α_s from τ decay data

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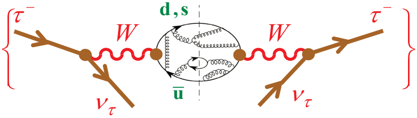
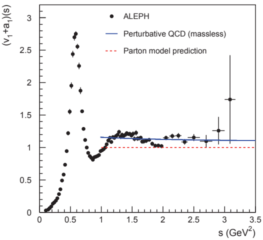
ECT* Trento, 31 January – 4 February 2022

τ Hadronic Width: R_τ

$$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}\left\{ \quad \right\}$$



Davier et al, 1312.1501



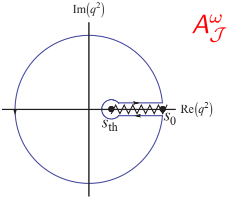
$$\Pi^{(J)}(s) \;\equiv\; |V_{ud}|^2\left(\Pi^{(J)}_{ud,V}(s)+\Pi^{(J)}_{ud,A}(s)\right) \;+\; |V_{us}|^2\,\Pi^{(J)}_{us,V+A}(s)$$

$$v_1 \;=\; 2\pi\;\text{Im}\Pi^{(1)}_{ud,V}(s) \quad , \quad a_1 \;=\; 2\pi\;\text{Im}\Pi^{(1)}_{ud,A}(s)$$

$$R_\tau \;=\; \frac{\Gamma[\tau^-\to\nu_\tau\,\text{hadrons}]}{\Gamma[\tau^-\to\nu_\tau e^-\overline{\nu}_e]} \;=\; R_{\tau,V}+R_{\tau,A}+R_{\tau,S}$$

$$=\; 12\pi\int_0^{m_\tau^2}\frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\text{Im}\,\Pi^{(0+1)}(s)-2\frac{s}{m_\tau^2}\,\text{Im}\,\Pi^{(0)}(s)\right]$$

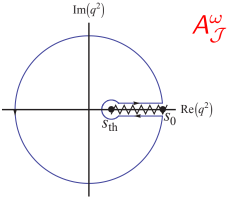
$$i\int d^4x\, e^{iqx}\,\langle 0|\,T\left[\mathcal{J}_{ij}^{\mu}(x)\,\mathcal{J}_{ij}^{\nu\,\dagger}(0)\right]\,|0\rangle \;=\; (-g^{\mu\nu}q^2+q^{\mu}q^{\nu})\,\Pi_{ij,\mathcal{J}}^{(1)}(q^2)+q^{\mu}q^{\nu}\;\Pi_{ij,\mathcal{J}}^{(0)}(q^2)$$



$$A_{\mathcal{J}}^{\omega}(s_0)\;\equiv\;\int_{s_{th}}^{s_0}\frac{ds}{s_0}\,\omega(s)\;\text{Im}\,\Pi_{\mathcal{J}}^{(J)}(s)\;=\;\frac{i}{2}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\,\Pi_{\mathcal{J}}^{(J)}(s)$$

$$\Pi_{\mathcal{J}}^{(J)}(s)\;\approx\;\Pi_{\mathcal{J}}^{(J)}(s)^{\mathrm{OPE}}\;=\;\sum_{D}\frac{\mathcal{O}_{D,\mathcal{J}}^{(J)}}{(-s)^{D/2}}$$

$$i \int d^4x\; e^{iqx}\; \langle 0|\; T\left[\mathcal{J}_{ij}^{\mu}(x)\,\mathcal{J}_{ij}^{\nu\,\dagger}(0)\right]\; |0\rangle \;=\; (-g^{\mu\nu}\, q^2 + q^\mu q^\nu)\; \Pi^{(1)}_{ij,\mathcal{J}}(q^2) + q^\mu q^\nu\; \Pi^{(0)}_{ij,\mathcal{J}}(q^2)$$



$$A^{\omega}_{\mathcal{J}}(s_0) \;\equiv\; \int_{s_{th}}^{s_0} \frac{ds}{s_0}\; \omega(s)\; \mathrm{Im}\,\Pi^{(J)}_{\mathcal{J}}(s) \;=\; \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\; \omega(s)\; \Pi^{(J)}_{\mathcal{J}}(s)$$

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$$\mathrm{R}_{\tau} \;=\; \mathrm{N_C}\, \mathrm{S_{EW}}\, \left(1 + \delta_{\mathrm{P}} + \delta_{\mathrm{NP}}\right)$$

Braaten-Narison-Pich ’92

$$= \; 6\pi i \oint_{|x|=1} (1-x)^2 \left[(1+2x)\; \Pi^{(0+1)}(m_\tau^2 x) - 2x\; \Pi^{(0)}(m_\tau^2 x)\right]$$

$$\delta_{\mathrm{P}} = a_\tau + 5.20\; a_\tau^2 + 26\; a_\tau^3 + 127\; a_\tau^4 + \cdots \approx 20\%$$

Baikov-Chetyrkin-Kühn ’08

$a_\tau \equiv \alpha_s(m_\tau^2)/\pi$, $\mathrm{S_{EW}} = 1.0201\;(3)$

Marciano-Sirlin, Braaten-Li, Erler

$\delta_{\mathrm{NP}} = -0.0064 \pm 0.0013$ (Fitted from data)

Davier et al ’14

Perturbative Contribution $(m_q = 0)$

$a_\tau \equiv \frac{\alpha_s(m_\tau^2)}{\pi}$

$$-s\,\frac{d}{ds}\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n\, a_s(-s)^n \quad \Longrightarrow \quad \boxed{\delta_{\mathrm{P}} = \underbrace{\sum_{n=1} K_n\, A^{(n)}(\alpha_s)}_{\text{CIPT}} = \underbrace{\sum_{n=1} r_n\, a_\tau^n}_{\text{FOPT}}}$$

$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\left(1-2x+2x^3-x^4\right)\left(\frac{\alpha_s(-m_\tau^2 x)}{\pi}\right)^n = a_\tau^n + \cdots$$

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1) The dominant corrections come from the contour integration

Large running of α_s along the circle $s = m_\tau^2\, e^{i\phi}$, $\phi \in [-\pi, \pi]$

n	1	2	3	4	5
K_n	1	1.6398	6.37101	49.0757	?
r_n	1	5.2023	26.3659	127.079	$307.78 + K_5$
$r_n - K_n$	0	3.5625	19.9949	78.0029	307.78

Baikov-Chetyrkin-Kühn '08

Le Diberder-Pich '92

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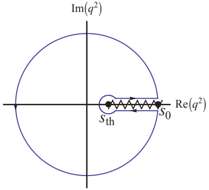
2) CIPT gives rise to a well-behaved perturbative series

$a_\tau = 0.11$	$A^{(1)}(\alpha_s)$	$A^{(2)}(\alpha_s)$	$A^{(3)}(\alpha_s)$	$A^{(4)}(\alpha_s)$	δ_{P}
$\beta_{n>1} = 0$	0.14828	0.01925	0.00225	0.00024	0.20578
$\beta_{n>2} = 0$	0.15103	0.01905	0.00209	0.00020	0.20537
$\beta_{n>3} = 0$	0.15093	0.01882	0.00202	0.00019	0.20389
$\beta_{n>4} = 0$	0.15058	0.01865	0.00198	0.00018	0.20273
$\beta_{n>5} = 0$	0.15041	0.01859	0.00197	0.00018	0.20232
$\mathcal{O}(a_\tau^4)$ FOPT	0.16115	0.02431	0.00290	0.00015	0.22665

FOPT overestimates δ_{P} by 11%

Non-Perturbative Contribution



$$A^{\omega}_{\mathcal{J}}(s_0) \equiv \int_{s_{th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\; \mathrm{Im}\,\Pi^{(J)}_{\mathcal{J}}(s) \;=\; \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\; \Pi^{(J)}_{\mathcal{J}}(s)$$

$$\Pi^{(J)}_{\mathcal{J}}(s) \;\approx\; \Pi^{(J)}_{\mathcal{J}}(s)^{\rm OPE} \;=\; \sum_D \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

$$A^{\omega,\rm NP}_{\mathcal{J}}(s_0) \;=\; \pi \sum_D a_{-1,D}\, \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{s_0^{D/2}} \qquad , \qquad \omega(-s_0 x) = \sum_n a_{n,D}\; x^{n+D/2}$$

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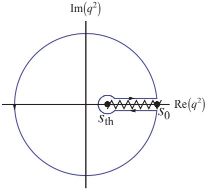

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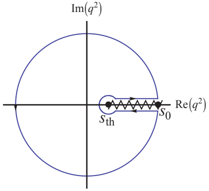
- **Strong power suppression at $s_0 = m_\tau^2$:** $\sim (\Lambda_{\mathrm{QCD}}/m_\tau)^D \quad , \quad D \geq 4$

$$\mathcal{O}_{4,V/A} \approx 4\pi^2 \left\{\tfrac{1}{12\pi} \langle\alpha_s GG\rangle + (m_u + m_d)\,\langle\bar{q}q\rangle\right\} \approx [(6.7 \pm 3.2) - 0.6] \cdot 10^{-3} \times m_\tau^4$$

Non-Perturbative Contribution



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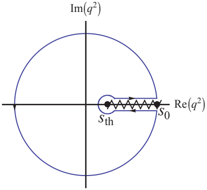
- R_τ: $\boldsymbol{\omega}(\mathrm{x}) = 1 - 3\mathrm{x}^2 + 2\mathrm{x}^3 \quad \Longrightarrow \quad \delta_{\rm NP} = -3\,\frac{\mathcal{O}_{6,\rm V+A}}{\mathrm{m}_\tau^6} - 2\,\frac{\mathcal{O}_{8,\rm V+A}}{\mathrm{m}_\tau^8}$

Additional chiral suppression in $|\mathcal{O}_{6,V+A}| \; < \; |\mathcal{O}_{6,V-A}| \approx (1.1 \pm 0.3)\cdot 10^{-4} \times m_\tau^6$

Non-Perturbative Contribution



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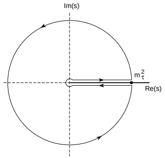
Additional chiral suppression in $|\mathcal{O}_{6,V+A}| \; < \; |\mathcal{O}_{6,V-A}| \approx (1.1 \pm 0.3) \cdot 10^{-4} \times m_\tau^6$

- **Sensitivity to $\mathcal{O}_D$ with different $\omega(x)$** $\Longrightarrow$ **Measure** $\delta_{\rm NP}$

R_τ suitable for a precise α_s determination



$$R_\tau \; = \; 6\pi i \oint_{|x|=1} (1-x)^2 \; \left[(1+2x)\, \Pi^{(0+1)}(m_\tau^2 x) - 2x\, \Pi^{(0)}(m_\tau^2 x) \right]$$

$$\Pi_{\mathcal{J}}^{(J)}(s) \; \approx \; \Pi_{\mathcal{J}}^{(J)}(s)^{\mathrm{OPE}} \; = \sum_D \frac{\mathcal{O}_{D,\mathcal{J}}^{(J)}}{(-s)^{D/2}}$$

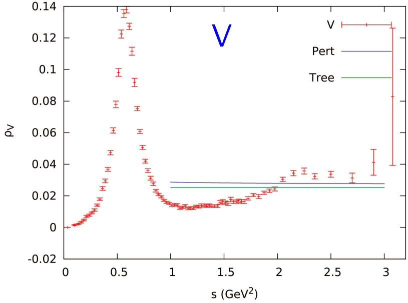
- **Known to $\mathcal{O}(\alpha_s^4)$. Sizeable $\delta_P \sim 20\%$. Strong sensitivity to α_s**
- **m_τ large enough to safely use the OPE. Flat $V + A$ distribution**
- **OPE only valid away from real axis: $(1-x)^2$ pinched at $s = m_\tau^2$**
- $m_{u,d} = 0$ ➔ $s\,\Pi^{(0)} = 0$ ➔ $R_{\tau,V+A} \; = \; 6\pi i \oint_{|x|=1} (1 - 3x^2 + 2x^3)\; \Pi_{ud,V+A}^{(0+1)}(m_\tau^2 x)$

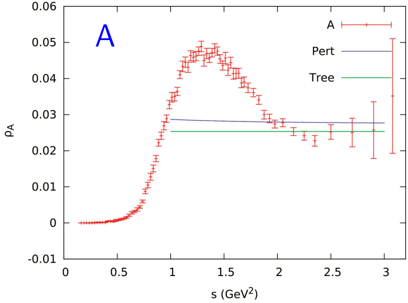
  ➔ $\delta_{\mathrm{NP}} \sim 1/m_\tau^6$ **Strong suppression of non-perturbative effects**
- **$D = 6$ OPE contributions have opposite sign for V & A. Cancellation**
- **δ_{NP} can be determined from data: $\delta_{\mathrm{NP}} = -0.0064 \pm 0.0013$** Davier et al

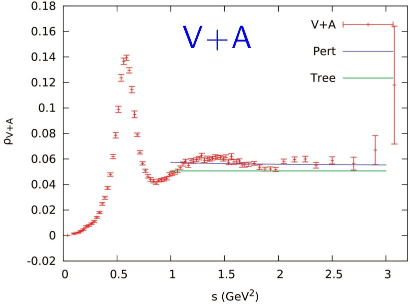
R_τ ➔ $\boxed{\alpha_s(m_\tau^2) = 0.331 \pm 0.013}$ Pich 2014

ALEPH Spectral Functions

Davier et al. 2014







— $\alpha_s(m_\tau^2) = 0.329$

— Parton Model

Detailed analysis of ALEPH data

Rodríguez-Sánchez, Pich, arXiv:1605.06830

Method (V + A)	$\alpha_s(m_\tau^2)$		
	CIPT	FOPT	Average
ALEPH moments[1]	$0.339\,^{+\,0.019}_{-\,0.017}$	$0.319\,^{+\,0.017}_{-\,0.015}$	$0.329\,^{+\,0.020}_{-\,0.018}$
Mod. ALEPH moments[2]	$0.338\,^{+\,0.014}_{-\,0.012}$	$0.319\,^{+\,0.013}_{-\,0.010}$	$0.329\,^{+\,0.016}_{-\,0.014}$
$A^{(2,m)}$ moments[3]	$0.336\,^{+\,0.018}_{-\,0.016}$	$0.317\,^{+\,0.015}_{-\,0.013}$	$0.326\,^{+\,0.018}_{-\,0.016}$
s_0 dependence[4]	0.335 ± 0.014	0.323 ± 0.012	0.329 ± 0.013
Borel transform[5]	$0.328\,^{+\,0.014}_{-\,0.013}$	$0.318\,^{+\,0.015}_{-\,0.012}$	$0.323\,^{+\,0.015}_{-\,0.013}$
Combined value	0.335 ± 0.013	0.320 ± 0.012	0.328 ± 0.013

$$\boxed{\alpha_s(M_Z^2) = 0.1197 \pm 0.0015}$$

1) $\omega_{kl}(x) = (1+2x)\,(1-x)^{2+k}x^l$ $\qquad (k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

2) $\tilde{\omega}_{kl}(x) = (1-x)^{2+k}x^l$ $\qquad (k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

3) $\omega^{(2,m)}(x) = (1-x)^2\sum_{k=0}^{m}(k+1)\,x^k = 1-(m+2)\,x^{m+1}+(m+1)x^{m+2}$, $\qquad 1 \le m \le 5$

4) $\omega^{(2,m)}(x)$ $\qquad 0 \le m \le 2$, 1 single moment in each fit

5) $\omega_a^{(1,m)}(x) = (1-x^{m+1})\,\mathrm{e}^{-ax}$ $\qquad 0 \le m \le 6$

α_s determination with ALEPH-like fit

Rodríguez-Sánchez, A.P.

$\omega_{kl}(x) \;=\; (1-x)^{2+k}\, x^l\, (1+2x)$, $x = s/m_\tau^2$, $(k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

Channel	$\alpha_s(m_\tau^2)$	$\langle \frac{\alpha_s}{\pi} GG\rangle$ ($10^{-3}\ \mathrm{GeV}^4$)	$\mathcal{O}_6$ ($10^{-3}\ \mathrm{GeV}^6$)	$\mathcal{O}_8$ $10^{-3}\ \mathrm{GeV}^8$)
V (FOPT)	$0.328\,^{+0.013}_{-0.007}$	$8\,^{+7}_{-14}$	$-3.2\,^{+0.8}_{-0.5}$	$5.0\,^{+0.4}_{-0.7}$
V (CIPT)	$0.352\,^{+0.013}_{-0.011}$	$-8\,^{+7}_{-7}$	$-3.5\,^{+0.3}_{-0.3}$	$4.9\,^{+0.4}_{-0.5}$
A (FOPT)	$0.304\,^{+0.010}_{-0.007}$	$-15\,^{+5}_{-8}$	$4.4\,^{+0.5}_{-0.4}$	$-5.8\,^{+0.3}_{-0.4}$
A (CIPT)	$0.320\,^{+0.011}_{-0.010}$	$-25\,^{+5}_{-5}$	$4.3\,^{+0.2}_{-0.2}$	$-5.8\,^{+0.3}_{-0.3}$
V+A (FOPT)	$0.319\,^{+0.010}_{-0.006}$	$-3\,^{+6}_{-11}$	$1.3\,^{+1.4}_{-0.8}$	$-0.8\,^{+0.4}_{-0.7}$
V+A (CIPT)	$0.339\,^{+0.011}_{-0.009}$	$-16\,^{+5}_{-5}$	$0.9\,^{+0.3}_{-0.4}$	$-1.0\,^{+0.5}_{-0.7}$

- **High sensitivity to α_s. Bad sensitivity to power corrections**
- **Cancellation in $\mathcal{O}_{6,V+A}$ confirmed. $V+A$ more reliable**
- $K_5 = 275 \pm 400,\ \mu^2 = (0.5, 2)\, m_\tau^2$
- Best values taken from $V+A$. Errors increased with sensitivity to $\mathcal{O}_{10}$

$$\alpha_s(m_\tau^2)^{\mathrm{CIPT}} \;=\; 0.339\,^{+0.019}_{-0.017}$$

$$\alpha_s(m_\tau^2)^{\mathrm{FOPT}} \;=\; 0.319\,^{+0.017}_{-0.015}$$

➡

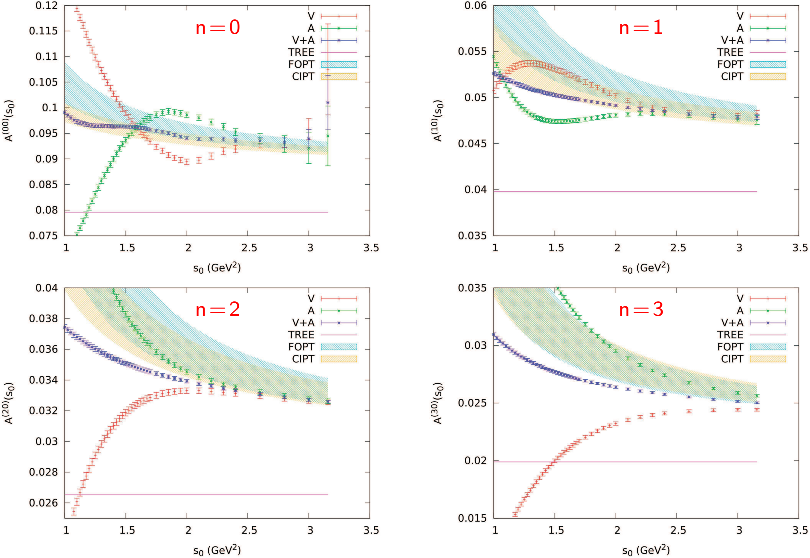
$$\boxed{\alpha_s(m_\tau^2) \;=\; 0.329\,^{+0.020}_{-0.018}}$$

Good agreement with Davier et al.: $\alpha_s(m_\tau^2) \;=\; 0.332 \pm 0.012$ (arXiv:1312.1501)

Experiment vs. (pinched) Perturbation Theory (only)

Rodríguez-Sánchez, A.P.

$\omega^{(n,0)}(s = s_0 x) \; = \; (1-x)^n \longrightarrow \mathcal{O}_{D\leq 2\,(n+1)}$, $\alpha_s(m_\tau^2) \; = \; 0.329\,^{+\,0.020}_{-\,0.018}$



Non-Perturbative Contributions Neglected

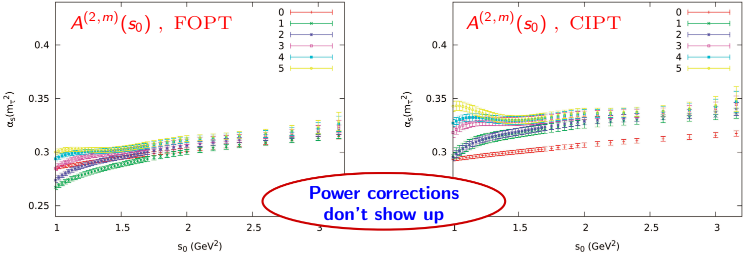
Rodríguez-Sánchez, A.P.

$$\omega^{(1,m)}(x) \;=\; 1 - x^{m+1} \;\rightarrow\; \mathcal{O}_{2m+4}$$

$$\omega^{(2,m)}(x) \;=\; (1-x)^2 \sum_{k=0}^{m} (k+1)\, x^k \;=\; 1 - (m+2)\, x^{m+1} + (m+1)\, x^{m+2} \;\rightarrow\; \mathcal{O}_{2m+4,2m+6}$$

Moment	$\alpha_s(m_\tau^2)$		Moment	$\alpha_s(m_\tau^2)$	
(n, m)	FOPT	CIPT	(n, m)	FOPT	CIPT
(1,0)	$0.315\,^{+0.012}_{-0.007}$	$0.327\,^{+0.012}_{-0.009}$	(2,0)	$0.311\,^{+0.015}_{-0.011}$	$0.314\,^{+0.013}_{-0.009}$
(1,1)	$0.319\,^{+0.010}_{-0.006}$	$0.340\,^{+0.011}_{-0.009}$	(2,1)	$0.311\,^{+0.011}_{-0.006}$	$0.333\,^{+0.009}_{-0.007}$
(1,2)	$0.322\,^{+0.010}_{-0.008}$	$0.343\,^{+0.012}_{-0.010}$	(2,2)	$0.316\,^{+0.010}_{-0.005}$	$0.336\,^{+0.011}_{-0.009}$
(1,3)	$0.324\,^{+0.011}_{-0.010}$	$0.345\,^{+0.013}_{-0.011}$	(2,3)	$0.318\,^{+0.010}_{-0.006}$	$0.339\,^{+0.011}_{-0.008}$
(1,4)	$0.326\,^{+0.011}_{-0.011}$	$0.347\,^{+0.013}_{-0.012}$	(2,4)	$0.319\,^{+0.009}_{-0.007}$	$0.340\,^{+0.011}_{-0.009}$
(1,5)	$0.327\,^{+0.015}_{-0.013}$	$0.348\,^{+0.014}_{-0.012}$	(2,5)	$0.320\,^{+0.010}_{-0.008}$	$0.341\,^{+0.011}_{-0.009}$

Amazing stability



Power corrections don't show up

V + A

Exp. errors only

Models of Duality Violation

$$\Delta A^{\omega}_{\mathcal{J}}(s_0) \; = \; \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0} \, \omega(s) \left\{ \Pi_{\mathcal{J}}(s) - \Pi_{\mathcal{J}}(s)^{\mathrm{OPE}} \right\} \; = \; -\pi \int_{s_0}^{\infty} \frac{ds}{s_0} \, \omega(s) \, \Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s)$$

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Ansatz: $\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s) \;=\; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\,s)}\,\sin\left(\alpha_{\mathcal{J}}+\beta_{\mathcal{J}}\,s\right)\quad , \quad s>\hat{s}_0$

1) Boito et al.: $\hat{s}_0\sim 1.55\ \mathrm{GeV}^2\quad , \quad \omega(x)=1$ (no OPE corrections)

- Fit s_0 dependence: ➔ $\{A^{(00)}(s_0)\,,\,\rho(s_0+\Delta s_0)\,,\,\cdots\,,\,\rho(s_0+(n-1)\Delta s_0)\}$
- Direct fit of $\rho(s)^{\mathrm{exp}}$ above $\hat{s}_0$ ➔ $\delta,\gamma,\alpha,\beta$ **OPE not valid**

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Models of Duality Violation

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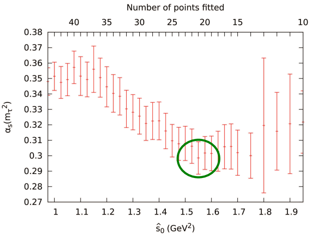
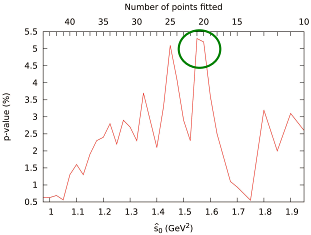




Rodríguez-Sánchez, A.P.

FOPT , V

(larger errors in A)

Bad fit (instabilities)

Model dependence?

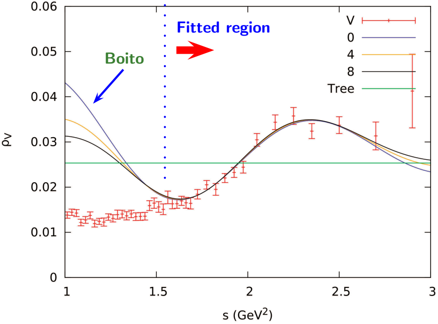
Boito et al. value

Ansatz: $\Delta\rho^{\mathrm{DV}}_{V/A}(s) \, = \, s^{\lambda_{V/A}} \, e^{-(\delta_{V/A}+\gamma_{V/A}\, s)} \, \sin\left(\alpha_{V/A}+\beta_{V/A}\, s\right) \quad , \quad s > \hat{s}_0$

2) $\boldsymbol{\lambda} \;\; \geq$ **0:** $\quad \hat{s}_0 \sim 1.55\ \mathrm{GeV}^2 \quad , \quad \omega(x) = 1$

Rodríguez-Sánchez, A.P.

	λ_V	$\alpha_s(m_\tau^2)^{\mathrm{FOPT}}$	δ_V	γ_V	α_V	β_V	p-value
Boito	0	0.298 ± 0.010	3.6 ± 0.5	0.6 ± 0.3	-2.3 ± 0.9	4.3 ± 0.5	5.3 %
	1	0.300 ± 0.012	3.3 ± 0.5	1.1 ± 0.3	-2.2 ± 1.0	4.2 ± 0.5	5.7 %
	2	0.302 ± 0.011	2.9 ± 0.5	1.6 ± 0.3	-2.2 ± 0.9	4.2 ± 0.5	6.0 %
	4	0.306 ± 0.013	2.3 ± 0.5	2.6 ± 0.3	-1.9 ± 0.9	4.1 ± 0.5	6.6 %
	8	0.314 ± 0.015	1.0 ± 0.5	4.6 ± 0.3	-1.5 ± 1.1	3.9 ± 0.6	7.7 %



- **Fitted $\boldsymbol{\alpha}$ is model dependent**
- $\boldsymbol{\lambda} \;\; = 0$ (Boito) **gives the worse fit**
- **Fit quality & $\boldsymbol{\alpha}$ increase with $\boldsymbol{\lambda}$**

  ➡ **closer to data at $\boldsymbol{s} \; < \hat{\boldsymbol{s}}_0$**
- $\Delta\hat{\boldsymbol{s}}_0$ ➡ **3 times larger errors**

Not competitive & unreliable

Ansatz: $\Delta\rho^{\mathrm{DV}}_{V/A}(s) \; = \; s^{\lambda_{V/A}}\, e^{-(\delta_{V/A}+\gamma_{V/A}\, s)}\, \sin\left(\alpha_{V/A}+\beta_{V/A}\, s\right) \quad , \quad s > \hat{s}_0$

$\boldsymbol{\lambda} \; \geq 0 \quad , \quad \hat{s}_0 \sim 1.55\ \mathrm{GeV}^2 \quad ,$ additional weights

Rodríguez-Sánchez, A.P.

Boito et al →

λ_V	$\mathcal{O}_{6,V}$	$\mathcal{O}_{8,V}$	$\mathcal{O}_{10,V}$	$\mathcal{O}_{12,V}$
0	-0.0082	0.014	-0.019	0.023
4	-0.0064	0.010	-0.012	0.014
8	-0.0037	0.004	0.001	-0.0099

GeV units

Fitted condensates are model dependent

The size of $\mathcal{O}_{2\mathrm{n,V}}$ decreases when α_s (& fit quality) increases

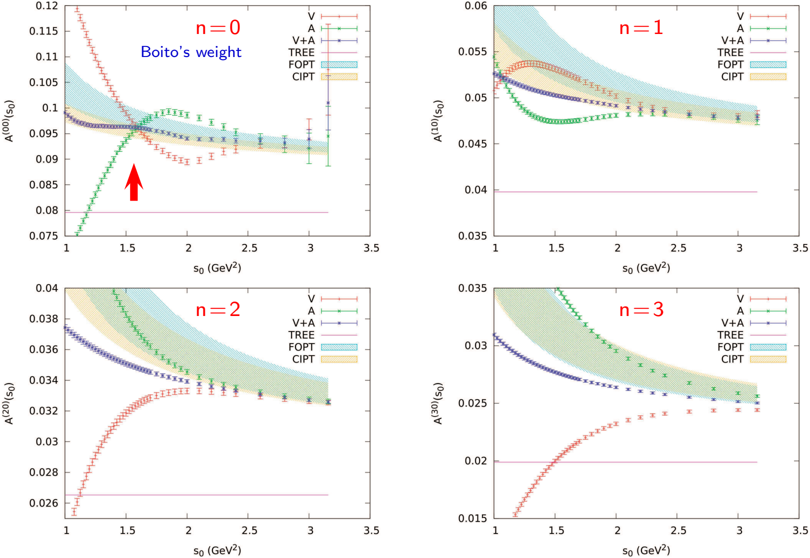
Huge condensates claimed by Boito et al (OPE breakdown), but:

- $|\mathcal{O}_{6,V/A}| < |\mathcal{O}_{6,V-A}| = |(-3.5 \pm 0.9)| \times 10^{-3}\ \mathrm{GeV}^6$
- Opposite signs found with e^+e^- data: $\mathcal{O}_6^{\mathrm{EM}} = +0.008 \quad , \quad \mathcal{O}_8^{\mathrm{EM}} = -0.03$
  (1805.08176)

Experiment vs. (pinched) Perturbation Theory (only)

Rodríguez-Sánchez, A.P.

$\omega^{(n,0)}(s = s_0 x) \; = \; (1-x)^n$ ➡ $\mathcal{O}_{D\le 2\,(n+1)}$, $\alpha_s(m_\tau^2) \; = \; 0.329\,^{+\,0.020}_{-\,0.018}$

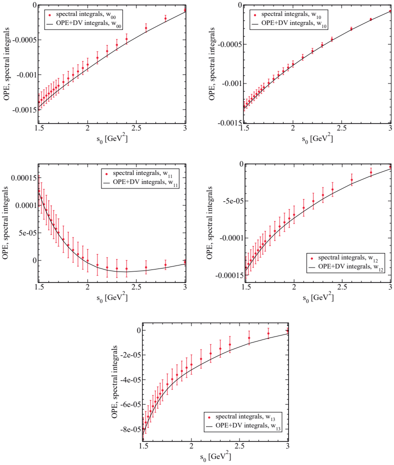


Creative Plotting

Boito et al, 1611.03457

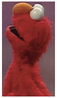
GOOD

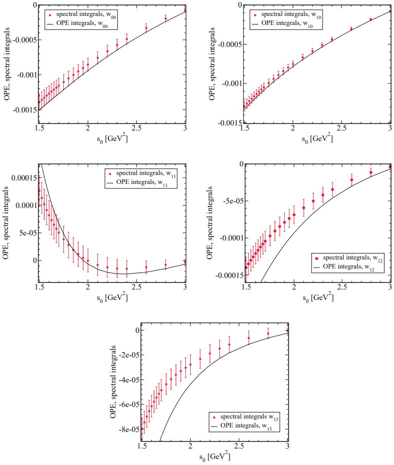
Miraculous DV ansatz



“Truncated OPE pitfalls”

BAD



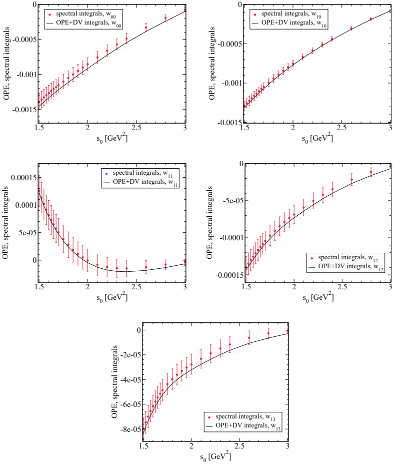


Creative Plotting

1. What is being plotted?
2. Why $s_0 \in [1.5, 3]$ GeV^2 only?
3. Where are the theoretical errors?
4. Why the vertical scale is so tiny?

GOOD

Miraculous DV ansatz



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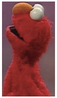

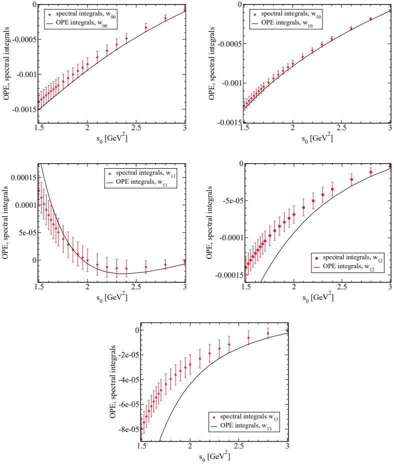

1) $\left[A^{\omega}(m_{\tau}^{2}) - A^{\omega}(s_0)\right]/\pi$

Determined by $A^{\omega}(\hat{s}_0)$ and fitted $\rho(s)$

2) **Only the fitted region is plotted!**

Left plots don't test α_s^{DV}

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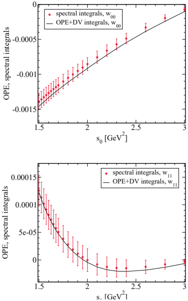

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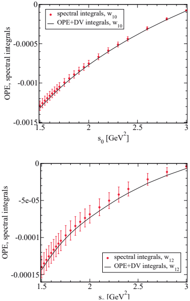

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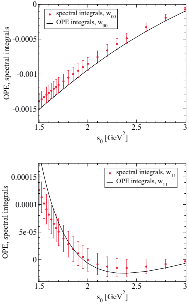
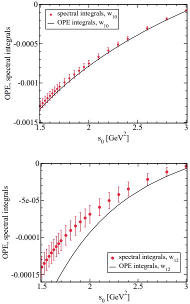
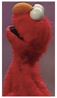
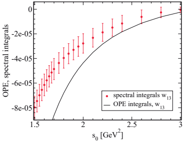


BAD







$\omega_n(x) = x^n$

$$A^{\omega_n}(s_0) = \left(\frac{\hat{s}_0}{s_0}\right)^{n+1} A^{\omega_n}(\hat{s}_0) + \frac{1}{s_0^{n+1}} \int_{\hat{s}_0}^{s_0} ds\, s^n \rho(s)$$

1) $\left[A^{\omega}(m_\tau^2) - A^{\omega}(s_0)\right] / \pi$

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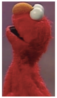
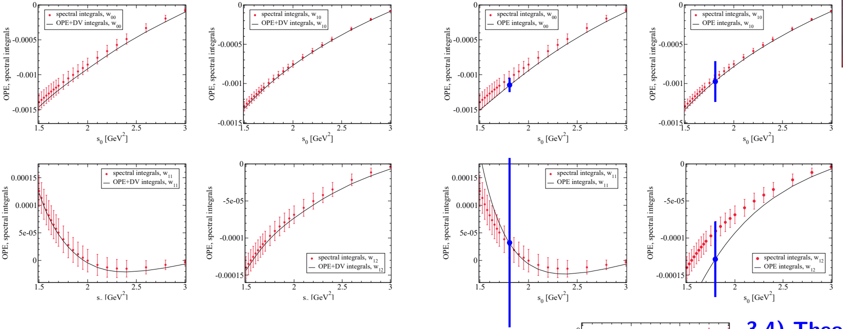
GOOD **Miraculous DV ansatz** **"Truncated OPE pitfalls"** **BAD**





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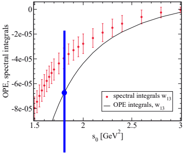

3,4) Theory:

Huge errors with these scales

$\mathcal{O}_8$ error only

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GOOD

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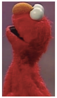

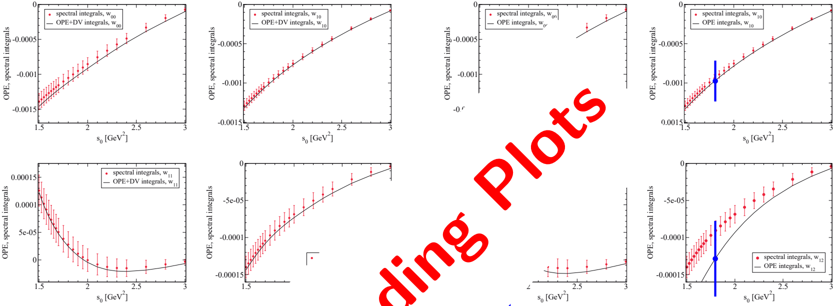

3,4) Theory:

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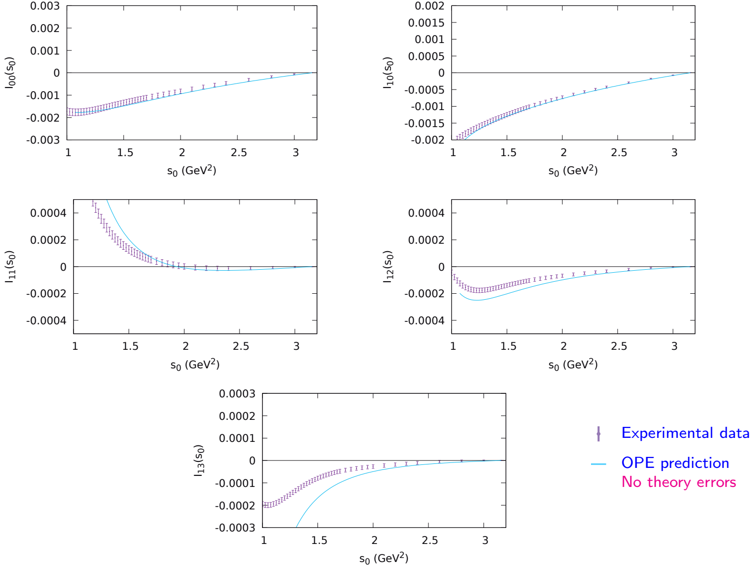
$$I_{kl}(s_0) \equiv \frac{1}{\pi}\left[A^{\omega^{kl}}(m_\tau^2) - A^{\omega^{kl}}(s_0)\right]$$

ω^{00} , ω^{10} , ω^{11} , ω^{12} , ω^{13}

Standard OPE Approach

Rodríguez-Sánchez, A.P.



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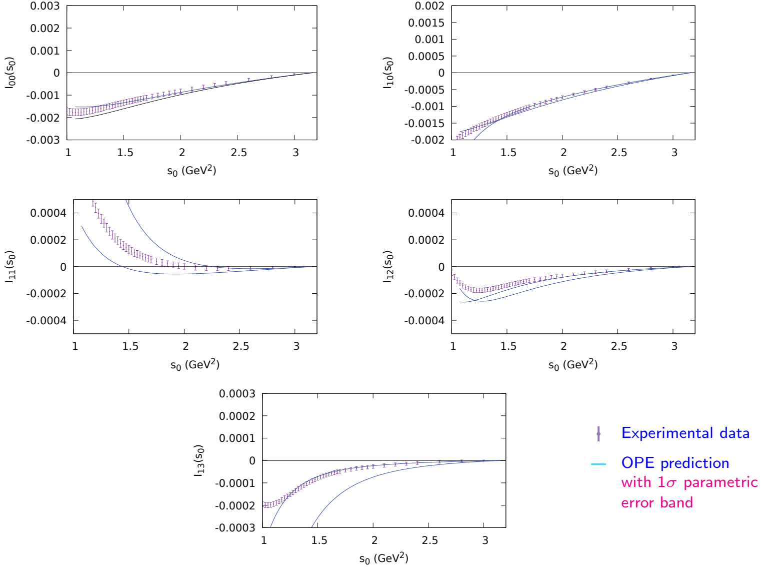

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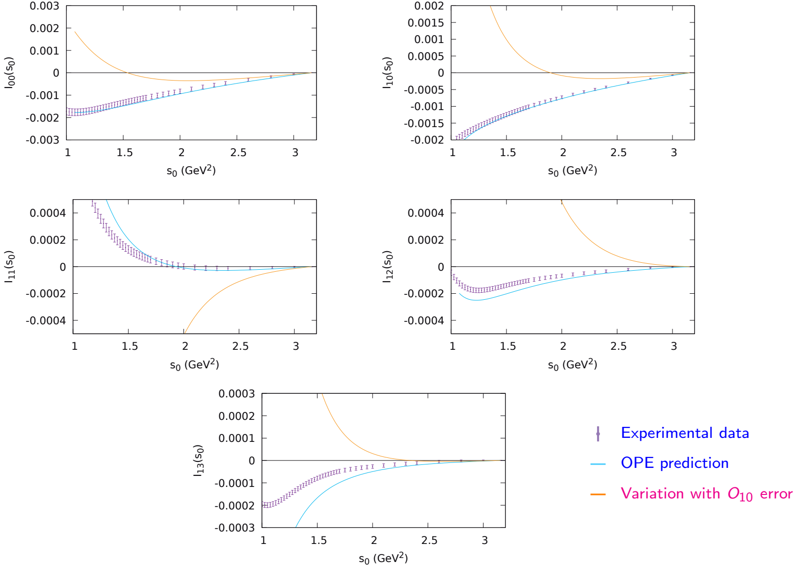


Summary

Precise determination of $\alpha_s(m_\tau^2)$: Rodríguez-Sánchez, Pich, 1605.06830

Method $(\mathrm{V}+\mathrm{A})$	$\alpha_s(m_\tau^2)$		
	CIPT	FOPT	Average
ALEPH moments [1]	$0.339\,^{+\,0.019}_{-\,0.017}$	$0.319\,^{+\,0.017}_{-\,0.015}$	$0.329\,^{+\,0.020}_{-\,0.018}$
Mod. ALEPH moments [2]	$0.338\,^{+\,0.014}_{-\,0.012}$	$0.319\,^{+\,0.013}_{-\,0.010}$	$0.329\,^{+\,0.016}_{-\,0.014}$
$A^{(2,m)}$ moments [3]	$0.336\,^{+\,0.018}_{-\,0.016}$	$0.317\,^{+\,0.015}_{-\,0.013}$	$0.326\,^{+\,0.018}_{-\,0.016}$
s_0 dependence [4]	0.335 ± 0.014	0.323 ± 0.012	0.329 ± 0.013
Borel transform [5]	$0.328\,^{+\,0.014}_{-\,0.013}$	$0.318\,^{+\,0.015}_{-\,0.012}$	$0.323\,^{+\,0.015}_{-\,0.013}$
Combined value	0.335 ± 0.013	0.320 ± 0.012	0.328 ± 0.013

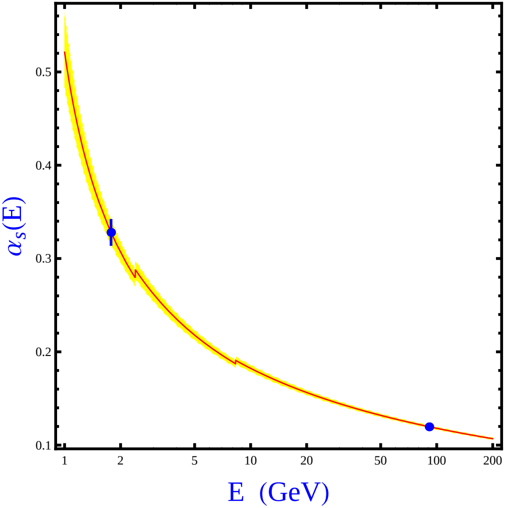
Good agreement with other analyses:

$$\alpha_s(m_\tau^2) = \begin{cases} 0.332 \pm 0.012 & (0.341_{\mathrm{CIPT}}\ ,\ 0.324_{\mathrm{FOPT}}) \\ 0.327 \pm 0.024 & (0.330_{\mathrm{PV}}\ ,\ 0.349_{\mathrm{CIPT}}\ ,\ 0.313_{\mathrm{FOPT}}) \end{cases}$$

Davier et al, 1312.1501

Ayala et al, 2112.01992

α_s at N^{3}LO from τ and Z



$$\alpha_s(m_\tau^2) = 0.328 \pm 0.013$$

$$\Downarrow$$

$$\alpha_s(M_Z^2) = 0.1197 \pm 0.0015$$

$$\alpha_s(M_Z^2)_{\text{Z width}} = 0.1199 \pm 0.0029$$

A very precise test of Asymptotic Freedom

$$\boxed{\alpha_s^\tau(M_Z^2) - \alpha_s^Z(M_Z^2) = 0.0002 \pm 0.0015_\tau \pm 0.0029_Z}$$