Les Diablerets Winter School in Mathematical Physics– Sheet 1

Vit Sriprachyakul

1 AdS$_3$ WZW model

Throughout the course, we describe string theory on the AdS$_3$ background by an $\mathfrak{sl}(2,\mathbb{R})^{(1)}_k$ WZW model. This algebra is generated by three bosonic fields, denoted J^+, J^-, and J^3, and three fermionic fields, denoted ψ^+, ψ^-, and ψ^3. Their modes satisfy the following (anti-) commutation relations

$$\begin{aligned} [J^+_m, J^-_n] &= -2J^3_{m+n} + km\delta_{m+n,0} & [J^3_m, J^\pm_n] &= \pm J^\pm_{m+n} & [J^3_m, J^3_n] &= -\frac{k}{2}m\delta_{m+n,0} \\ [J^\pm_m, \psi^3_r] &= \mp \psi^\pm_{m+r} & [J^3_m, \psi^\pm_r] &= \pm \psi^\pm_{m+r} & [J^\pm_m, \psi^\mp_r] &= \mp 2\psi^3_{m+r} \\ \{\psi^+_r, \psi^-_s\} &= k\delta_{r+s,0} & \{\psi^3_r, \psi^3_s\} &= -\frac{k}{2}\delta_{r+s,0}\ . \end{aligned} \tag{1}$$

Define

$$\begin{aligned} \mathcal{J}^\pm_m &= J^\pm_m \pm \frac{2}{k}\sum_r :\psi^3_{m-r}\psi^\pm_r: \\ \mathcal{J}^3_m &= J^3_m + \frac{1}{k}\sum_r :\psi^-_{m-r}\psi^+_r:\ . \end{aligned} \tag{2}$$

a) By considering the commutation relations $[\mathcal{J}^a_m, \psi^b_r]$, show that the $\mathcal{J}^a$ decouples from all the fermions, namely, the commutation relations between them are all zero. The $\mathcal{J}^a$'s are what we call the *decoupled* currents in the lectures. For simplicity, work in the NS sector.

   Hint: For the NS sector, the fermion modes in (2) are ordered such that $:\psi^a_r\psi^b_s: = \psi^a_r\psi^b_s$ if r is negative and $:\psi^a_r\psi^b_s: = -\psi^b_s\psi^a_r$ if r is positive.

b) The decoupled currents $\mathcal{J}^a$ also satisfy an affine $\mathfrak{sl}(2,\mathbb{R})$ algebra, but with a shifted level, namely $k_\mathcal{J} = k_J + 2 = k + 2$. Confirm this partially by calculating the commutator $[\mathcal{J}^3_m, \mathcal{J}^3_n]$ using eq. (2), i.e. expressing $\mathcal{J}^3_m$ in terms of J^3_m and $\psi^\pm_r$. Derive the same result by considering instead the commutator $[J^3_m, J^3_n]$, where you write J^3_m in terms of $\mathcal{J}^3_m$ and $\psi^\pm_r$.

c) The Virasoro algebra of this system is given by (assuming for simplicity that all fermions are in the NS sector)

$$L_m = \frac{1}{2k}\sum_n \left(:\mathcal{J}^+_{m-n}\mathcal{J}^-_n + :\mathcal{J}^-_{m-n}\mathcal{J}^+_n: -2:\mathcal{J}^3_{m-n}\mathcal{J}^3_n:\right) \tag{3}$$

$$+\frac{1}{2k}\sum_r (r+\tfrac{1}{2})\left(:\psi^+_{m-r}\psi^-_r: + :\psi^-_{m-r}\psi^+_r: -2:\psi^3_{m-r}\psi^3_r:\right)\ . \tag{4}$$

   Show that the central charge is

$$c = 3\left(\frac{k+2}{k} + \frac{1}{2}\right).$$

Hint: It is useful to observe that L_m is the sum of two Virasoro algebras: the one associated to the decoupled currents $\mathcal{J}$, and the one associated to the free fermions $\psi^\pm$ and ψ^3.

d) **Bonus**: The $\mathcal{J}$ part of the stress tensor is obtained by the *Sugawara* construction. This is a procedure that gives rise to a stress tensor with respect to which the currents all have weight 1. Show that this is indeed the case, namely, show that

$$[L_m, \mathcal{J}^a_n] = -n\mathcal{J}^a_{m+n} \ .$$

2 Spectrally flowed representaion of $\mathfrak{su}(2)_k$

The affine algebra $\mathfrak{su}(2)_k$ is generated by $K^\pm_n$ and K^3_n, subject to the commutation relations

$$[K^+_m, K^-_n] = 2K^3_{m+n} + km\delta_{m+n,0} \ , \qquad [K^3_m, K^\pm_n] = \pm K^\pm_{m+n} \ , \qquad [K^3_m, K^3_n] = \frac{k}{2}m\delta_{m+n,0} \ . \tag{5}$$

A highest weight representation of $\mathfrak{su}(2)_k$ is uniquely characterised by specifying the representation of the zero modes $\mathfrak{su}(2)$ on the (Virasoro) highest weight space, i.e. by specifying the spin j of these highest weight states. We shall denote the highest weight states in the spin j representation of $\mathfrak{su}(2)$ by $|j, m\rangle$ where $m = -j, -j+1, \ldots, j-1, j$ is the magnetic quantum number, i.e. the eigenvalue of K^3_0, while j is determined via the Casimir

$$\frac{1}{2}\Big(K^+_0 K^-_0 + K^-_0 K^+_0 + 2K^3_0 K^3_0\Big)|j, m\rangle = j(j+1)\,|j, m\rangle \ . \tag{6}$$

The fact that they are (Virasoro) highest weight states simply means that all positive modes annihilate all of these states,

$$K^a_n|j, m\rangle = 0 \ , \qquad n > 0 \ . \tag{7}$$

The full space of states is then generated by the action of the negative modes K^a_n with $n < 0$ on the $|j, m\rangle$. We denote the resulting Hilbert space of states (after quotienting out by the null-states) by $\mathcal{H}_j$. For a fixed k, the only allowed (unitary) representations of $\mathfrak{su}(2)_k$ appear for $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$.

The affine algebra $\mathfrak{su}(2)_k$ has an outer automorphism given by

$$\sigma(K^3_n) = K^3_n + \frac{k}{2}\delta_{n,0} \ , \qquad \sigma(K^\pm_n) = K^\pm_{n\pm 1} \ . \tag{8}$$

This induces an action of the automorphism σ on the representation $\mathcal{H}_j$. More specifically, we define $\sigma(\mathcal{H}_j)$ to be spanned by the states $[|\psi\rangle]^\sigma$, where ψ runs over a basis of $\mathcal{H}_j$, and the action of $\mathfrak{su}(2)_k$ on $[|\psi\rangle]^\sigma$ is defined by

$$K^a_n[|\psi\rangle]^\sigma = [\sigma(K^a_n)\,|\psi\rangle]^\sigma \ , \qquad \forall a, n \ . \tag{9}$$

a) Show that $\sigma([K^a_m, K^b_n]) = [\sigma(K^a_m), \sigma(K^b_n)]$, i.e. that σ is indeed an automorphism.

b) Show that $K^+_0[|j, m\rangle]^\sigma = 0$. Show also that $K^3_0[|j, m\rangle]^\sigma = \left(\frac{k}{2} + m\right)[|j, m\rangle]^\sigma$.

c) Show that the states $[|j, m\rangle]^\sigma$ are all annihilated by the positive modes of $K^{+,3}$.

d) Show that the states in $c)$ are also annihilated by $K^-_n, n \geq 2$. What is the neccesary and sufficient condition on m for K^-_1 to annihilate the state $[|j, m\rangle]^\sigma$. We denote the relevant value of m by m_0.

e) The state $[|j,m_0\rangle]^\sigma$ defines a new ground state (why?), namely

$$[|j,m_0\rangle]^\sigma = |j_{\rm new}, m_{\rm new}\rangle \ ,$$

with $m_{\rm new} = j_{\rm new}$ and find this common value.

f) From the results derived thus far, conclude that

$$\sigma\left(\mathcal{H}_j\right) = \mathcal{H}_{\frac{k}{2}-j} \ .$$

g) Show that applying a spectral twice maps the Hilbert space to itself, i.e., $\sigma^2(\mathcal{H}_j) = \mathcal{H}_j$.

For the case of $\mathfrak{su}(2)_k$, the representation $\sigma(\mathcal{H}_j)$ also contains a highest weight state, and hence is a highest weight representation. In the next exercise, we will see that this is generically not the case for $\mathfrak{sl}(2,\mathbb{R})$.

3 Spectrally flowed representaion of $\mathfrak{sl}(2,\mathbb{R})_k$

Now we repeat the above analysis for a highest weight representation of $\mathfrak{sl}(2,\mathbb{R})_k$, where the generators of $\mathfrak{sl}(2,\mathbb{R})_k$ satisfy

$$[J^+_m, J^-_n] = -2J^3_{m+n} + km\delta_{m+n,0} \ , \qquad [J^3_m, J^\pm_n] = \pm J^\pm_{m+n} \ , \qquad [J^3_m, J^3_n] = -\frac{k}{2}m\delta_{m+n,0} \ . \tag{10}$$

For $\mathfrak{sl}(2,\mathbb{R})_k$ the ground states are also of the form $|j,m\rangle$ but now $m \in \mathbb{Z}+\alpha$, while j parametrises the Casimir via

$$\frac{1}{2}\Big(J^+_0 J^-_0 + J^-_0 J^+_0 - 2J^3_0 J^3_0\Big)|j,m\rangle = -j(j-1)\,|j,m\rangle \ . \tag{11}$$

The states are again assumed to be highest weight, i.e. $J^a_n|j,m\rangle = 0$ for $n > 0$, and the resulting irreducible representation of the affine algebra $\mathfrak{sl}(2,\mathbb{R})_k$ is again denoted by $\mathcal{H}_j$. The spectral flow automorphism is now

$$\sigma(J^3_n) = J^3_n + \frac{k}{2}\delta_{n,0} \ , \qquad \sigma(J^\pm_n) = J^\pm_{n\mp 1} \ , \tag{12}$$

and the action on the spectrally flowed representation is defined as before, i.e.

$$J^a_n[|\psi\rangle]^\sigma = [\sigma(J^a_n)\,|\psi\rangle]^\sigma \ , \qquad \forall a, n \ . \tag{13}$$

a) Show that $\sigma([J^a_m, J^b_n]) = [\sigma(J^a_m), \sigma(J^b_n)]$, i.e. that σ is indeed an automorphism of $\mathfrak{sl}(2,\mathbb{R})_k$.

b) Show that $J^-_0[|j,m\rangle]^\sigma = 0$. Show also that $J^3_0[|j,m\rangle]^\sigma = \left(\frac{k}{2}+m\right)[|j,m\rangle]^\sigma$.

c) Show that the states $[|j,m\rangle]^\sigma$ are all annihilated by the positive modes of $J^{-,3}$.

d) Show that the states in *c*) are also annihilated by $J^+_n, n \geq 2$. Which states $[|j,m\rangle]^\sigma$ are also annihilated by J^+_1?

As you see from *d*), for general α and j, none of the states $[|j,m\rangle]^\sigma$ is annihilated by J^+_1, and hence the resulting representation is not highest weight.

4 More on spectral flow: Virasoro algebra

Spectral flow also has an effect on the Virasoro algebra whose modes satisfy the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \ . \tag{14}$$

Furthermore, since the stress tensor is constructed from the Sugawara construction, their modes also satisfy,

$$[L_m, J^a_n] = -nJ^a_{m+n} \ . \tag{15}$$

In this question, we will consider a general unit of flow, namely, σ^w with $w \in \mathbb{N}$.

$\underline{\mathfrak{su}(2)_k}$

a) Show that $\sigma^w(K^3_n) = K^3_n + \frac{kw}{2}\delta_{n,0}$ and $\sigma^w(K^\pm_n) = K^\pm_{n\pm w}$.

b) Assume that σ^w acts as an automorphism for all operators including the Virasoro generators, i.e., $\sigma^w([A,B]) = [\sigma^w(A), \sigma^w(B)]$. By considering $\sigma^w([L_m, K^3_n])$, show that this requires that $\sigma^w(L_m) = L_m + wK^3_m + C$, where C is some constant (that could depend on m, w and k).

   Hint: Use the automorphism property and compare terms. One could start from the general ansatz that (schematically) $\sigma^w(L_m) = \sum_n L_n + \sum_n K^a_n + C$, namely, $\sigma^w(L_m)$ is a linear combination of all the modes L_n and K^a_n. Then use the commutation relations to eliminate terms.

c) From b), by considering $[L_m, L_n]$, show that

$$\sigma^w(L_m) = L_m + wK^3_m + \frac{kw^2}{4}\delta_{m,0} \ . \tag{16}$$

d) **Bonus**: There is actually yet another way of deriving this result. Since spectral flow acts on products as

$$\sigma^w(AB) = \sigma^w(A)\sigma^w(B) \ ,$$

   we can directly apply spectral flow to the (bosonic part) of the Sugawara expression,

$$L_m = \frac{1}{2(k+2)}\sum_n \Big(:K^+_{m-n}K^-_n: + :K^-_{m-n}K^+_n: + 2:K^3_{m-n}K^3_n: \Big) \ , \tag{17}$$

   where normal ordering means that $:K^a_m K^b_n: := K^a_m K^b_n$ if $m \le -1$, and $:K^a_m K^b_n: := K^b_n K^a_m$ if $m \ge 0$. Using this approach and carefully keeping track of the normal ordering terms, derive (16).

$\underline{\mathfrak{sl}(2,\mathbb{R})_k}$

a) Show that $\sigma^w(J^3_n) = J^3_n + \frac{kw}{2}\delta_{n,0}$ and $\sigma^w(J^\pm_n) = J^\pm_{n\mp w}$.

b) By considering $\sigma^w([L_m, J^3_n])$, show that $\sigma^w(L_m) = L_m - wJ^3_m + C$, where C is some constant (that could depend on m, w and k).

c) From b), consider $[L_m, L_n]$ to show that

$$\sigma^w(L_m) = L_m - wJ^3_m - \frac{kw^2}{4}\delta_{m,0} \ . \tag{18}$$

d) **Bonus**: As in the case of $\mathfrak{su}(2)_k$, there is another way of deriving eq. (18) by using the Sugawara expression

$$L_m = \frac{1}{2(k-2)} \sum_n \Big(: J^+_{m-n} J^-_n : + : J^-_{m-n} J^+_n : -2 : J^3_{m-n} J^3_n : \Big) \ , \tag{19}$$

and applying spectral flow to the individual terms.