EXERCISE SESSION 11.01.2021

(1) Let $\mathfrak{t}_n$ be the Kohno–Drinfeld Lie algebra with generators $t_{ij} = t_{ji}$, $1 \leq i < j \leq n$ and relations

$$[t_{ij}, t_{kl}] = 0 \quad \text{for all distinct } i, j, k, l,$$
$$[t_{ij}, t_{ik} + t_{jk}] = 0 \quad \text{for all distinct } i, j, k.$$

(a) Show that $r_{ij}(z) = \frac{t_{ij}}{z}$ are solutions of the classical Yang–Baxter equation

$$\mathrm{cycl}_{ijk}[r_{ij}(z_i - z_j), r_{ik}(z_i - z_k)] = 0 \quad \text{for all distinct } i, j, k$$

(cycl is the sum over cyclic permutations) and $[r_{ij}(z), r_{kl}(w)] = 0$ for all distinct i, j, k, l.

(b) Deduce that for all $z \in \mathbb{C}^n$ with $z_i \neq z_j$ $(i \neq j)$ the Gaudin Hamiltonians

$$H_i = \sum_{j: j \neq i} r_{ij}(z_i - z_j)$$

form a commutative Lie subalgebra of $\mathfrak{t}_n$.

(c) Show that the following are examples of representations of $T_n \to \mathfrak{gl}(V)$.

(i) $V = V_1 \otimes \cdots \otimes V_n$ a tensor product of representations of a Lie algebra $\mathfrak{g}$ with an invariant[1] non-degenerate symmetric bilinear form

$$t_{ij} \mapsto \sum_{k=1}^{\dim \mathfrak{g}} e_k^{(i)} e_k^{(j)},$$

where $x^{(i)} = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ is the action on the ith factor, for any orthonormal basis (e_k) of $\mathfrak{g}$.

(ii) V a representation of $\mathfrak{gl}_n$,

$$t_{ij} \mapsto E_{ij}E_{ji} + E_{ji}E_{ij}.$$

(iii) $V = \mathbb{C}S_n$ the group algebra of the symmetric group,

$$t_{ij} \to (ij),$$

the transposition of i and j.

(2) Let $R(z) \in \mathrm{End}(V \otimes V)$ be a meromorphic invertible solution of the Yang–Baxter equation with transfer matrix $\tau_n(z) = \mathrm{tr}_0 R^{0n}(z - z_n) \cdots R^{01}(z - z_1) \in \mathrm{End}(V^{\otimes n})$. Assume that $R(0) = P$, the permutation $v \otimes w \mapsto w \otimes v$ (e.g. the normalized McGuire–Yang R-matrix $R(z) = \frac{z\mathrm{Id} + \hbar P}{z + \hbar}$). Show that the commuting q-Gaudin Hamiltonians $H_i = \tau_n(z_i)$ can be written as

$$H_i = R^{(i,i-1)}(z_i - z_{i-1}) \cdots R^{(i,1)}(z_i - z_1) \times R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+1)}(z_i - z_{i+1}),$$

[1] i.e., such that $\langle [a,b], c\rangle = \langle a, [b,c]\rangle$

(3) Let $V = \mathbb{C}^n$ be the vector representation of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ $(n \geq 2)$. Let $V(z)$ be the evaluation representation of the loop Lie algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ at $z \in \mathbb{C} \smallsetminus 0$: $(a \otimes f(t))v = f(z)av$, $a \in \mathfrak{g}$, $f(t) \in \mathbb{C}[t, t^{-1}]$, $v \in V$. Show that $V(z_1) \otimes \cdots \otimes V(z_n)$ is an irreducible representation of $L\mathfrak{g}$ if and only if $z_i \neq z_j$ for all $i \neq j$.