Les Diablerets Winter School in Mathematical Physics– Sheet 2

Vit Sriprachyakul

1 The free field construction of AdS$_3$ strings

We begin by considering the bosonic half of the free field construction of tensionless strings on AdS$_3$. This consists of two pairs, $(\xi^\alpha, \eta^\alpha)$ of free bosonic fields with $\alpha = \pm$, which satisfy the algebra

$$[\xi^\alpha_r, \eta^\beta_s] = \varepsilon^{\alpha\beta}\delta_{r+s,0} \ , \tag{1}$$

where $\epsilon^{+-} = -\epsilon^{-+} = 1$ is the antisymmetric tensor.

a) Show that the currents

$$\begin{aligned} J^+_m &= \sum_r \xi^+_{m-r}\eta^+_r \ , \quad J^-_m = \sum_r \xi^-_{m-r}\eta^-_r \ , \\ J^3_m &= -\tfrac{1}{2}\sum_r :(\xi^+_{m-r}\eta^-_r + \xi^-_{m-r}\eta^+_r): \ , \\ U_m &= \tfrac{1}{2}\sum_r :(\xi^+_{m-r}\eta^-_r - \xi^-_{m-r}\eta^+_r): \end{aligned} \tag{2}$$

satisfy the $\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{u}(1)$ algebra, namely

$$\begin{aligned} [J^+_m, J^-_n] &= m\delta_{m+n,0} - 2J^3_{m+n} \ , \quad [J^3_m, J^\pm_n] = \pm J^\pm_{m+n} \ , \\ [J^3_m, J^3_n] &= -\tfrac{1}{2}m\delta_{m+n,0} \ , \quad [U_m, U_n] = -\tfrac{1}{2}m\delta_{m+n,0} \ . \end{aligned} \tag{3}$$

For simplicity and concreteness, work in the NS sector, i.e. take $r \in \mathbb{Z}+\frac{1}{2}$. Here normal ordering is defined by $: a_m b_n := a_m b_n$ if $m < 0$ and $: a_m b_n := \pm b_n a_m$ if $m > 0$, where we have the minus sign in the second equation if both a and b are fermionic, and the plus sign otherwise.

b) Now, consider a representation of the R-sector of the symplectic boson algebra, defined through the zero mode operations

$$\begin{aligned} \xi^+_0 \left|m_1, m_2\right\rangle &= \left|m_1, m_2 + \tfrac{1}{2}\right\rangle \ , \quad \xi^-_0 \left|m_1, m_2\right\rangle = -\left|m_1 - \tfrac{1}{2}, m_2\right\rangle \\ \eta^+_0 \left|m_1, m_2\right\rangle &= 2m_1 \left|m_1 + \tfrac{1}{2}, m_2\right\rangle \ , \quad \eta^-_0 \left|m_1, m_2\right\rangle = -2m_2 \left|m_1, m_2 - \tfrac{1}{2}\right\rangle \ . \end{aligned} \tag{4}$$

Furthermore, all of these states are annihilated by the positive free field modes, i.e.

$$\xi^\pm_m|m_1, m_2\rangle = \eta^\pm_m|m_1, m_2\rangle = 0 \tag{5}$$

for $m > 0$. Show that these states sit in a representation of the zero mode algebra of $\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{u}(1)$ with the actions

$$\begin{aligned} J^+_0 \left|m_1, m_2\right\rangle &= 2m_1 \left|m_1 + \tfrac{1}{2}, m_2 + \tfrac{1}{2}\right\rangle \ , \quad J^-_0 \left|m_1, m_2\right\rangle = 2m_2 \left|m_1 - \tfrac{1}{2}, m_2 - \tfrac{1}{2}\right\rangle \ , \\ J^3_0 \left|m_1, m_2\right\rangle &= (m_1 + m_2) \left|m_1, m_2\right\rangle \ , \quad U_0 \left|m_1, m_2\right\rangle = \left(m_1 - m_2 - \tfrac{1}{2}\right) \left|m_1, m_2\right\rangle \ . \end{aligned} \tag{6}$$

Hint: In the R-sector, r runs over the integers. Normal ordering is defined as above, together with the prescription that

$$:\xi_0^+\eta_0^-: = \tfrac{1}{2}\left(\xi_0^+\eta_0^- + \eta_0^-\xi_0^+\right)$$
$$:\xi_0^-\eta_0^+: = \tfrac{1}{2}\left(\xi_0^-\eta_0^+ + \eta_0^+\xi_0^-\right) \tag{7}$$

c) The Casimir operator of the $\mathfrak{sl}(2,\mathbb{R})$ algebra is defined by

$$C^{\mathfrak{sl}(2,\mathbb{R}} = \frac{1}{2}\Big(J_0^- J_0^+ + J_0^+ J_0^- - 2J_0^3 J_0^3\Big) \ . \tag{8}$$

Using the free field realisation from above evaluate $C^{\mathfrak{sl}(2,\mathbb{R}}$ on $|m_1, m_2\rangle$, and show that

$$C^{\mathfrak{sl}(2,\mathbb{R})}\,|m_1,m_2\rangle = \left(\tfrac{1}{4} - U_0^2\right)|m_1,m_2\rangle \ . \tag{9}$$

Hence show that we may set (there are obviously two solutions for j)

$$j = m_1 - m_2 \ , \tag{10}$$

since $C^{\mathfrak{sl}(2,\mathbb{R})}\,|m_1,m_2\rangle = -j(j-1)\,|m_1,m_2\rangle$.

1.1 The fermionic fields in the free field construction

Next we consider the fermionic fields $\psi^\pm$ and $\chi^\pm$ with anti-commutation relations

$$\{\psi_r^\alpha, \chi_s^\beta\} = \epsilon^{\alpha\beta}\,\delta_{r,-s} \ . \tag{11}$$

The $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ generators are defined via

$$\begin{aligned} K_m^+ &= \sum_r \chi_{m-r}^+\psi_r^+ \ , \quad K_m^- = -\sum_r \chi_{m-r}^-\psi_r^- \ , \\ K_m^3 &= -\tfrac{1}{2}\sum_r :(\chi_{m-r}^+\psi_r^- + \chi_{m-r}^-\psi_r^+): \ , \\ V_m &= -\tfrac{1}{2}\sum_r :(\chi_{m-r}^+\psi_r^- - \chi_{m-r}^-\psi_r^+): \ , \end{aligned} \tag{12}$$

i.e. these generators satisfy

$$\begin{aligned} [K_m^+, K_n^-] &= m\delta_{m+n,0} + 2K_{m+n}^3 \ , \quad [K_m^3, K_n^\pm] = \pm K_{m+n}^\pm \ , \\ [K_m^3, K_n^3] &= \tfrac{1}{2}m\delta_{m+n,0} \ , \quad [V_m, V_n] = \tfrac{1}{2}m\delta_{m+n,0} \ . \end{aligned} \tag{13}$$

The ground states $|m_1, m_2\rangle$ are characterised by

$$\begin{aligned} \psi_n^+|m_1,m_2\rangle &= \chi_n^+|m_1,m_2\rangle = 0 \ , \qquad n \geq 0 \ . \\ \psi_n^-|m_1,m_2\rangle &= \chi_n^-|m_1,m_2\rangle = 0 \ , \qquad n > 0 \ . \end{aligned} \tag{14}$$

Using the normal ordering prescription from above together with

$$\begin{aligned} :\psi_0^+\chi_0^-: &= \tfrac{1}{2}\left(\psi_0^+\chi_0^- - \chi_0^-\psi_0^+\right) \\ :\psi_0^-\chi_0^+: &= \tfrac{1}{2}\left(\psi_0^-\chi_0^+ - \chi_0^+\psi_0^-\right) \end{aligned} \tag{15}$$

show that the ground states $|m_1, m_2\rangle$ satisfy

$$K_0^3\,|m_1,m_2\rangle = \tfrac{1}{2}\,|m_1,m_2\rangle \ , \qquad V_0\,|m_1,m_2\rangle = 0 \ . \tag{16}$$

1.1.1 Combining the free field constructions

In order to go from $\mathfrak{u}(1,1|2)_1$ to $\mathfrak{psu}(1,1|2)_1$ we have to consider only the states for which $Z_n\phi = 0$ with $n \geq 0$, where

$$Z_n = U_n + V_n \ . \tag{17}$$

Combining the results from the previous two sections and demanding that $Z_0 = 0$ deduce that the states $|m_1, m_2\rangle$ generate the representation

$$\big(C^{1/2}_{\alpha}, \mathbf{2}\big) \ , \tag{18}$$

with respect to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$, where $C^{1/2}_{\alpha}$ is the continuous series representation with $j = \frac{1}{2}$, while $\mathbf{2}$ is the spin-$\frac{1}{2}$ representation of $\mathfrak{su}(2)$. Here α is determined to equal

$$\alpha = m_1 + m_2 \pmod 1 \ . \tag{19}$$

1.1.2 The fermionic generators

Finally, the fermionic generators of $\mathfrak{u}(1,1|2)_1$ are determined in terms of the free fields via

$$S^{\alpha\beta +}_m = \sum_r \xi^{\alpha}_{m-r} \chi^{\beta}_r \ , \qquad S^{\alpha\beta -}_m = - \sum_r \eta^{\alpha}_{m-r} \psi^{\beta}_r \ . \tag{20}$$

Determine the action of the fermionic zero modes $S^{\alpha\beta\pm}_0$ on the states $|m_1, m_2\rangle$, and show that they generate the remaining two summands in the representation of $\mathfrak{psu}(1,1|2)_1$ (in addition to (18)), namely

$$\big(C^0_{\alpha+\frac{1}{2}}, \mathbf{1}\big) \oplus \big(C^1_{\alpha+\frac{1}{2}}, \mathbf{1}\big) \ . \tag{21}$$