520\zeta_2-1071360\zeta_3]C_A^3n_f+[106520-149760\zeta_2]C_A^2n_f^2+14080n_f^3C_A \\
& -1200n_f^4)\frac{1}{\xi^2}(S^{-1}-1-2\xi-6\xi^2)-([2095591+158976\zeta_2-1140480\zeta_3+331776\zeta_2^2]C_A^4 \\
& +[61560+396288\zeta_2-207360\zeta_3]C_A^3n_f-[83352-64512\zeta_2]C_A^2n_f^2+224n_f^3C_A+880n_f^4) \\
& \cdot 5\frac{1}{\xi^2}(S^{-3}-1-6\xi-30\xi^2)-10([198803-209088\zeta_2]C_A^4+[69872-76032\zeta_2]C_A^3n_f \\
& -[600+6912\zeta_2]C_A^2n_f^2-2368n_f^3C_A-208n_f^4)\frac{1}{\xi^2}(S^{-5}-1-10\xi-70\xi^2) \\
& -25(11C_A+2n_f)^4\frac{1}{\xi^2}(S^{-7}-1-14\xi-126\xi^2)\Big\}+C_A^2a_s^2\bar{N}(S+S^{-1}-2)B_{\mathrm{gg}}^{S(3)}, \qquad \text{(B.11)}
\end{aligned}
$$

Conclusions

- The anomalous dimension was obtained from from the mixed resummation of LO expanded splitting functions and NLO corrections to the splitting functions.
- The moments, multiplicity, maximum peak position, dispersion, skewness and kurtosis of the DG were obtained at the same level of precision as a function of the only free parameter $\Lambda_{\rm QCD}$.
- The theoretical mean multiplicity was absolutely normalized to match the data (LPTHE): care about shape, NOT absolute normalization.
- Very good agreement between e^+e^-, DIS and theory for peak position and dispersion.
- Novel high precision of α_s at NNLL+NLO*:
  $\alpha_{\mathbf{s}} = \mathbf{0.1205 \pm 0.0010^{+0.0022}_{-0.0000}}$.
- Outlook: matching the NMLLA resummation with the $\overline{\rm MS}$ anomalous dimension from Vogt's resummation and exact diagonalisation at NNLO!