SOLUTIONS TO EXERCISES TO GIOVANNI'S COURSE

(1) Let $\mathfrak{t}_n$ be the Kohno–Drinfeld Lie algebra with generators $t_{ij} = t_{ji}$, $1 \leq i < j \leq n$ and relations

$$\begin{aligned} [t_{ij}, t_{kl}] &= 0 \quad \text{for all distinct } i, j, k, l, \\ [t_{ij}, t_{ik} + t_{jk}] &= 0 \quad \text{for all distinct } i, j, k. \end{aligned} \tag{1}$$

(a) Show that $r_{ij}(z) = \frac{t_{ij}}{z}$ are solutions of the classical Yang–Baxter equation

$$\mathrm{cycl}_{ijk}[r_{ij}(z_i - z_j), r_{ik}(z_i - z_k)] = 0 \quad \text{for all distinct } i, j, k \tag{2}$$

(cycl is the sum over cyclic permutations) and $[r_{ij}(z), r_{kl}(w)] = 0$ for all distinct i, j, k, l.

(b) Deduce that for all $z \in \mathbb{C}^n$ with $z_i \neq z_j$ $(i \neq j)$ the Gaudin Hamiltonians

$$H_i = \sum_{j:j\neq i} r_{ij}(z_i - z_j)$$

form a commutative Lie subalgebra of $\mathfrak{t}_n$.

(c) Show that the following are examples of representations of $\mathfrak{t}_n \to \mathfrak{gl}(V)$.

(i) $V = V_1 \otimes \cdots \otimes V_n$ a tensor product of representations of a Lie algebra $\mathfrak{g}$ with an invariant[1] non-degenerate symmetric bilinear form

$$t_{ij} \mapsto \sum_{k=1}^{\dim \mathfrak{g}} e_k^{(i)} e_k^{(j)},$$

where $x^{(i)} = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ is the action on the ith factor, for any orthonormal basis (e_k) of $\mathfrak{g}$.

(ii) V a representation of $\mathfrak{g}l_n$,

$$t_{ij} \mapsto E_{ij}E_{ji} + E_{ji}E_{ij}.$$

(iii) $V = \mathbb{C}S_n$ the group algebra of the symmetric group,

$$t_{ij} \to (ij),$$

the transposition of i and j.

(2) Let $R(z) \in \mathrm{End}(V \otimes V)$ be a meromorphic invertible solution of the Yang–Baxter equation with transfer matrix $\tau_n(z) = \mathrm{tr}_0 R^{0n}(z - z_n) \cdots R^{01}(z - z_1) \in \mathrm{End}(V^{\otimes n})$. Assume that $R(0) = P$, the permutation $v \otimes w \mapsto w \otimes v$ (e.g. the normalized McGuire–Yang R-matrix $R(z) = \frac{z\mathrm{Id} + \hbar P}{z + \hbar}$). Show that the commuting q-Gaudin Hamiltonians $H_i = \tau_n(z_i)$ can be written as

$$\begin{aligned} H_i = {} & R^{(i,i-1)}(z_i - z_{i-1}) \cdots R^{(i,1)}(z_i - z_1) \\ & \times R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+1)}(z_i - z_{i+1}), \end{aligned}$$

[1] i.e., such that $\langle [a,b], c\rangle = \langle a, [b,c]\rangle$

(3) Let $V = \mathbb{C}^n$ be the vector representation of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ $(n \geq 2)$. Let $V(z)$ be the evaluation representation of the loop Lie algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ at $z \in \mathbb{C} \setminus 0$: $(a \otimes f(t))v = f(z)av$, $a \in \mathfrak{g}$, $f(t) \in \mathbb{C}[t, t^{-1}]$, $v \in V$. Show that $V(z_1) \otimes \cdots \otimes V(z_m)$ is an irreducible representation of $L\mathfrak{g}$ if and only if $z_i \neq z_j$ for all $i \neq j$.

1. Solutions

1.1. **Solution to 1.a.** Set $z_{ij} \coloneqq z_i - z_j$ for $i, j \in \{ 1, \dots, n \}$. Taking the cyclic sum yields:

$$\begin{aligned}\big[r_{ij}(z_{ij}), r_{ik}(z_{ik})\big] &+ \big[r_{ki}(z_{ki}), r_{kj}(z_{kj})\big] + \big[r_{jk}(z_{jk}), r_{ji}(z_{ji})\big] \\ &= \frac{[t_{ij}, t_{ik}]}{z_{ij}z_{ik}} + \frac{[t_{ki}, t_{kj}]}{z_{ki}z_{kj}} + \frac{[t_{jk}, t_{ji}]}{z_{jk}z_{ji}} \\ &= \frac{z_{kj}[t_{ij}, t_{ik}] + z_{ji}[t_{ki}, t_{kj}] + z_{ik}[t_{jk}, t_{ji}]}{z_{ij}z_{ik}z_{kj}},\end{aligned}$$

using $z_{ij} + z_{ji} = 0$. Then using $t_{ij} - t_{ji} = 0$ (etc), and the bilinearity of the Lie bracket, the numerator expands to

$$\begin{aligned}z_i\big([t_{jk}, t_{ji}] - [t_{ki}, t_{kj}]\big) &+ z_j\big([t_{ki}, t_{kj}] - [t_{ij}, t_{ik}]\big) + z_i\big([t_{ij}, t_{ik}] - [t_{jk}, t_{ji}]\big) \\ &= z_i[t_{jk}, t_{ji} + t_{ki}] + z_i[t_{jk}, t_{ji} + t_{ki}] + z_i[t_{jk}, t_{ji} + t_{ki}],\end{aligned}$$

and the three summands vanish by (1) (which also directly implies the vanishing over distinct indices).

1.2. **Solution to 1.b.** By definition $r_{ij} \colon \mathbb{C}^* \to \mathfrak{t}_n$ (with a simple pole at $z = 0$), for $i, j \in \{ 1, \dots, n \}$. Then $z_{ij} \mapsto r_{ij}(z_{ij})$ is the composition of $\boldsymbol{z} = (z_1, \dots, z_n) \mapsto z_{ij} = z_i - z_j$ with "r", for $\boldsymbol{z} \in \mathbb{C}^n$ out of the diagonals. The Lie bracket of such $\mathfrak{t}_n$-valued functions is induced from that of the target, hence all the evaluations commute inside $\mathfrak{t}_n$—for any choice of configuration of points on the complex affine line.

Now for $i, j \in \{ 1, \dots, n \}$ with $i \neq k$ the Lie bracket expands as

$$[H_i, H_j] = \sum_{k \colon k \neq i} \sum_{l \colon l \neq j} \big[r_{ik}(z_{ik}), r_{jl}(z_{jl})\big].$$

By 1.a nonvanishing term only arise if $\{ i, j, k, l \} \subseteq \{ 1, \dots, n \}$ has three elements (it cannot have less): this leaves a sum where either $k = l$, or $k = j$, or $l = i$, i.e.

$$\begin{aligned}[H_i, H_j] &= \sum_m \big[r_{im}(z_{im}), r_{jm}(z_{jm})\big] + \sum_l \big[r_{ij}(z_{ij}), r_{jl}(z_{jl})\big] + \sum_k \big[r_{ik}(z_{ik}), r_{ji}(z_{ji})\big] \\ &= \sum_k \Big(\big[r_{ik}(z_{ik}), r_{jk}(z_{jk})\big] + \big[r_{ij}(z_{ij}), r_{jk}(z_{jk})\big] + \big[r_{ik}(z_{ik}), r_{ji}(z_{ji})\big]\Big),\end{aligned}$$

and all addends vanish by (2).[2]

[2] To get the exact same formula use $r_{ij}(z_{ij}) + r_{ji}(z_{ji}) = 0$, and the skew-symmetry of the Lie bracket.

1.3. **Solution to 1.c.** In all cases we must show $t_{ij} = t_{ji}$ and (1).

i) Actions on different slots commute, hence $e_k^{(i)} e_k^{(j)} = e_k^{(j)} e_k^{(i)}$ for $i \neq j$: this yields $t_{ij} = t_{ji}$. For the same reason $[t_{ij}, t_{kl}] = 0$ if i, j, k, l are distinct.

Consider then the element $t = \sum_{k=1}^{\dim \mathfrak{g}} e_k \otimes e_k \in \mathfrak{g} \otimes \mathfrak{g}$.

Lemma 1.1. *The element "t" is invariant for the $\mathfrak{g}$-action on $\mathfrak{g} \otimes \mathfrak{g}$ induced from the adjoint action* $\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Proof. Intrinsically $t \in \mathfrak{g} \otimes \mathfrak{g}$ corresponds to $\mathrm{Id}_{\mathfrak{g}} \in \mathfrak{g}^\vee \otimes \mathfrak{g}$ under the duality $\langle \cdot \mid \cdot \rangle^\flat : \mathfrak{g} \to \mathfrak{g}^\vee$ induced by the invariant nondegenerate symmetric bilinear form $\langle \cdot \mid \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. But the invariance of $\langle \cdot, \cdot \rangle$ implies $\langle \cdot \mid \cdot \rangle^\flat$ is a morphism of $\mathfrak{g}$-modules, hence t is $\mathfrak{g}$-invariant since clearly the identity is. ☐

Remark 1.2. For a more concrete approach fix $i \in \{ 1, \dots, \dim \mathfrak{g} \}$ and compute:

$$\mathrm{ad}_{e_i}(t) = \left[e_i^{(1)} + e_i^{(2)}, t\right] = \sum_{k=1}^{\dim \mathfrak{g}} [e_i, e_k] \otimes e_k + e_k \otimes [e_i, e_k] \, ,$$

then expand $[e_i, e_k] = \sum_{l=1}^{\dim \mathfrak{g}} c_{ik}^l e_l$: finally use the invariance of the nondegenerate symmetric bilinear form to show a relation among the coefficients "c_{ik}^l (which implies the sum vanishes).

Note by definition $t_{ij} = t^{(ij)} \in \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$. Then fix distinct indices i, j, k and note that

$$\left[t^{(ij)}, t^{(ik)}\right] = \sum_{l=1}^{\dim \mathfrak{g}} \left[t^{(ij)}, e_l^{(i)} e_l^{(k)}\right] = \sum_{l=1}^{\dim \mathfrak{g}} \left[t^{(ij)}, e_l^{(i)}\right] e_l^{(k)} \, ,$$

using again the commutativity of actions on disjoint slots, and analogously

$$\left[t^{(ij)}, t^{(jk)}\right] = \sum_{l=1}^{\dim \mathfrak{g}} \left[t^{(ij)}, e_l^{(j)}\right] e_l^{(k)} \, .$$

Hence

$$\left[t^{(ij)}, t^{(ik)} + t^{(jk)}\right] = \sum_{l=1}^{\dim \mathfrak{g}} \left[t^{(ij)}, e_l^{(i)} + e_l^{(j)}\right] e_l^{(k)} = \sum_{l=1}^{\dim \mathfrak{g}} \left[t, e_l\right]^{(ij)} e_l^{(k)} = 0 \, ,$$

by Lem. 1.1.

ii) Clearly the anticommutator is symmetric in $i, j \in \{ 1, \dots, n \}$.

The commutation relations of $\mathfrak{gl}_n$ are

$$\left[E_{ij}, E_{kl}\right] = \delta_{jk} E_{il} - \delta_{il} E_{kj} \, ,$$

which immediately yields $[t_{ij}, t_{kl}] = 0$ for i, j, k, l distinct. Then expanding the commutator leads to

$$\begin{aligned}
[t_{ij}, t_{kl}] &= \big[E_{ij}E_{ji} + E_{ji}E_{ij}, t_{kl}\big] \\
&= E_{ij}\big[E_{ji}, t_{kl}\big] + E_{ji}\big[E_{ij}, t_{kl}\big] + \big[E_{ji}, t_{kl}\big]E_{ij} + \big[E_{ij}, t_{kl}\big]E_{ij} \\
&= E_{ij}\Big(E_{kl}\big[E_{ji}, E_{lk}\big] + E_{lk}\big[E_{ji}, E_{kl}\big] + \big[E_{ji}, E_{lk}\big]E_{kl} + \big[E_{ji}, E_{kl}\big]E_{lk}\Big) \\
&+ E_{ji}\Big(E_{kl}\big[E_{ij}, E_{lk}\big] + E_{lk}\big[E_{ij}, E_{kl}\big] + \big[E_{ij}, E_{lk}\big]E_{kl} + \big[E_{ij}, E_{kl}\big]E_{lk}\Big) \\
&+ \Big(E_{kl}\big[E_{ji}, E_{lk}\big] + E_{lk}\big[E_{ji}, E_{kl}\big] + \big[E_{ji}, E_{lk}\big]E_{kl} + \big[E_{ji}, E_{kl}\big]E_{lk}\Big)E_{ij} \\
&+ \Big(E_{kl}\big[E_{ij}, E_{lk}\big] + E_{lk}\big[E_{ij}, E_{kl}\big] + \big[E_{ij}, E_{lk}\big]E_{kl} + \big[E_{ij}, E_{kl}\big]E_{lk}\Big)E_{ji} \\
&= E_{ij}\Big(E_{kl}\big(\delta_{il}E_{jk} - \delta_{jk}E_{li}\big) + E_{lk}\big(\delta_{ik}E_{jl} - \delta_{jl}E_{ki}\big) + \big(\delta_{il}E_{jk} - \delta_{jk}E_{li}\big)E_{kl} + \big(\delta_{ik}E_{jl} - \delta_{jl}E_{ki}\big)E_{lk}\Big) \\
&+ E_{ji}\Big(E_{kl}\big(\delta_{jl}E_{ik} - \delta_{ik}E_{lj}\big) + E_{lk}\big(\delta_{jk}E_{il} - \delta_{il}E_{kj}\big) + \big(\delta_{jl}E_{ik} - \delta_{ik}E_{lj}\big)E_{kl} + \big(\delta_{jk}E_{il} - \delta_{il}E_{kj}\big)E_{lk}\Big) \\
&+ \Big(E_{kl}\big(\delta_{il}E_{jk} - \delta_{jk}E_{li}\big) + E_{lk}\big(\delta_{ik}E_{jl} - \delta_{jl}E_{ki}\big) + \big(\delta_{il}E_{jk} - \delta_{jk}E_{li}\big)E_{kl} + \big(\delta_{ik}E_{jl} - \delta_{jl}E_{ki}\big)E_{lk}\Big)E_{ij} \\
&+ \Big(E_{kl}\big(\delta_{jl}E_{ik} - \delta_{ik}E_{lj}\big) + E_{lk}\big(\delta_{jk}E_{il} - \delta_{il}E_{kj}\big) + \big(\delta_{jl}E_{ik} - \delta_{ik}E_{lj}\big)E_{kl} + \big(\delta_{jk}E_{il} - \delta_{il}E_{kj}\big)E_{lk}\Big)E_{ji} \,.
\end{aligned}$$

In particular if $i = k$ (with i, j, l distinct):

$$\begin{aligned}
[t_{ij}, t_{il}] &= \big[E_{ij}E_{ji} + E_{ji}E_{ij}, E_{il}E_{li} + E_{li}E_{il}\big] \\
&= E_{ij}\big(E_{li}E_{jl} + E_{jl}E_{li}\big) - E_{ji}\big(E_{il}E_{lj} + E_{lj}E_{il}\big) \\
&+ \big(E_{li}E_{jl} + E_{jl}E_{li}\big)E_{ij} - \big(E_{il}E_{lj} + E_{lj}E_{il}\big)E_{ji} \,;
\end{aligned}$$

while if $k = j$:[3]

$$\begin{aligned}
[t_{ij}, t_{jl}] &= \big[E_{ij}E_{ji} + E_{ji}E_{ij}, E_{jl}E_{lj} + E_{lj}E_{jl}\big] \\
&= -E_{ij}\big(E_{jl}E_{li} + E_{li}E_{jl}\big) + E_{ji}\big(E_{il} + E_{lj} + E_{il}E_{lj}\big) \\
&- \big(E_{jl}E_{li} + E_{li}E_{jl}\big)E_{ij} + \big(E_{lj}E_{il} + E_{il}E_{lj}\big)E_{ji} \,,
\end{aligned}$$

and their sum vanishes.

iii) Recall $\mathbb{C}S_n$ is the complex vector space generated by elements of S_n, and S_n acts on the generators by (left) multiplications.

Clearly transposing i and j is the same as transposing j and i, and transpositions of disjoint pairs commute.

Finally note that

$$(ij)(ik) = (ikj)\,, \qquad (ik)(ij) = (ijk)\,,$$

while

$$(ij)(jk) = (ijk)\,, \qquad (jk)(ij) = (ikj)\,,$$

whence

$$\big[(ij), (ik) + (jk)\big] = (ikj) - (ijk) + (ijk) - (ikj) = 0\,.$$

[3] It's smarter to just do these latest two computations of course.

1.4. **Solution to 2.** Recall if $T \in \operatorname{End}(V_0 \otimes W) \simeq \operatorname{End}(V_0) \otimes \operatorname{End} W$ then its trace "over V_0" is $\operatorname{tr}_0(T) = (\operatorname{tr} \otimes \operatorname{Id})(T) \in \operatorname{End}(W)$.

In particular for $n = 1$ we must the q-Gaudin Hamiltonian

$$H_1 = \tau_1(z_1) = \operatorname{tr}_0 R^{(01)}(0) = \operatorname{tr}_0 P^{(01)} \in \operatorname{End}(V_0 \otimes V_1) = \operatorname{End}\big(V^{\otimes 2}\big)\,,$$

is the identity of V (the empty product).

For this let $(e_i)_i$ be a basis of V, with dual basis $(e_i^\vee)_i$. Recall the trace of $T \in \operatorname{End}(V)$ is obtained by

$$\operatorname{tr} T = \sum_i e_i^\vee T(e_i)\,,$$

i.e. composing evaluations and coevaluations. Analogously if $v \in V$ and $T \in \operatorname{End}(V \otimes V)$ by definition one has

$$\operatorname{tr}_0 T(v) = \sum_i (e_i^\vee \otimes \operatorname{Id}) T(e_i \otimes v)\,.$$

In particular for $T = P$ and $v = e_j$:

$$\operatorname{tr}_0 P(e_j) = \sum_i (e_i^\vee \otimes \operatorname{Id}) P(e_i \otimes e_j) = \sum_i (e_i^\vee \otimes \operatorname{Id})(e_j \otimes e_i) = \sum_i \delta_{ij} e_i = e_j\,.$$

Now let us consider the general case $n \geq 1$. We have

$$\begin{aligned}\tau_n(z_i) &= \operatorname{tr}_0 R^{(0n)}(z_{in}) \cdots R^{(0,i+1)}(z_{i,i+1}) R^{(0i)}(0) R^{(0,i-1)}(z_{i,i-1}) \cdots R^{(01)}(z_{i1}) \\ &= \operatorname{tr}_0 R^{(0n)}(z_{in}) \cdots R^{(0,i+1)}(z_{i,i+1}) P^{(0i)} R^{(0,i-1)}(z_{i,i-1}) \cdots R^{(01)}(z_{i1})\,.\end{aligned}$$

Let us first commute the flip $P^{(0i)}$ of V_0 and V_i to the left, past the R-matrices; for this we should use the identities

$$P^{(0i)} T^{(0j)} = T^{(ij)} P^{(0i)}\,, \qquad T^{(0j)} P^{(0i)} = P^{(0i)} T^{(ij)}\,, \tag{3}$$

valid for any $T = \sum_k T'_k \otimes T''_k \in \operatorname{End}(V \otimes V) \simeq \operatorname{End}(V) \otimes \operatorname{End}(V)$. Indeed if $i < j$:

$$\begin{aligned}P^{(0i)} T^{(0j)}(v_0 \otimes v_1 \otimes \cdots \otimes v_n) &= \sum_k P^{(0,i)}(T'_k v_0 \otimes v_1 \otimes \cdots \otimes v_{j-1} \otimes T''_k v_j \otimes v_{j+1} \otimes \cdots \otimes v_n) \\ &= \sum_k v_i \otimes v_1 \otimes \cdots \otimes v_{i-1} \otimes T'_k v_0 \otimes v_{i+1} \otimes \cdots \otimes v_{j-1} \otimes T''_i v_j \otimes v_{j+1} \otimes \cdots \otimes v_n \\ &= T^{(ij)}(v_i \otimes v_1 \otimes \cdots \otimes v_{i-1} \otimes v_0 \otimes v_{i+1} \otimes \cdots \otimes v_n) = T^{(ij)} P^{(0i)}(v_0 \otimes v_1 \otimes \cdots \otimes v_n)\,,\end{aligned}$$

and analogously for the second identity.

Hence using (3)

$$\begin{aligned}\tau_n(z_i) &= \operatorname{tr}_0 P^{(0i)} R^{(in)}(z_{in}) \cdots R^{(i,i+1)}(z_{i,i+1}) \cdot R^{(0,i-1)}(z_{i,i-1}) \cdots R^{(01)}(z_{i1}) \\ &= \operatorname{tr}_0 P^{(0i)} R^{(0,i-1)}(z_{i,i-1}) \cdots R^{(01)}(z_{i1}) \cdot R^{(in)}(z_{in}) \cdots R^{(i,i+1)}(z_{i,i+1})\,,\end{aligned}$$

using that the action on different slots commute. Now we commute again $P^{(0i)}$ to the right, using again (3):

$$\begin{aligned}\tau_n(z_i) &= \operatorname{tr}_0 R^{(i,i-1)}(z_{i,i-1}) \cdots R^{(i1)}(z_{i1}) P^{(0i)} R^{(in)}(z_{in}) \cdots R^{(i,i+1)}(z_{i,i+1}) \\ &= R^{(i,i-1)}(z_{i,i-1}) \cdots R^{(i1)}(z_{i1}) \cdot \operatorname{tr}_0(P^{(0i)}) \cdot R^{(in)}(z_{in}) \cdots R^{(i,i+1)}(z_{i,i+1}) \\ &= R^{(i,i-1)}(z_{i,i-1}) \cdots R^{(i1)}(z_{i1}) R^{(in)}(z_{in}) \cdots R^{(i,i+1)}(z_{i,i+1})\,,\end{aligned}$$

using $\operatorname{tr}_0(ST) = S \operatorname{tr}_0(T) \in \operatorname{End}(W)$ for endomorphisms $S \in \operatorname{End}(W)$ and $T \in \operatorname{End}(V_0 \otimes W)$, as well as the result from $n = 1$.

1.5. **Solution to 3.** Here we consider the Lie-algebra evaluation morphism

$$\mathrm{ev}_z \colon L\mathfrak{g} \to \mathfrak{g}\,, \qquad a \otimes f(t) \mapsto f(z)a\,, \qquad a \in \mathfrak{g}, f \in \mathbb{C}[t^{\pm 1}]\,.$$

Every $\mathfrak{g}$-module becomes an $L\mathfrak{g}$-module via pullback along this morphism, with $\mathrm{Ker}(\mathrm{ev}_z)$ acting trivially (and we have Lie-algebra isomorphisms $L\mathfrak{g}\big/\mathrm{Ker}(\mathrm{ev}_z) \simeq \mathfrak{g}$).

Hence $V(z) = V$ as a vector space, and $\mathfrak{g} \subseteq L\mathfrak{g}$ taking constant functions: the result follows for $m = 1$, since $\mathfrak{g}v = V$ for $v \in V \setminus (0)$.

Consider now the case $m = 2$ with $z_1 = z_2 = z$. Then $L\mathfrak{g}$ acts on $V(z) \otimes V(z)$ with trivial action of the evaluation kernel @ $t = z$: hence $V(z) \otimes V(z) \simeq V \otimes V$ as $\mathfrak{g}$-modules, and we can show reducibility here. To this end recall the $\mathfrak{g}$-action on $V^{\otimes 2}$ commutes with the flip $P \colon v \otimes w \mapsto w \otimes v$, i.e. $P \in \mathrm{Aut}_{\mathfrak{g}}(V \otimes V)$: hence P-invariant tensor yields a *proper* submodule.[4] Now for generic $m \geq 2$ suppose wlog. that $z_1 = z_2 = z$, and split

$$\bigotimes_{i=1}^{m} V(z_i) = V(z)^{\otimes 2} \otimes V(z_3) \otimes \cdots \otimes V(z_m)\,.$$

By the "$m = 2$" case this contains the proper submodule

$$\big(V(z) \otimes V(z)\big)^P \otimes V(z_3) \otimes \cdots \otimes V(z_m)\,.$$

Now get back to the case $m = 2$ with $z_1 - z_2 \neq 0$ (as a warm-up). For $k \in \mathbb{Z}$ and $a \in \mathfrak{g}$ consider the element $a_k \coloneqq a \otimes t^k \in L\mathfrak{g}$; its action on pure tensors is

$$a_k(v_1 \otimes v_2) = \big(z_1^k a^{(1)} + z_2^k a^{(2)}\big) v_1 \otimes v_2\,, \qquad v_1 \in V(z_1), v_2 \in V(z_2)\,. \tag{4}$$

It follows that

$$\frac{a_1 - z_2 a_0}{z_1 - z_2} = a^{(1)} = a \otimes 1\,,$$

in the given representation, and analogously

$$\frac{a_1 - z_1 a_0}{z_2 - z_1} = a^{(2)} = 1 \otimes a\,.$$

Consider now a nonzero vector $v = \sum_{i=1}^r u_i \otimes v_i \in V(z_1) \otimes V(z_2)$. This can be turned into a pure tensor by the action, as follows. Choose $a \in \mathfrak{g}$ such that $u_1 - u_2 \in \mathrm{Ker}(a)$, then

$$a \otimes 1(v) = \sum_{i=1}^{r} (a u_i) \otimes v_i = u \otimes (v_1 + v_2) + \sum_{i=3}^{r} a(u_i) \otimes v_i\,,$$

with $u = a u_1 = a u_2$, getting to a sum of pure tensors with fewer summands; we conclude recursively.

Finally suppose $v = u \otimes w$ is a nonzero pure tensor, with $u \in V(z_1)$ and $v \in V(z_2)$: the action of $\mathfrak{g} \otimes 1 + 1 \otimes \mathfrak{g} \subseteq \mathrm{End}\big(V(z_1) \otimes V(z_2)\big)$ moves this to any other pure tensor, so in conclusion there are no nontrivial submodules.

The idea in the general case $m \geq 2$ is thus to express $a^{(i)} \in \mathrm{End}\big(V(z_1) \otimes \cdots \otimes V(z_m)\big)$ as a linear combination of $a_0, \ldots, a_{m-1} \in L\mathfrak{g}$, for all $a \in \mathfrak{g}$ and $i \in \{\, 1, \ldots, m \,\}$. For this we use

$$a_k(v_1 \otimes \cdots \otimes v_m) = \sum_{l=1}^{m} z_l^k a^{(l)} (v_1 \otimes \cdots\cdots \otimes v_m)\,, \qquad v_l \in V(z_l)\,,$$

[4]This defines symmetric tensors $S^2 V \subseteq V \otimes V$. Alternating tensors $\bigwedge^2 V \subseteq V \otimes V$ are defined analogously (as the (-1)-eigenspace for P), and $V \otimes V \simeq S^2 V \oplus \bigwedge^2 V$.

generalising (4).

Hence for $\boldsymbol{v} = v_1 \otimes \cdots \otimes v_m$, with $v_i \in V(z_i)$, we impose/compute:

$$a^{(i)}\boldsymbol{v} \overset{!}{=} \sum_{k=1}^{m} c_{ik} a_k \boldsymbol{v} = \sum_{k,l=1}^{m} c_{ik} z_l^k a^{(l)} \boldsymbol{v} \in V(z_1) \otimes \cdots \otimes V(z_m)\,,$$

i.e.

$$\sum_{k=1}^{m} c_{ik} z_l^k = \delta_{il} \in \mathbb{C}\,.$$

This means the matrix $C = (c_{ik})_{ik}$ (if it exists) is the inverse of the size-m Vandermonde matrix $V(z_1, \dots, v_m)$ with parameters $z_1, \dots, z_m \in \mathbb{C}$, i.e.

$$V(z_1, \dots, z_m) = \begin{pmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{m-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_m & z_m^2 & \cdots & z_m^{m-1} \end{pmatrix} \in \mathfrak{gl}_m(\mathbb{C})\,.$$

But this matrix is invertible precisely because the parameters are all distinct.[5]

[5] The determinant is $\det\bigl(V(z_1, \dots, z_m)\bigr) = \prod_{1 \leq i < j \leq m} (z_i - z_j)$.