Update on α_s from FLAG

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FLAG WG on α_s: Roger Horsley, Peter Petreczky and Stefan Sint

alphas-22: workshop on precision measurements of the QCD coupling constant

Trento/Dublin, 31 January 2022

Outline

- FLAG
- General observations on determinations of $\alpha_s(m_Z)$
- Definitions of α_{eff}, $\Lambda_{\overline{\text{MS}}}$
- FLAG criteria for determinations of $\alpha_s(m_Z)$
- The problem with large scale differences
- $N_{\text{f}} = 3$ new results which pass the FLAG criteria
- Decoupling strategy by ALPHA collaboration
- $N_{\text{f}} = 0$ results
- Conclusions

FLAG = FLavour Lattice Averaging Group

- FLAG is an effort by the international lattice QCD community to provide the wider high energy physics community with lattice results for quantities of phenomenological interest, satisfying clearly defined quality criteria
- Original focus was on flavour physics, but now FLAG includes also sections on α_s, nucleon matrix elements and scale setting.
- FLAG website: `flag.unibe.ch`
- Besides the quality criteria FLAG requires acceptance by/publication in a peer reviewed journal.
- Cutoff date for FLAG 2021 was 30 April 2022, link to report: `https://arxiv.org/pdf/2111.09849.pdf`
- Links to previous FLAG 2019 report `https://arxiv.org/pdf/1902.08191.pdf`, published in Eur. Phys. J. C 80 (2020) 2, 113

N.B. FLAG insists that anyone using FLAG results should cite the original sources which enter the averages.

List of new publications for α_s I

- Static energy/force:
  1. $N_\mathrm{f} = 2+1$: [TUMQCD 19]: Bazavov et al. (TUMQCD collab.) Phys. Rev. D 100 (2019) 114511 [1907.11747]
  2. $N_\mathrm{f} = 2+1$: [Ayala 20]: C. Ayala, X. Lobregat and A. Pineda, JHEP 09 (2020) 016 [2005.12301]
  3. $N_\mathrm{f} = 0$: [Husung 20]: N. Husung, A. Nada and R. Sommer, PoS LATTICE2019 (2020) 263.
- Heavy quark current 2-point functions:
  1. $N_\mathrm{f} = 2+1$ [Petrezcky 19]: P. Petreczky and J. Weber, Phys. Rev. D 100 (2019) 034519 [1901.06424].
  2. $N_\mathrm{f} = 2+1$ [Boito 20]: D. Boito and V. Mateu, Phys. Lett. B 806 (2020) 135482 [1912.06237]
- QCD vertices:

  $N_\mathrm{f} = 2+1$: [Zafeiropoulos 19], Zafeiropoulos et al., Phys. Rev. Lett. 122 (2019) 162002 [1902.08148].
- Light quark current 2-point function in position space (/vacuum polarization):

  $N_\mathrm{f} = 2+1$, [Cali 20]: S. Cali et al, eprint 2003.05781, accepted for publication in Phys. Rev. Lett.

List of new results/publications for α_s II (cutoff date April 30,2021)

- Decoupling strategy

  Dalla Brida et al. (ALPHA collab.), Phys. Lett. B 807 (2020) 135571 [1912.06001].

- Step-scaling in finite volume:

  1. $N_\mathrm{f} = 0$ [Dalla Brida 19], M. Dalla Brida and A. Ramos, Eur. Phys. J. C 79 (2019) 720 [1905.05147].
  2. $N_\mathrm{f} = 0$ [Nada 20]: A. Nada and A. Ramos, [2007.12862].

Some general observations on determinations of $\alpha_s(m_Z)$

- FLAG 19 average: $\alpha_s(m_Z) = 0.1182(8)$, the uncertainty is 0.7%
- All but one determinations: $N_\mathrm{f} = 2+1$, combined with 4-loop matching across charm and bottom thresholds
- A 1% error on α_s requires $\Delta\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f}=3)} < 5\%$

$\Rightarrow$ isospin breaking due to electromagnetism + mass differences is not yet relevant for α_s

Majority of determinations affected predominatly by systematics, in particular:

- Perturbative truncation errors: requires $\mu \gg \Lambda_{\overline{\mathrm{MS}}}$
- continuum limit: requires, $\mu \ll 1/a$

Note: given the very good quantitative perturbative description of decoupling across charm and bottom threshold [cf. Athenodorou et al (ALPHA '18)] the determination of α_s is equivalent to a non-perturbative result for the Λ-parameter with $N_\mathrm{f} = 3, 4$

Reminder: definition of effective coupling, α_{eff}

Starting point for all α_s determinations: Euclidean short distance quantity Q, that

- can be measured in a lattice simulation
- has a perturbative expansion, $Q = c_0 + c_1\alpha + c_2\alpha^2 + \ldots$.

We associate an effective coupling to Q, by normalizing

$$\alpha_{\text{eff}} = (Q - c_0)/c_1$$

- Advantage: no need to refer to a particular scale, α_{eff} is measured, possibly after chiral and continuum extrapolations (exception: couplings at $1/a$, e.g. from small Wilson loops).
- Loop counting: Relate to the $\overline{\text{MS}}$ scheme:

$$\alpha_{\text{eff}} = \alpha_{\overline{\text{MS}}} + d_1\alpha_{\overline{\text{MS}}}^2 + d_2\alpha_{\overline{\text{MS}}}^3 + d_3\alpha_{\overline{\text{MS}}}^4 + \ldots$$

  If d_k are known up to $k = n_l$ the loop order is n_l. Currently best cases have $n_l = 3$ (plus partial information on $n_l = 4$ for static potential/force)

Relation to Λ-parameter

Λ-parameter in mass-independent renormalization scheme:

$$\begin{aligned}\Lambda_{\overline{\mathrm{MS}}} &= \mu\varphi\left(\bar{g}(\mu)\right) \\ \varphi\left(\bar{g}\right) &= \left[b_0\bar{g}^2\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2}} \exp\Big\{\underbrace{-\int_0^{\bar{g}} dg\left[\frac{1}{\color{red}{\beta(g)}} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]}_{=I[\bar{g};\beta]}\Big\}\end{aligned}$$

Non-perturbatively defined coupling $\alpha(\mu) = \bar{g}^2(\mu)/(4\pi)$ implies a non-perturbative β-function:

$$\beta(\bar{g}) \stackrel{\text{def}}{=} \mu\frac{\partial\bar{g}(\mu)}{\partial\mu}, \qquad \beta(g) = -b_0 g^3 - b_1 g^5 + \ldots$$

with universal 1- and 2-loop coefficients b_0, b_1:

$$b_0 = (11 - \tfrac{2}{3}N_{\mathrm{f}})/(4\pi)^2, \qquad b_1 = (102 - \tfrac{38}{3}N_{\mathrm{f}})/(4\pi)^4\,.$$

At large μ, use perturbative β-function to n_l loops (b_{n_l+1} known) to replace $\beta->\beta^{n_l}$

$$I[g;\beta] \quad \stackrel{g\to 0}{\simeq} \quad I[g,\beta^{(n_l)}] + O\left(g^{2n_l)}\right)$$

$\Rightarrow$ For large μ expect remaining μ-dependence $\Lambda_{\overline{\mathrm{MS}}}^{\text{estimated}}/\Lambda_{\overline{\mathrm{MS}}} = 1 + O\left(\alpha^{n_l}(\mu)\right)$

FLAG criteria for α_s-determinations

The α_s WG for FLAG 2021 opted to keep the FLAG 2019 criteria without change:

Renormalization scale

- ★ all points in the analysis have $\alpha_{\text{eff}} < 0.2$
- ○ all points have $\alpha_{\text{eff}} < 0.4$ and at least 1 with $\alpha_{\text{eff}} < 0.25$
- ■ otherwise

Perturbative behaviour

- ★ verified over a range of a factor 4 change in $\alpha_{\text{eff}}^{n_l}$ (= parametric uncertainty in Λ) without power corrections or alternatively $\alpha_{\text{eff}}^{n_l} < \frac{1}{2}\Delta\alpha_{\text{eff}}/(8\pi b_0 \alpha_{\text{eff}}^2)$ is reached.

$$\Delta\Lambda|_{\Delta\alpha} = \Delta\alpha \frac{\partial\Lambda}{\partial\alpha} = \frac{2\pi\Delta\alpha}{-g\beta(g)}\Lambda \approx \frac{\Delta\alpha}{8\pi b_0 \alpha^2}\Lambda$$

- ○ verified over a range of a factor $(3/2)^2$ change in $\alpha_{\text{eff}}^{n_l}$ possibly fitting with power corrections or alternatively $\alpha_{\text{eff}}^{n_l} < \Delta\alpha_{\text{eff}}/(8\pi b_0 \alpha_{\text{eff}}^2)$ is reached.
- ■ otherwise

FLAG criteria for α_s-determinations

Continuum limit: at a reference point of $\alpha_{\text{eff}} = 0.3$ (or less) require

- ★ three lattice spacings with $\mu a < 1/2$ and full O(a) improvement, or three lattice spacings with $\mu a \leq 1/4$ and 2-loop O(a) improvement, or $\mu a \leq 1/8$ and 1-loop O(a) improvement
- ○ three lattice spacings with $\mu a < 3/2$ reaching down to $\mu a = 1$ and full O(a) improvement, or three lattice spacings with $\mu a \leq 1/4$ and 1-loop O(a) improvement
- ■ otherwise

plus convention for μ in different quantities (e.g. $\mu = q$ in momentum space observables, or $\mu = 1/L$ for step-scaling)

The problem with large scale differences

$$\Lambda = \mu\varphi(\bar{g}(\mu)) = \mu\left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} dg\left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- Continuum relation, exact at any scale μ:
  - require large μ to evaluate integral perturbatively
  - require small μ to match hadronic scale

$\Rightarrow$ problem of large scale differences:

  - The scale μ must reach the perturbative regime: $\mu \gg \Lambda$
  - lattice cutoff must still be larger: $\mu \ll a^{-1}$
  - spatial volume must be large enough to contain pions: $L \gg 1/m_\pi$
  - Taken together this means

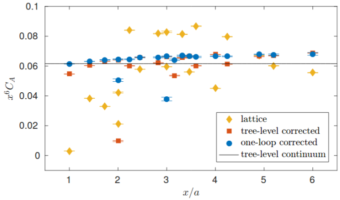
$$L/a \gg \mu L \gg m_\pi L \gg 1 \quad \Rightarrow L/a \quad \text{must be very large!}$$

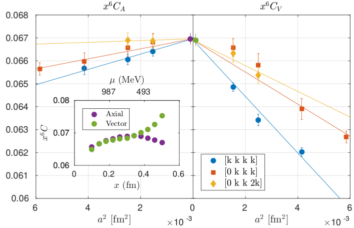
$\Rightarrow$ widely different scales cannot be resolved simultaneously on a single lattice!

- Strategy: step-scaling (multi-lattice) method or various compromises.

$\Lambda_{\overline{\mathrm{MS}}}$ from light quark current 2-point functions in position space: Cali 20

$$C_{\mathsf{A,V}}(x) = \sum_{\mu} \left\langle J^{\mu}_{\mathsf{A,V}}(x) J^{\mu}_{\mathsf{A,V}}(0) \right\rangle = \frac{6}{\pi^4 (x^2)^3} \left(1 + \frac{\alpha}{\pi} + O(\alpha^2)\right)$$





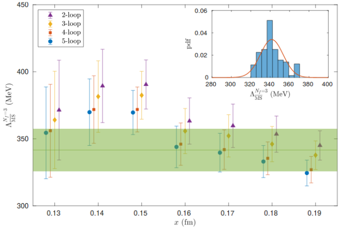
- $\alpha_{\mathsf{eff}}(\mu = 1/|x|) = \pi[(x^2)^3(\pi^4/6)C_{\mathsf{A,V}}(x) - 1]$
- use $|x| = 0.13 - 0.19\,\mathrm{fm}$, CLS, lattice spacings $a = 0.039 - 0.076\,\mathrm{fm}$, $\alpha_{\mathsf{eff}} = 0.235 - 0.308$, extrapolated to chiral limit.
- Non-perturbatively O(a) improved with 3 lattice spacings at $\mu^{-1} = |x| = 0.13\mathrm{fm}$ with $a\mu < 1/2$ and $\alpha_{\mathsf{eff}} \approx 0.3$

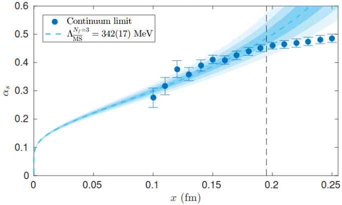
$\Rightarrow$ ★ for continuum extrapolation

- HOWEVER: 1-loop subtraction (using NSPT) of hypercubic lattice artifacts crucial!

$\Lambda_{\overline{\mathrm{MS}}}$ from light quark current 2-point functions in position space (Cali 20)

Result $\Lambda_{\overline{\mathrm{MS}}}^{N_\mathrm{f}=3} = 342(17)\,\mathrm{MeV}$ from weighted average over $|x| = 0.13 - 0.19\,\mathrm{fm}$:





Apply FLAG criteria:

- <u>Perturbative behaviour</u>: α_{eff}^3 covers a range of 2.2 close to $(3/2)^2$. Also the error $\Delta\alpha_{\mathrm{eff}} \approx 4 - 6\%$ are comfortably larger than the relative parametric uncertainty of Λ, with the stronger criterion still ok, so ★
- <u>Renormalization scale</u>: α_{eff} reaches $0.235 < 0.25 \Rightarrow$ ○

<u>Conclusion</u>: ★ in continuum limit and perturbative behaviour and ○ for the renormalization scale.
$\Rightarrow$ passes all FLAG criteria!

Discussion of Cali 20

Nevertheless the error estimate seems rather optimistic:

- Cali 20 show that vector and axial vector correlators yield compatible results within errors

$\Rightarrow$ absence of chirality breaking effects, however, one may expect other non-perturbative effects at these low energies.

- Cali 20 convert to the $\alpha_{\overline{\rm MS}}$ by solving numerically the truncated expansion

$$\alpha_{\rm eff}(\mu = 1/|x|) = \alpha_{\overline{\rm MS}}(\mu) + c_1\alpha^2_{\overline{\rm MS}}(\mu) + c_2\alpha^3_{\overline{\rm MS}}(\mu) + c_3\alpha^4_{\overline{\rm MS}}(\mu)$$

- If one instead inverts perturbatively, one obtains $\Lambda_{\overline{\rm MS}}$ in the range $409 - 468$ MeV, i.e. $15 - 30\%$ higher!
- The difference decreases roughly proportionally to the expected $\alpha^3_{\rm eff}$.

In FLAG we decided:

- to estimate a systematic error as the difference 54 MeV between $\Lambda_{\overline{\rm MS}}$ estimates at $\mu = 1.5$ GeV
- take as the range for vacuum polarization/light quark current correlation functions $\Lambda^{(3)}_{\overline{\rm MS}} = 342(54)$ MeV,

$\Lambda_{\overline{\mathrm{MS}}}^{(N_f=3)}$ from static energy or force: TUMQCD 19 and Ayala 20

- TUMQCD 19 extends and supersedes Bazavov 14
  - 3 finer lattice spacings, $a = 0.035\,\mathrm{fm}, 0.030\,\mathrm{fm}$ and $0.025\,\mathrm{fm}$
  - topology freezing: occurs, but found to be irrelevant numerically at the current level of precision
  - renormalization scale $2/r$ now reaches $5.4\,\mathrm{GeV}$ where $\alpha_{\overline{\mathrm{MS}}} = 0.198$
  - Scale variation by a factor of 2 up and 1/2 down (previously by $\sqrt{2}$ and $1/\sqrt{2}$)
  - Result: $\Lambda_{\overline{\mathrm{MS}}}^{N_f=3} = 314^{+16}_{-8}\,\mathrm{MeV}$
  - FLAG criteria: ★ for renormalization scale and perturbative behaviour, ○ for continuum extrapolation
  - Perform Cross check with free energy, assuming that thermal effects are negligible so the PT of static energy can be applied; find $\Lambda_{\overline{\mathrm{MS}}}^{N_f=3} = 311(13)\,\mathrm{MeV}$
- Ayala 20: use (subset of) the lattice data of TUMQCD 19
  - use different distance range than TUMQCD 19, i.e. somewhat lower energy.
  - use ultrasoft log resummation
  - Quote result $\Lambda_{\overline{\mathrm{MS}}}^{N_f=3} = 338(12)\,\mathrm{MeV}$
  - FLAG criteria: ★ for perturbative behaviour, ○ for renormalization scale and continuum limit

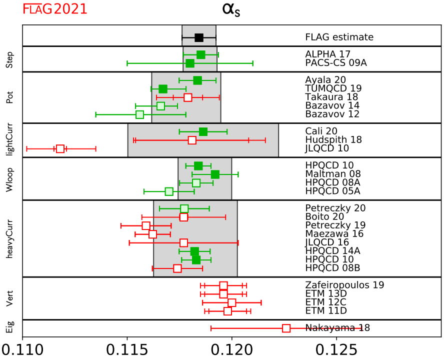
$\Rightarrow$ very different central values, mainly due to different treatment of ultrasoft log's.
We perform a weighted average and use the difference in central values as error.

$$\Lambda_{\overline{\mathrm{MS}}} = 330(24)\,\mathrm{MeV} \qquad \Rightarrow \qquad \alpha_s(m_Z) = 0.11782(165)$$

FLAG average for α_s

We form a range for each of the 5 categories and then combine:

$$\alpha_s(m_Z) = 0.1184(8)$$



N.B. Dominance systematic errors $\Rightarrow$ we used the only statistics dominated error (step-scaling) as proxy for the total error (i.e. errors are NOT added in quadrature)

Discussion and further developments

- The FLAG average has barely moved since FLAG 2019, from $\alpha_s = 0.1182(8)$ to $\alpha_s = 0.1184(8)$
- Systematic errors dominate most results, due to using perturbation theory at rather low scales. (cf. also review by Del Debbio and Ramos '20)
- The current FLAG criteria (taken over from FLAG 2019) may not be stringent enough anymore, as the example of Cali 20 demonstrates; however, this may also be the price to pay for trying to apply the same criteria across the board;
- Except for the continuum limit criterion, the FLAG criteria could also be applied to non-lattice determinations of α_s; indeed, this would be quite interesting!
- Several lattice methods are stuck with systematics that cannot be easily reduced

$\Rightarrow$ It would be important to have more results using the step-scaling solution to avoid perturbation theory at low energies. A single scale factor of $1.5 - 2$ might make a big difference and might be feasible in large/infinite volume.

- A promising new method based on non-perturbative decoupling of $N_\mathrm{f} = 3$ quarks in combination with step-scaling in the $N_\mathrm{f} = 0$ theory. (cf. talks by M.Dalla Brida and Alberto Ramos, and review by Dalla Brida '21)

Decoupling strategy: ALPHA 19-22

ALPHA 17: step-scaling with 2 renormalized finite volume couplings for $N_\mathrm{f} = 2+1$:

- GF coupling for $\mu = 0.2 - 2\,\mathrm{GeV}$ + non-perturbative matching $\bar{g}_\mathsf{GF}(2\,\mathrm{GeV}) \leftrightarrow \bar{g}_\mathsf{SF}(4\,\mathrm{GeV})$
- SF coupling for $\mu = 4 - 120\,\mathrm{GeV}$.

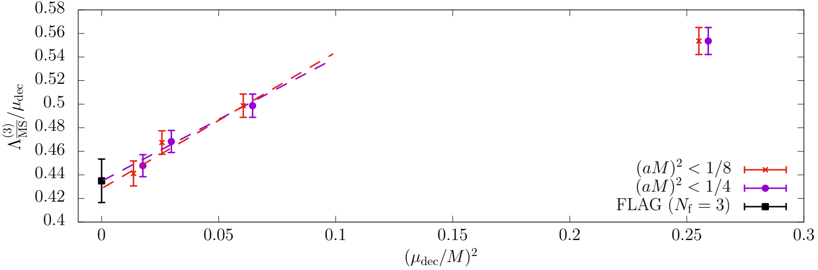
$\Rightarrow$ error dominated by high energy running of the SF coupling!

Decoupling strategy (here for $N_\mathrm{f} = 3$ and $N_\mathrm{f} = 0$):

- use $N_\mathrm{f} = 0$ results for very precise running to high energies
- Match 3-flavour QCD to 0-flavour QCD by simultaneous decoupling of 3 degenerate heavy flavours:
  1. define decoupling scale μ_dec through $\bar{g}^2(\mu_\mathsf{dec}, M=0) = 3.95$; ALPHA 17 $\Rightarrow$ $\mu_\mathsf{dec} = 789(15)\,\mathrm{MeV}$.
  2. $\bar{g}_s(\mu_\mathsf{dec}, M)$ is then traced for $M/\mu_\mathsf{dec} = 2, 4, 6, 8$, using non-perturbative renormalization and O(a) improvement, with M the RGI mass reaching $> 6\,\mathrm{GeV}$.
  3. Establish relation between Λ-parameters for $N_\mathrm{f} = 3$ and $N_\mathrm{f} = 0$ in the form

$$\underbrace{\frac{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}{\mu_\mathsf{dec}}}_{\text{target}} \times \underbrace{P\left(\frac{M}{\mu_\mathsf{dec}}\,\frac{\mu_\mathsf{dec}}{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}\right)}_{\text{pert. in } \alpha_{\overline{\mathrm{MS}}}(m_\star)} = \underbrace{\frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\Lambda^{(0)}_s}}_{\text{known constant}} \times \underbrace{\varphi^{(0)}_s\Big(\underbrace{\bar{g}^{(3)}_s(\mu_\mathsf{dec}, M)}_{\text{massive } \bar{g}}\Big)}_{\substack{\text{step-scaling } N_\mathrm{f}=0 \\ \text{(Dalla Brida \& Ramos '19)} \\ \text{(Nada \& Ramos '20)}}} + O(M^{-2})$$

Decoupling strategy: ALPHA 19



- The large mass introduces a 2-scale problem but this appears still manageable.
- For details and first results cf. talks by A. Ramos and M. Dalla Brida later today.

<u>Nota Bene</u>: Decoupling promotes $N_{\mathrm{f}} = 0$ results from mere test cases to physically relevant inputs! Precision $N_{\mathrm{f}} = 0$ results are needed and FLAG must continue to monitor and assess them!

New step-scaling studies for $N_\mathrm{f} = 0$, Dalla Brida 19 and Nada 20

Dalla Brida 19

- lattice sizes $L/a = 8 - 48$, covering a factor up to 3 in lattice spacings for step-scaling functions.
- use 2 finite volume GF couplings (colour magnetic and electric comp'ts) with SF boundary conditions, flow time t fixed by $c = \sqrt{8t}/L = 0.3$, projection to $Q = 0$ (cf. ALPHA 17)
- 3-loop β-function for GF schemes known from NSPT (only $N_\mathrm{f} = 0$)
- trace couplings over wide range, α_eff varies by a factor 12, reaching 0.08.
- Non-perturbative matching to SF coupling for a whole interval from mid to high energies

$\Rightarrow$ infer non-perturbative β-function for SF coupling from GF calculation

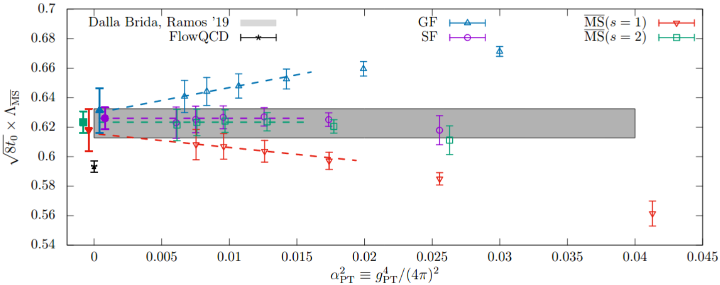
Findings:

- GF couplings are very slow to reach asymptotic perturbative behaviour; an extrapolation to $\alpha^2 = 0$ yields consistent results within errors and is advocated by the authors.
- Alternatively, non-perturbative matching to the SF scheme shows that the SF scheme reaches the perturbative region much earlier. Fully compatible results.
- FLAG criteria: ★ in all categories.

New step-scaling studies for $N_\mathrm{f} = 0$, Dalla Brida 19 and Nada 20

Nada 20:

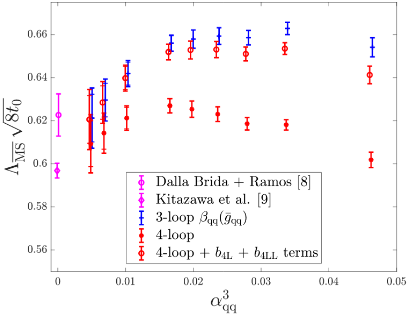
- same lattice configurations as in Dalla Brida 19 are used but at smaller flow time, $c = 0.2$.
- The step-scaling function is computed as a composition of 2 functions thereby avoiding the large lattice artifacts which render a direct computation at $c = 0.2$ unreliable.
- $\Rightarrow$ provides highly non-trivial cross check of Dalla Brida 19 with similar results and errors.
- FLAG criteria: ★in all 3 categories (at low energies this uses Dalla Brida 19)



Husung 20: static force, $N_\mathrm{f} = 0$

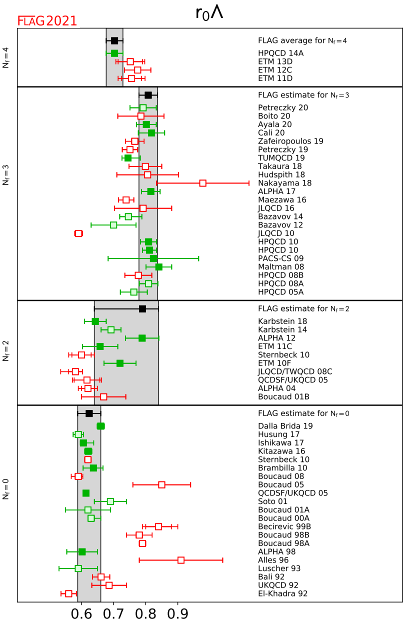
extensive study on very fine lattices in pure gauge theory, follows up on Husung 17:

- Wilson action to avoid unitarity violations (important for ground state extraction, but paid with larger lattice artifacts)
- lattice spacings down to $a = 0.01\,\mathrm{fm}$, lattice size up to $L/a = 192$.
- open b.c.'s, to avoid topology freezing
- step scaling with $s = 3/2$ in infinite volume to compare with β-function
- RG improved subtraction of O(a^2) lattice effects. An ansatz for remnant O(a^4) effects propagated as systematic error on the data.



Husung 20 do not quote a value $\Lambda_{\overline{\mathrm{MS}}}$, conclude:

- perturbative uncertainties small where the error gets large, mainly due to O(a^4) effects.
- cannot distinguish between Kitazawa 16 and Dalla Brida 19/Nada 20
- would require even finer lattices to nail down $\Lambda_{\overline{\mathrm{MS}}}$ to better than 5%.



Ranges of $\Lambda_{\overline{\rm MS}}$ in units of r_0;

for $N_{\rm f} = 3$ we use $r_0 = 0.472$ fm:

- $N_{\rm f} = 3$: $r_0 \Lambda^{(3)}_{\overline{\rm MS}} 0.808(29)$

$\Leftrightarrow$ $\Lambda^{(3)}_{\overline{\rm MS}} = 338(12)$ MeV

- For $N_{\rm f} = 0$, Dalla Brida 19 obtain:

$$[r_0 \Lambda_{\overline{\rm MS}}]^{(0)} = 0.660(11)$$

$\Rightarrow$ we take as range $r_0 \Lambda^{(0)}_{\overline{\rm MS}} = 0.624(36)$ which doubles the error from FLAG 19 $(0.615(18))$.

Conclusions

- Quality criteria unchanged from FLAG 2019, revision left to the future.
- 3 new publications for $N_f = 2+1$ which pass the FLAG criteria. Ayala 20 use the published lattice data by TUMQCD 19 for further perturbative analysis.

$\Rightarrow$ The average over the ranges formed for each of the 5 categories which pass the criteria leads to $\alpha_s = 0.1184(8)$. Note that the error is NOT the error combined in quadrature, but a conservative estimate.

- new evidence that errors in most results are dominated by systematics, due to the use of perturbation theory at low scales.

$\Rightarrow$ Further applications of the step-scaling method would be highly welcome; this seems mandatory to reduce systematic below statistical errors.

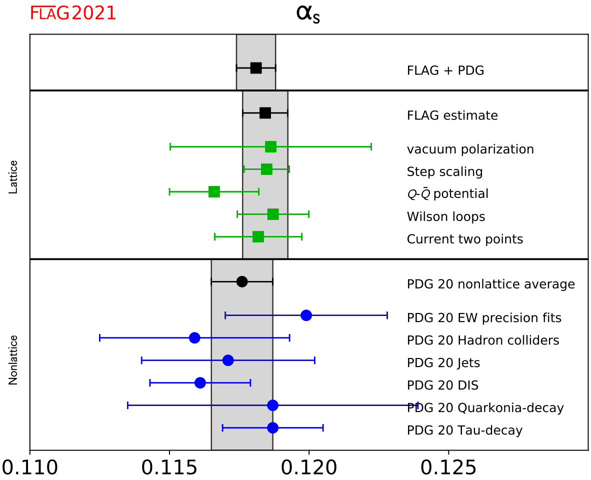
- A new strategy based on simultaneous decoupling of 3 heavy quarks has been proposed by the ALPHA collaboration

$\Rightarrow$ new emphasis on precision $N_f = 0$ results; $N_f = 0$ not just a test bed for methods but acquires physical significance!

- New $N_f = 0$ result by Dalla Brida 19 and Nada 20 significantly larger than FLAG 19 estimate. Further $N_f = 0$ results at the same level of reliability would be very welcome!

For progress since the deadline of FLAG 2021 (April '21) cf. talks by J. Weber, N. Brambilla, A. Ramos, M. Dalla Brida

α_s plot with PDG results



Step scaling function

- Widely different scales cannot be resolved simultaneously on a *single* lattice

$\Rightarrow$ break calculation up in steps [Lüscher, Weisz, Wolff '91; Jansen et al. '95]:

1. define $\bar{g}^2(L)$ that runs with the space-time volume, i.e. $\mu = 1/L$
2. construct the step-scaling function

$$\sigma(u) = \bar{g}^2(2L)\Big|_{u=\bar{g}^2(L)}$$

   for a range of values $u \in [u_{\mathrm{min}}, u_{\mathrm{max}}]$
3. iteratively step up/down in scale by factors of 2:

$$\bar{g}^2(L_{\mathrm{max}}) = u_{\mathrm{max}} \equiv u_0\,, \quad u_k = \sigma(u_{k+1}) = \bar{g}^2(2^{-k}L_{\mathrm{max}}), \quad k = 0, 1, \ldots$$

4. match to hadronic input at a hadronic scale L_{max}, i.e. $F_K L_{\max} = \mathrm{O}(1)$
5. once arrived in the perturbative regime $L_{\mathrm{pert}} = 2^{-n}L_{\mathrm{max}}$ one now knows $u_n = \bar{g}^2(L_{\mathrm{pert}})$; determine $L_{\mathrm{pert}}\Lambda$ and combine to obtain Λ/F_K.

Lattice approximants $\Sigma(u, a/L)$ for $\sigma(u)$

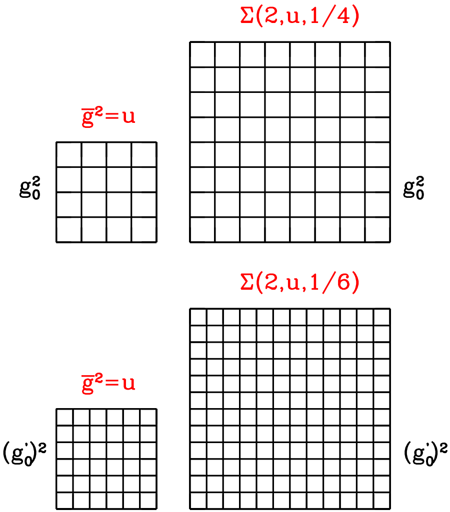
- choose g_0 and $L/a = 4$, measure $\bar{g}^2(L) = u$ (this sets the value of u)
- double the lattice and measure

$$\Sigma(u, 1/4) = \bar{g}^2(2L)$$

- now choose $L/a = 6$ and tune g_0' such that $\bar{g}^2(L) = u$ is satisfied
- double the lattice and measure

$$\Sigma(u, 1/6) = \bar{g}^2(2L)$$

- $\sigma(u) = \lim_{a/L \to 0} \Sigma(u, a/L)$.
- change u and repeat...



Continuum limit $\sigma(u) = \lim_{a/L\to 0} \Sigma(u, a/L)$