EXERCISE SESSION 13.01.2021

(1) Recall that $U_q sl_2$ is the algebra with generators e, f, k, k^{-1} and relations $kek^{-1} = qe$, $kfk^{-1} = q^{-1}f$, $[e, f] = \frac{k-k^{-1}}{q-q^{-1}}$. It has a coassociative coproduct such that $\Delta(e) = e \otimes 1 + k \otimes e$, $\Delta(f) = f \otimes k^{-1} + 1 \otimes f$, $\Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1}$. The vector representation $\rho\colon U_q sl_2 \to \mathrm{End}(V)$ on $V = \mathbb{C}e_1 + \mathbb{C}e_2$ is $e \mapsto E_{12}$, $f \mapsto E_{21}$, $k \mapsto qE_{11} + q^{-1}E_{22}$.

(a) Show that $\bigwedge_q^2 V = \mathbb{C}(e_1 \otimes e_2 - qe_2 \otimes e_1)$ is a subrepresentation of $V \otimes V$.

(b) Show that $e_1 \otimes e_1$ is a highest weight vector and generates a 3-dimensional subrepresentation $\mathrm{Sym}_q^2 V$.

(c) Show that

$$T = q \sum E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ij} \otimes E_{ji} + (q - q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj}$$

acts on these subrepresentations as a multiple of the identity.

(d) Deduce that T commutes with the action of $U_q sl_2$ and obeys the Hecke relation $(T - q)(T + q^{-1}) = 0$.

(2) Suppose T_i, $i = 1, 2, 3$ are invertible endomorphisms of a vector space obeying the Hecke relations $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $(T_i - q)(T_i + q^{-1}) = 0$ for some $q \in \mathbb{C} \setminus \{0\}$. Show that $\check{R}_{i,i+1}(z) = zT_i - T_i^{-1}$ is a solution of the Yang–Baxter equation

$$\check{R}_{12}(z)\check{R}_{23}(zw)\check{R}_{12}(w) = \check{R}_{23}(w)\check{R}_{12}(zw)\check{R}_{23}(z).$$

(3) The exterior algebra $\bigwedge V$ of a complex vector space V is the algebra generated by V with relations $v^2 = 0$, $v \in V$. It is characterized as an algebra with a linear map $i\colon E \to \bigwedge E$ by the following universal property: every linear map $f\colon E \to A$ to a unital algebra A so that $f(v)^2 = 0$ factors uniquely as $V \to \bigwedge V \to A$.

(a) Show that $\bigwedge V$ has a unique $\mathbb{Z}$-grading $\bigwedge V = \oplus_{k=0}^n \bigwedge^k V$ such that $\deg v = 1$ and is graded commutative: $ab = (-1)^{\deg a \deg b} ba$.

(b) Show that $\bigwedge \mathbb{C}$ is isomorphic to $\mathbb{C}[\theta]/\theta^2\mathbb{C}[\theta]$.

(c) Show with the universal property that one has an isomorphism $\bigwedge(V_1 \oplus V_2) \to \bigwedge V_1 \otimes \bigwedge V_2$ to the tensor product of graded commutative algebras (defined by $(a \otimes b)(c \otimes d) = (-1)^{\deg b \deg c} ac \otimes bd$). Deduce that any basis $\theta_1, \ldots, \theta_n$ of V defines a basis $\theta_{i_1} \cdots \theta_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq n)$ of $\bigwedge V$.

(d) Show that the group $GL(V)$ of automorphisms of V acts on $\bigwedge V$ by degree preserving automorphisms.

(4) Let $C(E)$ be the Clifford algebra of a vector space E with a non-degenerate bilinear form $\langle -, - \rangle$. It is generated by E with relations $v^2 = \langle v, v \rangle 1$, $v \in E$. Denote by $\gamma(v)$ the image in $C(E)$ of $v \in E$.

(a) If you care, formulate the universal property as in the previous exercise and show $C(E)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded commutative algebra.

(b) The Lie algebra $\mathfrak{o}(E)$ of the orthogonal group of E is isomorphic to $\wedge^2 E$, acting on E via $(u \wedge v) \cdot w = \langle v, w\rangle u - \langle v, w\rangle v$. Let $\rho\colon \mathfrak{o}(E) = \wedge^2 E \to C(E)$ be the map

$$u \wedge v \mapsto \frac{1}{2}(\gamma(u)\gamma(v) - \gamma(v)\gamma(u))$$

Show that $\rho(x)\gamma(v) = \gamma(x \cdot v)$, $(x \in \mathfrak{o}(E),\, v \in E)$ and that ρ is a Lie algebra homomomorphism (for the commutator bracket on $C(E)$).

(c) Let $E = V \oplus V^*$ with bilinear form

$$\langle v \oplus \alpha, w \oplus \beta\rangle = \beta(v) + \alpha(w).$$

Then there is a unique $C(E)$-module structure on $\bigwedge V^*$ such that $\gamma(v)1 = 0$ if $v \in V$. If $\alpha \in V^*$, $\gamma(\alpha)$ acts by multiplication and for $v \in V$,

$$\gamma(v) \cdot (\alpha_1 \cdots \alpha_k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha_i(v) \alpha_1 \cdots \alpha_{i-1}\alpha_{i+1} \cdots \alpha_k$$

(omitting $\wedge$ for TEXnical reasons).