A.V.Kotikov, JINR, Dubna

(in collab. with V.G. Krivokhizhin and B.G.Shaikhatdenov, JINR, Dubna).

"alphas-2022: Workshop on precision measurements of the strong coupling constant"

ECT*-Trento in the first week of February 2022

PRELIMINARY

some discussions are based on old analyses in

Phys. Rev. D81, 034008 (2010) [0912.4672[hep-ph]];

Phys.Lett. B333 (1994) 190-195 [hep-ph/9405290];

Z.Phys.C 58 (1993) 465-470

α_s from DIS data via a large x resummations

OUTLINE

1. Introduction
2. Results
3. Conclusions and Prospects

1. Introduction

The search for new physics at the LHC accelerator requires a threshold resummation of the cross sections. (see, for example, the revent papers (Beneke, Broggio, Hasner, Urban, Vollmann:2019), (Beneke, Broggio, Jaskiewicz, Vernazza: 2020)).
It would also be good to consider extracting from the deep-inelastic scattering (DIS) data (for the structure function (SF) $F_2(x,Q^2)$), taking into account resumming at large values of x (= threshold resumming), where x is Bjorken's variable.

Usually the function $F_2(x,Q^2)$ is represented as a sum of the leading twist $F_2^{pQCD}(x,Q^2)$ and the twist four terms

$$F_2(x,Q^2) = F_2^{pQCD}(x,Q^2)\left(1+\frac{\tilde{h}_4(x)}{Q^2}\right).$$

While analysing experimental data various corrections should be taken into account: nuclear effects, target mass corrections, heavy quark threshold corrections and higher twist terms are considered.

As is known there are at least two ways to perform QCD analysis over DIS data:
the first one deals with Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) integro-differential equations and let the data be examined directly, whereas the second one involves the SF moments and permits performing an analysis in analytic form as opposed to the former option.

We take on the way in-between these two latter, i.e. analysis is carried out over the moments of SF $F_2^k(x,Q^2)$ defined as follows

$$M_n^{pQCD/twist2/\ldots}(Q^2) = \int_0^1 x^{n-2}\, F_2^{pQCD/twist2/\ldots}(x,Q^2)\, dx$$

and then reconstruct SF for each Q^2 by using Jacobi polynomial expansion method. (Parisi, Sourlas:1979), (Barnet et al.: 1981,1983,1984), (Krivokhizhin et al.: 1987,1990)

2. A brief theoretical input

The twist-two DIS SF can be represented as a sum of two terms: $F_2^{twist2}(x,Q^2) = F_2^{NS}(x,Q^2) + F_2^{S}(x,Q^2)\,,$ the nonsinglet (NS) and singlet (S) parts.

Let's introduce PDFs, the gluon distribution function $f_G(x,Q^2)$ and the singlet and nonsinglet quark distribution functions $\mathbf{f}_S(x,Q^2)$ and $\mathbf{f}_{NS}(x,Q^2)$ (hereafter, unlike the standard case, here PDFs are multiplied by x):

$$\mathbf{f}_S(x,Q^2) \equiv \sum_q^f \mathbf{f}_q(x,Q^2) = V(x,Q^2) + S(x,Q^2)\,,$$
$$\mathbf{f}_{NS}(x,Q^2) = \mathbf{u}_v(x,Q^2) - \mathbf{d}_v(x,Q^2)\,,$$

where f is the number of quark flavors ($\mathbf{u}$p, $\mathbf{d}$own, $\mathbf{s}$trange,. . .), $V(x,Q^2) = \mathbf{u}_v(x,Q^2) + \mathbf{d}_v(x,Q^2)$ is the distribution of valence quarks and $S(x,Q^2)$ is a sum of sea parton distributions set equal to each other.

At large x values, no gluons and we can confine ourselves to considering only the NS contribution.

There is a direct relation between SF moments $M_n(Q^2)$ and those of PDFs

$$\mathbf{f}_{NS}(n,Q^2) \;=\; \int_0^1 dx x^{n-2} \mathbf{f}_{NS}(x,Q^2).$$

For example, in the nonsinglet case it looks:

$$M_n^{NS}(Q^2) = R_{NS}(f) \times C_{NS}^{twist2}(n, a_s(Q^2)) \times \mathbf{f}_{NS}(n,Q^2)\,,$$

with

$$a_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi}$$

and $C_{NS}^{twist2}(n, a_s(Q^2))$ are the Wilson coefficient functions. The constant $R_{NS}(f)$ depends on the weak and electromagnetic charges and is fixed to be one sixth for $f = 4$.

2.1 Strong coupling constant

The strong coupling constant is determined from the corresponding solution of the renormalization group equation.
At NLO level:

$$\frac{1}{a_1(Q^2)} - \frac{1}{a_1(M_Z^2)} + b_1 \ln\left[\frac{a_1(Q^2)}{a_1(M_Z^2)} \frac{(1+b_1 a_1(M_Z^2))}{(1+b_1 a_1(Q^2))}\right] = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right),$$

where hereafter

$$a_1(Q^2) = a_s^{NLO}(Q^2), \quad a_1(Q^2) = a_s^{NNLO}(Q^2)\,.$$

At NNLO level:

$$\frac{1}{a_2(Q^2)} - \frac{1}{a_2(M_Z^2)} + b_1 \ln\left[\frac{a_2(Q^2)}{a_2(M_Z^2)}\sqrt{\frac{1+b_1 a_2(M_Z^2)+b_2 a_2^2(M_Z^2)}{1+b_1 a_2(Q^2)+b_2 a_2 2(Q^2)}}\right]$$
$$+ \left(b_2 - \frac{b_1^2}{2}\right) \times (I(Q^2) - I((M_Z^2)) = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right).$$

The expression for I looks:

$$I(Q^2) = \begin{cases} \frac{2}{\sqrt{\Delta}} \arctan \frac{b_1 + 2b_2 a_2(Q^2)}{\sqrt{\Delta}} & \text{for } f = 3,4,5; \Delta > 0, \\ \frac{1}{\sqrt{-\Delta}} \ln \left[\frac{b_1 + 2b_2 a_2(Q^2) - \sqrt{-\Delta}}{b_1 + 2b_2 a_2(Q^2) + \sqrt{-\Delta}} \right] & \text{for } f = 6; \quad \Delta < 0, \end{cases}$$

where $\Delta = 4b_2 - b_1^2$ and $b_i = \frac{\beta_i}{\beta_0}$ are read off from the QCD β-function:

$$\beta(a_s) \; = \; -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 + \ldots$$

These equations allow us to eliminate QCD parameter Λ_{QCD} from the analysis.

2.2 Q^2-dependence of SF moments

The coefficient functions $C_{NS}^{twist2}(n, a_s(Q^2))$ is further expressed through the functions $B_{NS}^{j}(n)$ which are known exactly

$$C_{NS}^{twist2}(n, a_s(Q^2)) = 1 + a_s(Q^2)B_1(n) + a_s^2(Q^2)B_2(n) + \mathcal{O}(a_s^3(Q^2))\,,$$

where

$$B_1(n) = B_{NS}^{NLO}(n), \quad B_2(n) = B_{NS}^{NNLO}(n)\,.$$

The Q^2-evolution of the PDF moments can be calculated within the framework of perturbative QCD:

$$\frac{\mathbf{f}_{NS}(n, Q^2)}{\mathbf{f}_{NS}(n, Q_0^2)} = \left[\frac{a_s(Q^2)}{a_s(Q_0^2)}\right]^{\frac{\gamma_0(n)}{2\beta_0}} \times H^{NS}(n, Q^2, Q_0^2)\,,$$

where

$$\gamma_i(n) = \gamma_{NS}^{(i)}(n),\ Z_1(n) = Z_{NS}^{NLO}(n), \quad Z_2(n) = Z_{NS}^{NNLO}(n)$$

The function $H^{NS}(n,Q^2,Q_0^2)$ up to NNLO may be represented as

$$H^{NS}(n,Q^2,Q_0^2) = \frac{h^{NS}(n,Q^2)}{h^{NS}(n,Q_0^2)},$$
$$h^{NS}(n,Q^2) = 1 + a_s(Q^2)Z_1(n) + a_s^2(Q^2)Z_2(n) + \mathcal{O}\left(a_s^3(Q^2)\right) ,$$

where

$$Z_1(n) \;=\; \frac{1}{2\beta_0}[\gamma_1(n) - \gamma_0(n)\, b_1]\,,$$
$$Z_2(n) \;=\; \frac{1}{4\beta_0}\left[\gamma_2(n) - \gamma_1(n)b_1 + \gamma_0(n)(b_1^2 - b_2)\right] + \frac{1}{2}Z_1^2(n)\,.$$

Here $\gamma_k(n)$ are anomalous dimensions of Wilson operators.

2.3 Factorization μ_F and renormalization μ_R scales

It is good to consider the dependence of results on the factorization μ_F and renormalization μ_R scales, caused by the truncation of a perturbative series while doing the calculus.
A modification is achieved by replacing a_s with the expressions in which the scales were accounted in the following way: $\mu_F^2 = k_F Q^2, \quad \mu_R^2 = k_R \mu_F^2 = k_R k_F Q^2$.

Then,

$$M_n^{NS}(Q^2) = R_{NS}(f) \times \hat{C}_{NS}^{twist2}(n, a_s(k_F Q^2)) \times \mathbf{f}_{NS}(n, k_F Q^2),$$

and

$$\frac{\mathbf{f}_{NS}(n, k_F Q^2)}{\mathbf{f}_{NS}(n, k_F Q_0^2)} = [\frac{a_s(k_F k_R Q^2)}{a_s(k_F k_R Q_0^2)}]^{\gamma_0(n)/2\beta_0} \times \hat{H}^{NS}(n, k_F k_R Q^2, k_F k_R Q_0^2)\,.$$

The functions $\hat{C}_{NS}, \hat{H}^{NS}$ are

$$
\begin{aligned}
a_s(Q^2) \;&\to\; a_s(k_F Q^2)\,, \quad B_1(n) \to B_1(n) + \frac{1}{2}\gamma_0(n)\ln k_F\,, \\
B_2(n) \;&\to\; B_2(n) + \frac{1}{2}\gamma_1(n)\ln k_F + \left(\frac{1}{2}\gamma_0 + \beta_0\right) B_1 \ln k_F \\
&+\; + \frac{1}{8}\gamma_0\,(\gamma_0 + 2\beta_0)\ln^2 k_F\,,
\end{aligned}
$$

and

$$
\begin{aligned}
a_s(Q^2) \;&\to\; a_s(k_F k_R Q^2), \quad a_s(Q_0^2) \;\to\; a_s(k_F k_R Q_0^2)\,, \\
Z_1(n) \;&\to\; Z_1(n) + \frac{1}{2}\gamma_0(n)\ln k_R \\
Z_2(n) \;&\to\; Z_2(n) + \frac{1}{2}\gamma_1(n)\ln k_R + \frac{1}{2}\gamma_0(n) Z_1 \ln k_R
\end{aligned}
$$

Below we put $k_R = 1$ and use n-dependent k_F.

At large n values, $Z_i(n) \sim \ln n$ and $B_i(n) \sim \ln^{2k} n$.
So, $B_i(n)$ rise strongly at large n values (i.e. at large x values.

We would like to resum the large terms and to study a dependence of the resummation on the $\alpha_S(M_Z)$.

To study it, it is possible to use the scheme-invariant (SI) perturbation theory (SIPT) (Grunberg:1981,1982), DIS scheme (a former Λ_n-scheme (Bace:1978), (Bardeen et al.:1978), (Buras:1980)). and W^2-evolution: $W^2 = Q^2 \frac{1-x}{x} + M_p^2$.

2.4 SIPT= Grunber effective charge (GEC) method

The GEC method:

An observable

$$m(Q^2) = a_s(Q^2)[1 + \tilde{c}_1 a_s(Q^2) + \tilde{c}_2 a_s^2(Q^2) + \tilde{c}_3 a_s^3(Q^2) + ...]$$

can be considered as a new coupling constant, containing the new scale $Q^2 \to e^{-\tilde{c}_1/\beta_0} Q^2$ and new coefficients β_i $(i \geq 2)$ of β-function: $\beta_i \to \tilde{\beta}_i$ and $\tilde{\beta}_i$ depend on $\tilde{c}_i$

.

In our case,

$$M_n^{NS}(Q^2) \sim \left(a_s(Q^2)\right)^{d(n)} [1 + ...]$$

and we can use $m_n(Q^2) = \left(M_n(Q^2)\right)^{1/d(n)}$, where $d(n) = \gamma_{\rm NS}^{(0)}/(2\beta_0)$.

Consider

$$M_n^{NS}(Q^2) = \frac{\overline{C}_{NS}^{twist2}(n, a_s(Q^2))}{\overline{C}_{NS}^{twist2}(n, a_s(Q_0^2))} \times M_n^{NS}(n, Q_0^2)\,,$$

where the function

$$\overline{C}_{NS}^{twist2}(n, a_s(Q^2)) = a_s^{d(n)}(Q^2)\left[1 + C_1 a_s(Q^2) + C_2 a_s^2(Q^2) + ...\right]$$

contain all Q^2-dependence and

$$d(n) = \frac{\gamma_0(n)}{2\beta_0}, \quad C_1(n) = B_1(n) + Z_1(n),$$
$$C_2(n) = B_2(n) + Z_1(n)B_1(n) + Z_2(n)\,.$$

The normalization $M_n^{NS}(Q_0^2)$ relates with the one $\mathbf{f}_{NS}(n, Q_0^2)$:

$$M_n^{NS}(Q_0^2) = R_{NS}(f) \times C_{NS}^{twist2}(n, a_s(Q_0^2)) \times \mathbf{f}_{NS}(n, Q_0^2)\,.$$

2.4.1 NLO

In the case

$$a_1(Q^2) \to a_1(k_{\rm SI}(n)Q^2) \equiv a_n(Q^2),$$
$$k_{\rm SI}(n) = exp\left(\frac{-2C_1(n)}{\gamma_0(n)}\right) = exp\left(\frac{-r_1(n)}{\beta_0}\right) ,$$

where

$$r_1(n) = \frac{2C_1(n)\beta_0}{\gamma_0} = \frac{C_1(n)}{d(n)}$$

With the above choice of the scale, we have

$$C_1^{\rm SI}(n) = 0 \,,$$

i.e.

$$\overline{C}_{NS}^{\rm SI}(n) = a_n^{d(n)}(Q^2)\left[1 + O(a_n^2)\right].$$

We would like to note that the NLO coupland $a_n(Q^2)$ obey the following equation (here NLO $a_s(Q^2) \equiv a_1(Q^2)$)

$$\frac{1}{a_n(Q^2)} - \frac{1}{a_1(M_Z^2)} + b_1 \ln\left[\frac{a_n(Q^2)}{a_1(M_Z^2)}\frac{(1+b_1 a_1(M_Z^2))}{(1+b_1 a_n(Q^2))}\right]$$
$$= \beta_0 \ln\left(\frac{k_{\rm SI}(n)Q^2}{M_Z^2}\right) = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right) - r_1(n) \qquad (1)$$

2.4.2 NNLO

In the case

$$a_2(Q^2) \to a_2(k_{\rm SI}(n)Q^2)) \equiv a_n(Q^2),$$

where the NNLO coupland $a_n(Q^2)$ obeys the following equation

$$\frac{1}{a_n(Q^2)} - \frac{1}{a_2(M_Z^2)} + b_1 \ln\left[\frac{a_n(Q^2)}{a_2(M_Z^2)}\sqrt{\frac{1+b_1a_2(M_Z^2)+b_2a_2^2(M_Z^2)}{1+b_1a_n(Q^2)+\tilde{b}_2a_n(Q^2)}}\right]$$

$$+\left(\tilde{b}_2 - \frac{b_1^2}{2}\right)\tilde{I}(Q^2) - \left(b_2 - \frac{b_1^2}{2}\right)I(M_Z^2) = \beta_0 \ln\left(\frac{Q^2}{M_Z^2}\right) - r_1(n)\,, \quad (2)$$

were

$$\tilde{I} = I(b_2 \to \tilde{b}_2), \quad \tilde{b}_2 = \frac{\tilde{\beta}_2}{\beta_0}, \quad \tilde{\beta}_2 = \beta_2 - r_1(n)\,\beta_1 + (r_2(n) - r_1^2(n))\,\beta_0,$$

and

$$r_2(n) = \frac{C_2(n)}{d(n)} - \frac{d(n)-1}{2}\,r_1^2(n)\,.$$

With the above choice of the scale $k_{\rm SI}(n)Q^2$, we have

$$C_1(n) \to C_1^{\rm SI}(n) = 0, \quad C_2(n) \to C_2^{\rm SI}(n) = 0,$$

i.e.

$$\overline{C}^{\rm SI}_{NS}(n) = a_n^{d(n)}(Q^2)\left[1 + O(a_n^3)\right].$$

The NLO and NNLO SI analyses were performed long ago (Kotikov,Parente,Sanchez Guillen: 1992), (Parente,Kotikov,Krivokhizhin: 1994)

The main results are: $\alpha_s(M_Z^2)$ **is stable during resummation and HTs are strongly decreased in SIPT.** We will confirm it (see below).

2.5 (Generalized) W^2 evolution

Here consider the renormalization scale will be equal to Q^2, i.e. $k_R = 1$, and the nonzero (n-dependent) factorization scale, which is used to resumm the large x logarithms.

It is conveninet to transfor above transformations (from nonzero k_F values) in x-space:

$$a_s(Q^2) \to a_s(k_F Q^2)\,, \quad B_1(x) \to B_1(x) + \frac{1}{2}\gamma_0(x)\ln k_F(x)\,,$$

$$\begin{aligned} B_2(x) &\to B_2(x) + \left(\frac{1}{2}[\gamma_0(x)\otimes B_1(x)] + \frac{1}{2}\gamma_1(x) + \beta_0 B_1(x)\right)\ln k_F(x) \\ &+ \left(\frac{1}{8}[\gamma_0(x)\otimes\gamma_0(x)] + \frac{1}{4}\gamma_0(x)\beta_0\right)\ln^2 k_F(x)\,, \end{aligned}$$

where

$$B_i(n) = \int_0^1 dx x^{n-1} B_i(x), \quad \gamma_{i-1}(n) = \int_0^1 dx x^{n-1}\gamma_{i-1}(x), (i = 1, 2)$$

and the simbol $\otimes$ marks the usual Mellin convolution as

$$[f_1(x)\otimes f_2(x)] = \int_x^1 \frac{dy}{y} f_1\left(\frac{x}{y}\right) f_2(y)\,.$$

Now we consider the following form of $k_F(x)$:

$$k_F(x) = e^{\delta} \frac{(1-x)}{x^m}, \qquad \text{a generalized } W^2 \text{ scale}$$

and, respectively,

$$\ln k_F(x) = \ln (1-x) - m \ln x + \delta \, .$$

The result of application of this $k_F(x)$:

$$\frac{1}{2}\gamma_0(x)\ln k_F(x) \to B_{1a}(n) + mB_{1b}(n) + \delta B_{1c}(n)\,,$$
$$\frac{1}{2}[\gamma_0(x)\otimes B_1(x)]\ln k_F(x) \to B_{21a}(n) + mB_{21b}(n) + \delta B_{21c}(n)\,,$$
$$B_1(x)\ln k_F(x) \to B_{22a}(n) + mB_{22b}(n) + \delta B_{22c}(n)\,,$$
$$\frac{1}{4}[\gamma_0(x)\otimes\gamma_0(x)]\ln^2 k_F(x) \to B_{23aa}(n) + 2mB_{23ab}(n) + m^2B_{23bb}(n)$$
$$+2\delta(B_{23ac}(n) + mB_{23bc}(n)) + \delta^2 B_{23cc}(n)\,,$$
$$\frac{1}{4}\gamma_0(x)\ln^2 k_F(x) \to B_{24aa}(n) + 2mB_{24ab}(n) + m^2B_{24bb}(n)$$
$$+2\delta(B_{24ac}(n) + mB_{24bc}(n)) + \delta^2 B_{24cc}(n)\,,$$
$$\frac{1}{2}\gamma_1(x)\ln k_F(x) \to B_{25a}(n) + mB_{25b}(n) + \delta B_{25c}(n)\,,$$

and, respectively,

$$
\begin{aligned}
&B_1(n) \to B_1(n) + \left[B_{1a}(n) + mB_{1b}(n) + \delta B_{1c}(n)\right],\\
&B_2(n) \to B_2(n) + \left[B_{21a}(n) + mB_{21b}(n) + \delta B_{21c}(n)\right]\\
&\quad +\beta_0\left[B_{22a}(n) + mB_{22b}(n) + \delta B_{22c}(n)\right] + \frac{1}{2}\{B_{23aa}(n)\\
&+2mB_{23ab}(n) + m^2 B_{23bb}(n) + 2\delta(B_{23ac}(n) + mB_{23bc}(n)) + \delta^2 B_{23cc}(n)\}\\
&+\frac{\beta_0}{2}\{B_{24aa}(n) + 2mB_{24ab}(n) + m^2 B_{24bb}(n)\\
&+2\delta(B_{24ac}(n) + mB_{24bc}(n)) + \delta^2 B_{24cc}(n)\}\\
&+\left[B_{25a}(n) + mB_{25b}(n) + \delta B_{25c}(n)\right],
\end{aligned}
$$

where

$$
\beta_0 = \frac{1}{3}\left(11C_A - 2f\right).
$$

Let us to consider the four cases:
1. $\mu_F^2(x) = Q^2(1-x) \equiv \mu_1^2(x)$;
2. $\mu_F^2(x) = Q^2(1-x)/x \equiv \mu_2^2(x)$;
3. $\mu_F^2(x) = Q^2(1-x)/x * e^{\delta} \equiv \mu_3^2(x)$;
4. $\mu_F^2(x) = W^2 \equiv Q^2(1-x)/x + m_p^2 \equiv \mu_4^2(x)$, where m_p is the proton mass.

After trasformation to Mellin space, the $\mu_i^2(x)$ scales will transform to $\mu_i^2(n) = k_i(n)Q^2$.

2.5.1 NLO

Here the NLO coefficient faction and the LO anomalous dimension have the following form

$$B_1 = 2C_F[S_1^2(n) - S_2(n) + \left(\frac{3}{2} - \frac{1}{n(n+1)}\right) S_1(n) - \frac{9}{2}$$
$$+\frac{3}{2n} + \frac{2}{n+1} + \frac{1}{n^2}] \sim \ln^2 n \,,$$
$$\gamma_0 = 8C_F\left[S_1(n) - \frac{3}{4} - \frac{1}{2n(n+1)}\right] \sim \ln n \,.$$

The case 1.

$$a_s(Q^2) \to a_s(k_1(n)Q^2), \quad k_1(n) = exp\left(\frac{2\overline{B}_{1a}(n)}{\gamma_0(n)}\right),$$

and

$$B_1(n) \to \tilde{B}_{1a}(n) = B_1(n) + \overline{B}_{1a}(n),$$

where

$$\overline{B}_{1a}(n) = -2C_F[S_1^2(n) + S_2(n) - \frac{1}{n(n+1)}S_1(n) - \frac{7}{4}$$
$$+\frac{1}{(n+1)^2}] \sim \ln^2 n\,,$$
$$\tilde{B}_{1a}(n) = 2C_F[-2S_2(n) + \frac{3}{2}S_1(n) - \frac{11}{4} + \frac{3}{2n} + \frac{2}{n+1}$$
$$+\frac{1}{n^2} - \frac{1}{(n+1)^2}] \sim \ln n\,,$$ **So, the terms $\sim \ln^2 n$ are cancelled.**

For the coupling constant $a_s(k_1(n)Q^2)$ we can use also Eq. (1), with the replace $r_1(n) \to -\overline{B}_{1a}(n)/d(n)$.

The case 2.

$$a_s(Q^2) \to a_s(k_2(n)Q^2), \quad k_2(n) = k_1(n)\, exp\left(\frac{2\overline{B}_{1b}(n)}{\gamma_0(n)}\right),$$

and

$$B_1(n) \to \tilde{B}_{1b}(n) = \tilde{B}_{1a}(n) + \overline{B}_{1b}(n),$$

where

$$\overline{B}_{1b}(n) = 2C_F\left[2S_2(n) - \frac{5}{4} - \frac{1}{n^2} + \frac{1}{(n+1)^2}\right],$$
$$\tilde{B}_{1b}(n) = 3C_F\left[S_1(n) - \frac{8}{3} + \frac{1}{n} + \frac{4}{3(n+1)}\right] \sim \ln n\,.$$

For the coupling constant $a_s(k_2(n)Q^2)$ we can use also Eq. (1), Ewith the replace $r_1(n) \to -(\overline{B}_{1a}(n) + \overline{B}_{1b}(n))/d(n)$.

The case 3.

$$a_s(Q^2) \to a_s(k_3(n)Q^2), \quad k_3(n) = k_2(n)\, e^{\delta}\,,$$

and

$$B_1(n) \to \tilde{B}_{1c}(n) = \tilde{B}_{1b}(n) + \frac{\delta}{2}\gamma_0(n).$$

Wnen we put $\delta = -3/4$, then

$$\tilde{B}_c(n) = \frac{C_F}{2}\left[\frac{9}{n} + \frac{5}{n+1} - \frac{23}{2}\right] \sim \ln^0 n\,.$$

For the coupling constant $a_s(k_3(n)Q^2)$ we can use also Eq. (1) with the replace $r_1(n) \to \delta - (\overline{B}_{1a}(n) + \overline{B}_{1b}(n))/d(n)$.

The case 4.

The case is equal to the case 2 but

$$a_s(Q^2) \to a_s(k_2(n)Q^2 + m_p^2),$$

In a sence, here the coupling constant with the proton mass m_p in its argument can be considered as a "frozen" coupling constant.

2.5.2 NNLO. A large n analysis

Consider the large n asymptotics as

$$S_1(n) = \ln n + \gamma_E + ... \equiv S + ..., \;\; S_m(n) = \zeta_m + ..., \;\; (m \geq 2) \quad (3)$$

For applications it is convenient to put $S = S_1(n-1)$. It leads to zero values for all considered variables at $n = 1$.

Here the NLO and NNLO coefficient factions and the LO and NLO anomalous dimensions have the following form:

$$\frac{\gamma_0}{2} = 4C_F \left[S - \frac{3}{4}\right] \sim S, \quad B_1 = 2C_F \left[S^2 + \frac{3}{2} S - \zeta_2 - \frac{9}{2}\right] \sim S^2,$$

$$\frac{\gamma_1}{2} = 4C_F \{ C_A \left[(\frac{67}{9} - 2\zeta_2) S + 3\zeta_3 - \frac{11}{3}\zeta_2 - \frac{17}{24} \right] + \frac{2f}{3} \left[-5S + \zeta_2 - \frac{3}{8} \right]$$

$$+ C_F \left[-6\zeta_3 + 3\zeta_2 - \frac{3}{8} \right] \} \sim S,$$

$$B_2 = 2C_F \{ C_F \left[S^4 + 3S^3 - (\frac{27}{4} + 2\zeta_2) S^2 + (12\zeta_3 - 18\zeta_2 - \frac{51}{4}) S \right]$$

$$+ C_A \left[\frac{11}{9} S^3 + (\frac{367}{36} - 2\zeta_2) S^2 - (20\zeta_3 + \frac{11}{3}\zeta_2 - \frac{3155}{108}) S \right]$$

$$+ f \left[-\frac{2}{9} S^3 - \frac{29}{18} S^2 + (\frac{2}{3}\zeta_2 - \frac{247}{54}) S \right] \} \sim S^4 .$$

The NLO coefficients coming from 9generalized) W^2-evolution are

$$B_{1c} = \frac{\gamma_0}{2}, \quad B_{1b} = 4C_F\left[\zeta_2 - \frac{5}{8}\right], \quad B_{1a} = 2C_F\left[\frac{7}{4} - S^2 - \zeta_2\right].$$

For the NNLO coefficients, which are $\sim \beta_0$, we have

$$B_{24cc} = \frac{\gamma_0}{2}, \quad B_{24bc} = B_{1b}, \quad B_{24ac} = B_{1a}, \quad B_{24bb} = 8C_F\left[\zeta_3 - \frac{9}{16}\right],$$

$$B_{24ab} = 2C_F\left[3 - \frac{3}{2}\zeta_2 - 2\zeta_3\right], \quad B_{24aa} = \frac{4C_F}{3}\left[S^3 + 3\zeta_2 S + 2\zeta_3 - \frac{45}{8}\right],$$

$$B_{22c} = B_1, \quad B_{22b} = 2C_F\left[\frac{17}{4} + 3\zeta_2 - 4\zeta_3\right],$$

$$B_{22a} = C_F\left[-\frac{2}{3}S^3 - \frac{3}{4}S^2 - 2\zeta_2 S + \frac{2}{3}\zeta_3 + \frac{3}{4}\zeta_2 + \frac{21}{4}\right].$$

The other NNLO coefficients have the following form:

$$B_{23cc} = \frac{\gamma_0^2}{4}, \quad B_{23bc} = O(S^{-1}) \sim 0, \quad B_{23bb} = -2B_{1b}^2,$$

$$B_{24ab} = 8C_F^2\left[-7\zeta_4 - \frac{3}{2}\zeta_3 - \frac{11}{4}\zeta_2 + \frac{89}{16}\right],$$

$$B_{24ac} = 16C_F^2\left[-\frac{2}{3}S^3 - 2\zeta_2 S - \frac{1}{3}\zeta_3 + \frac{9}{8}\zeta_2 + \frac{11}{8}\right],$$

$$B_{23aa} = 8C_F^2[S^4 - S^3 + 6\zeta_2 S^2 + (8\zeta_3 - 3\zeta_2)S + \frac{19}{2}\zeta_4$$
$$-\frac{13}{2}\zeta_3 - \frac{21}{4}\zeta_2 - \frac{41}{16}],$$

$$B_{22c} = \frac{\gamma_0}{2}B_1, \quad B_{21b} = O(S^{-1}) \sim 0,$$

$$B_{21a} = 8C_F^2[-\frac{3}{4}S^4 - \frac{1}{2}S^3 + (\frac{45}{16} - \frac{5}{2}\zeta_2)S^2 - (\frac{3}{2}\zeta_2 + 6\zeta_3)S$$
$$+\frac{19}{8}\zeta_4 + \frac{23}{4}\zeta_3 + \frac{103}{16}\zeta_2 + \frac{51}{4}].$$

The case 1.

$$a_s(Q^2) \to a_s(k_1(n)Q^2), \quad k_1(n) = exp\left(\frac{2\overline{B}_{1a}(n)}{\gamma_0(n)}\right),$$

and

$$B_2(n) \to \tilde{B}_{2a}(n) = B_2(n) + \overline{B}_{2a}(n),$$

where

$$\overline{B}_{2a}(n) = B_{21a}(n) + \beta_0\, B_{22a}(n) + \frac{1}{2}\left[B_{23aa}(n) + \beta_0\, B_{24aa}(n)\right] + B_{25a}(n)\,.$$

Using above results, we have for large n

$$\overline{B}_{2a} = 2C_F\{C_F\left[-S^4 - 4S^3 + (\frac{45}{4} + 2\zeta_2)S^2 - (8\zeta_3 + 12\zeta_2)S\right]$$
$$+C_A\left[-\frac{11}{9}S^3 - (\frac{367}{36} - 2\zeta_2)S^2 - \frac{11}{3}\zeta_2 S\right] + f\left[\frac{2}{9}S^3 + \frac{29}{18}S^2 + \frac{2}{3}\zeta_2 S\right]\} \sim S^4$$

and

$$\tilde{B}_{2a} = 2C_F\{C_F\left[-S^3 + \frac{9}{2}S^2 + (4\zeta_3 - 30\zeta_2 - \frac{51}{4})S\right]$$
$$-C_A\left[20\zeta_3 + \frac{22}{3}\zeta_2 - \frac{3155}{108}\right]S + f\left[\frac{4}{3}\zeta_2 - \frac{247}{54}\right]S\} \sim S^3\,,$$

i.e. the basic contribution $\sim S^4$ is cancelled.

The case 2.

$$a_s(Q^2) \to a_s(k_2(n)Q^2), \quad k_2(n) = k_1(n)\, exp\left(\frac{2\overline{B}_{1b}(n)}{\gamma_0(n)}\right),$$

and

$$B_2(n) \to \tilde{B}_{2b}(n) = \tilde{B}_{2a}(n) + \overline{B}_{2b}(n),$$

where

$$\overline{B}_{2b}(n) = B_{21b}(n) + \beta_0\, B_{22b}(n) + \frac{1}{2}\left[2B_{23ab}(n) + B_{23bb}(n)\right.$$
$$\left. +\beta_0\,(2B_{24ab}(n) + B_{24bb}(n))\right] + B_{25b}(n)\,.$$

Using above results, we have $\overline{B}_{2b}(n) = O(S^0) \sim 0$ and, thus,

$$\tilde{B}_{2b} \sim \tilde{B}_{2a} = 2C_F\{C_F\left[-S^3 + \frac{9}{2}S^2 + (4\zeta_3 - 30\zeta_2 - \frac{51}{4})S\right]$$
$$-C_A\left[20\zeta_3 + \frac{22}{3}\zeta_2 - \frac{3155}{108}\right]S + f\left[\frac{4}{3}\zeta_2 - \frac{247}{54}\right]S\} \sim S^3\,,$$

The case 3.

$$a_s(Q^2) \to a_s(k_3(n)Q^2), \quad k_3(n) = k_2(n)\, e^{\delta}\,,$$

and

$$B_2(n) \to \tilde{B}_{2c}(n) = \tilde{B}_{2b}(n) + \delta\, \overline{B}_{2c}(n),$$

where

$$\overline{B}_{2c}(n) = B_{21c}(n) + B_{23ac}(n) + B_{23bc}(n) +$$
$$\beta_0 \left[B_{22c}(n) + B_{24ac}(n) + B_{24bc}(n)\right]$$
$$+B_{25bc}(n) + \frac{\delta}{2}[B_{23cc}(n) + \beta_0\, B_{24cc}(n)] = \overline{B}^{(1)}_{2c}(n) + \frac{\delta}{2}\,\overline{B}^{(2)}_{2c}(n)\,.$$

Using above results, we have for large n

$$\overline{B}^{(1)}_{2c} = 4C_F\{2C_F\left[-\frac{1}{3}S^3 + \frac{3}{4}S^2 - 5(\zeta_2 + \frac{9}{8})S\right]$$
$$+\left[C_A(\frac{67}{9} - 2\zeta_2) - \frac{10}{9}f\right]S\},$$
$$\overline{B}^{(2)}_{2c} = 4C_F\{4C_F\left[S^2 - \frac{3}{2}S\right] + \left[\frac{11}{3}\,C_A - \frac{2}{3}f\right]S\}\,.$$

For the case $\delta = -3/4$, we have

$$\tilde{B}_{2c} = 2C_F\{C_F\left[\frac{9}{2}S^2 + (4\zeta_3 - 15\zeta_2 + \frac{3}{4})S\right] \\ +C_A\left[-20\zeta_3 + \frac{31}{3}\zeta_2 - \frac{16553}{432}\right]S + f\left[\frac{4}{3}\zeta_2 - \frac{709}{216}\right]S\} \sim S^2,$$

i.e. the second basic contribution $\sim S^3$ is cancelled, too.

The case 4. (same as in NLO)

2.6 DIS scheme

$$a_s(Q^2) \to a_s(k_{\rm DIS}(n)Q^2) \equiv a_n^{\rm DIS}(Q^2),$$
$$k_{\rm DIS}(n) = exp\left(\frac{-2B_1(n)}{\gamma_0(n)}\right) = exp\left(\frac{-r_1^{\rm DIS}(n)}{\beta_0}\right),$$

where

$$r_1^{\rm DIS}(n) = \frac{2B_1(n)\beta_0}{\gamma_0} = \frac{B_1(n)}{d(n)}$$

and

$$B_1(n) \to B_1^{\rm DIS}(n) = 0,$$
$$B_2(n) \to B_2^{\rm DIS}(n) = B_2(n) - \left(\frac{1}{2} + \frac{\beta_0}{\gamma_0(n)}\right) B_1^2(n) - \frac{\gamma_1(n)}{\gamma_0(n)} B_1(n)\,.$$

We would like to that the larger terms $\sim \ln^4(n)$ are cancelled in $B_1^{\rm DIS}(n)$.

Moreover, the coupland $a_n^{\rm DIS}(Q^2)$ obeys at NLO and NNLO level to Eq. (1) and (2) with

$$r_1(n) \to r_1^{\rm DIS}(n), \quad \tilde{\beta}_2 \to \beta_2\,.$$

3. A fitting procedure

With the QCD expressions for the Mellin moments $M_n(Q^2)$ analytically calculated according to the formulas, given above the SF $F_2(x,Q^2)$ is reconstructed by using the Jacobi polynomial expansion method:

$$F_2(x,Q^2) = x^a(1-x)^b \sum_{n=0}^{N_{max}} \Theta_n^{a,b}(x) \sum_{j=0}^{n} c_j^{(n)}(\alpha,\beta) M_{j+2}(Q^2)\,,$$

where $\Theta_n^{a,b}$ are the Jacobi polynomials, a, b are the parameters fitted. A condition is the requirement of the error minimization while reconstructing the structure functions.

Since a twist expansion starts to be applicable only above $Q^2 \sim 1$ GeV^2 the cut $Q^2 \geq 1$ GeV^2 on data is applied throughout.

MINUIT program is used to minimize the variable

$$\chi^2_{SF} = \left|\frac{F_2^{exp} - F_2^{th}}{\Delta F_2^{exp}}\right|^2 .$$

4. Results

We use free normalizations of the data for different experiments. For a reference set, the most stable deuterium BCDMS data at the value of the beam initial energy $E_0 = 200$ GeV is used. With the other data sets taken to be a reference one the variation in the results is still negligible. In the case of the fixed normalization for each and all data sets the fits tend to yield a little bit worse χ^2, just as before.

The starting point of the evolution is taken to be $Q_0^2 = 90$ GeV^2. These Q_0^2 values are close to the average values of Q^2 spanning the corresponding data.

On grounds of previous knowledge the maximal value of the number of moments to be accounted for is $N_{max} = 8$ (Krivokhizhin et al.: 1987,1990) and the cut $0.25 \leq x \leq 0.8$ is imposed everywhere.

Table 1. Parameter values of the twist-four term in different cases obtained in the analysis of deutron data (288 points) carried out within a fixed-flavor-number scheme (FFNS) with $n_f = 4$.

x	Standard $\chi^2 = 180$	Case 1 $\chi^2 = 191$	Case 2 $\chi^2 = 190$	Case 3 $\chi^2 = 188$	Case 4 $\chi^2 = 180$
0.275	-0.223	-0.164	-0.169	-0.146	-0.111
0.35	-0.181	-0.145	-0.141	-0.105	-0.031
0.45	-0.005	-0.100	-0.075	-0060	0.141
0.55	0.244	-0.027	0.029	-0.030	0.398
0.65	0.558	0.013	0.011	-0.105	0.658
0.75	0.747	-0.192	-0.064	-0.496	0.704

The NLO $\alpha_s(M_Z^2) = 0.1180 \pm 0.0021$ in all cases excepting the case 3, where $\alpha_s(M_Z^2) = 0.1179 \pm 0.0021$.

Table 2. Parameter values of the twist-four term in different cases obtained in the analysis of data (314 points, the cut: $Q^2 \geq 2$ Gev^2) carried out within a fixed-flavor-number scheme (FFNS) with $n_f = 4$.

x	NLO Standard $\chi^2 = 259$	NLO SIPT $\chi^2 = 245$	NLO DIS $\chi^2 = 251$	NNLO Standard $\chi^2 = 254$	NNLO SIPT $\chi^2 = 249$	NNLO DIS $\chi^2 = 249$
0.275	-0.264	-0.223	-0.174	-0.204	-0.162	-0.170
0.35	-0.252	-0.153	-0.134	-0.193	-0.138	-0.149
0.45	-0.187	-0.073	-0.094	-0.158	-0.101	-0.104
0.55	0.096	-0.121	-0.088	-0.137	-0.093	-0.084
0.65	0.118	-0.144	-0.094	-0.051	-0.123	-0.100
0.75	0.477	-0.509	-0.442	0.648	-0.383	-0.314

The NLO $\alpha_s(M_Z^2) = 0.1192 \pm 0.0021$ **in standard case,** $\alpha_s(M_Z^2) = 0.1179 \pm 0.0021$ **in SIPT and DIS.**

The NNLO $\alpha_s(M_Z^2) = 0.1170 \pm 0.0021$ **in standard case,** $\alpha_s(M_Z^2) = 0.1178 \pm 0.0021$ **in SIPT and** $\alpha_s(M_Z^2) = 0.1171 \pm 0.0021$ **DIS.**

7. Summary

We did fits of experimental data for DIS SF $F_2(x, Q^2)$ by resumming large logariths at large x values in the corresponding coefficient functions in the framework of SIPT, DIS scheme and taking into account the (generalized) W^2-scale. A close case (so-called Λ_n-scheme, which is equal to so-called DIS scheme, where the NLO coefficient $B_1(n) \to 0$) of such form has been used already long time ago. (Bace:1978), (Bardeen et al.:1978), (Buras:1980).

We see that the resummation does not change values of strong coupling constant $\alpha_s(M_Z^2)$ but the values of HTs are decreased and, sometimes, HT-values change sign at large x values.

These results for W^2-evolution were obtained at NLO level. The extension to the NNLO level (and to VFNS) will be done in nearest future (I hope at February). We hope that the basic properties of the NLO analyses will be recovered (primarily) at the NNLO level.