Details on the calculation of NRQCD matching coefficients for $\eta_c \to \gamma\gamma$ using FeynCalc+FeynOnium

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- Open-source and publicly available: `https://feyncalc.github.io`
- Toolbox-oriented approach to symbolic Feynman diagram calculations
- Not foolproof: correctness of the results $\propto$ user's understanding of QFT
- Most useful at tree- and 1-loop level (WIP: extension to more loops using a **FORM** [Vermaseren, 2000] library)
- Since 2020 native support for noncovariant calculations (Cartesian vectors, Pauli matrices etc.) $\Rightarrow$ NRQCD, pNRQCD, NR Dark matter

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  - **FeynCalc** is difficult to master, the crash course here will not be enough
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- Following [Jia, Yang, Sang and Xu, 2011] (arXiv:1104.1418, very pedagogical)
- NRQCD factorization for the decay $\eta_c \to \gamma\gamma$ at relative order-v^2

$$\Gamma(\eta_c \to \gamma\gamma) = \frac{F(^1S_0)}{m_c^2} \langle \eta_c|\psi^\dagger \chi|0\rangle \langle 0|\chi^\dagger \psi|\eta_c\rangle + \frac{G(^1S_0)}{m_c^4} \left(\frac{1}{2} \langle \eta_c|\psi^\dagger (-\tfrac{i}{2}\overleftrightarrow{\boldsymbol{D}})^2 \chi|0\rangle \langle 0|\chi^\dagger \psi|\eta_c\rangle + \text{h.c.} \right)$$

- ψ (χ): Pauli field annihilating (creating) a quark (antiquark); $\psi^\dagger \overleftrightarrow{\boldsymbol{D}} \chi = \psi^\dagger (D^i \chi) - (D^i \psi)^\dagger \chi$
- **Wilson coefficients** $F(^1S_0)$ and $G(^1S_0)$ from the *perturbative* matching between QCD and NRQCD
- Exclusive electromagnetic process $\Rightarrow$ factorization holds at the *amplitude level*
- NRQCD amplitude for $\eta_c \to \gamma\gamma$

$$\mathcal{A}_{\text{NRQCD}} = \hat{\boldsymbol{k}}_1 \cdot \varepsilon_1^* \times \varepsilon_2^* \left[c_0 \langle 0|\chi^\dagger \psi|\eta_c\rangle + \frac{c_2}{m_c^2} \langle 0|\chi^\dagger (-\tfrac{i}{2}\overleftrightarrow{\boldsymbol{D}})^2 \psi|\eta_c\rangle + \mathcal{O}(v^2) \right]$$

- $\hat{\boldsymbol{k}}_1$ is the direction of the 3-momentum of the 1st photon, ε_i^* are the photon polarization vectors.
- c_0 and c_2 are **short distance coefficients**.

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- Relation between **Wilson** and **short distance coefficients**?
- Square $\mathcal{A}_{\mathrm{NRQCD}}$ and integrate over the $1 \to 2$ phase space. Then compare the result to $\Gamma(\eta_c \to \gamma\gamma)$:

$$F(^1S_0) = \frac{m_c^2}{8\pi}|c_0|^2, \quad G(^1S_0) = \frac{m_c^2}{4\pi}\,\mathrm{Re}(c_0 c_2^*)$$

- So our goal is to calculate c_0 and c_2!
- The matching is done between perturbative QCD and perturbative NRQCD amplitudes in the limit $v \ll c$
- Perturbative NRQCD amplitude: replace η_c (nonpert. bound state) by $c\bar{c}(^1S_0)$ (S-wave charm-anticharm pair in the spin and color singlet configuration)
- Tree-level perturbative NRQCD amplitude

$$\mathcal{A}_{\mathrm{NRQCD}}^{\mathrm{pert.},(0)} = \hat{\boldsymbol{k}}_1 \cdot \varepsilon_1^* \times \varepsilon_2^* \left[c_0 \langle 0|\chi^\dagger \psi | c\bar{c}(^1S_0)\rangle + \frac{c_2}{m_c^2} \langle 0|\chi^\dagger (-\tfrac{i}{2}\overleftrightarrow{\boldsymbol{D}})^2 \psi | c\bar{c}(^1S_0)\rangle \right]$$
$$= \sqrt{2N_c}\,\hat{\boldsymbol{k}}_1 \cdot \varepsilon_1^* \times \varepsilon_2^* \,(\eta^\dagger \xi) \left[c_0 + c_2 \frac{\boldsymbol{q}^2}{m_c^2} \right]$$

with $\eta^1 = \xi^1 = (1,0)^T, \eta^2 = \xi^2 = (0,1)^T$

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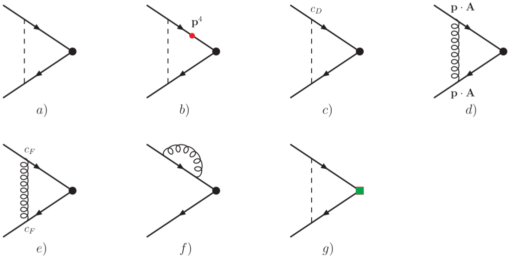
- In principle, when doing matching at NLO, we need NLO corrections both in QCD and NRQCD!
- There are indeed 1-loop corrections to the tree-level NRQCD amplitude [Jia, Yang, Sang and Xu, 2011]



- However, 1-loop contributions to the UV-renormalized NRQCD amplitude do not contribute to the matching coefficients!

- Perturbative NRQCD 1-loop amplitude

$$\mathcal{A}^{\text{pert.},(1)}_{\text{NRQCD}} = \sqrt{2N_c}\,\hat{\boldsymbol{k}}_1 \cdot \varepsilon_1^* \times \varepsilon_2^* \,(\eta^\dagger \xi) \left[c_0 + c_2 \frac{\boldsymbol{q}^2}{m_c^2} + \frac{4 C_F e_Q^2 \alpha \alpha_s}{m_c} \left\{ \frac{2v^2}{3} \left(\frac{1}{\varepsilon_{\text{IR}}} - \gamma_E + \ln(4\pi) \right) + \frac{1}{4v} \left(1 + \frac{5}{6} v^2 \right) \left[\pi^2 + i\pi \left(-\frac{1}{\varepsilon_{\text{IR}}} + \gamma_E + \ln \frac{\boldsymbol{q}^2}{\pi \mu^2} \right) \right] \right\} \right]$$

- Still contains $1/v$-Coulomb singularities and IR divergences: both must cancel in the matching!
- The explicit cancellation is a nontrivial cross check for the correctness of the calculation
- However, one can also turn the argument around and use it to simplify the calculation. Effectively

$$\mathcal{A}^{\text{pert.},(1)}_{\text{NRQCD}} \to \sqrt{2N_c}\,\hat{\boldsymbol{k}}_1 \cdot \varepsilon_1^* \times \varepsilon_2^* \,(\eta^\dagger \xi) \left[c_0 + c_2 \frac{\boldsymbol{q}^2}{m_c^2} \right]$$

- On the QCD side we can completely avoid the Coulomb singularities by expanding in $\boldsymbol{q}$ before calculating the loop integrals [Butenschön 2009]
- Quick and dirty: drop the remaining IR-divergences in the UV-renormalized QCD amplitude and match that to $\mathcal{A}^{\text{pert.},(1)}_{\text{NRQCD}}$ to get our short distance coefficients c_0 and c_2

- Kinematics for the QCD process $c(p_1)\bar{c}(p_2) \to \gamma(k_1)\gamma(k_2)$

$$p_1 = \frac{1}{2}P + q, \quad p_2 = \frac{1}{2}P - q$$

with

$$P = (2E, 0)^T, \quad q = (0, \boldsymbol{q})^T, \quad k_1 = E(1, \hat{\boldsymbol{k}}_1)^T, \quad k_2 = E(1, -\hat{\boldsymbol{k}}_1)^T$$

and $E = \sqrt{m_c^2 + \boldsymbol{q}^2}$

- We need to expand the 1-loop amplitude up to $\mathcal{O}(\boldsymbol{q}^2)$

- We also need to project out the spin singlet color singlet S-wave component [Bodwin and Petrelli, 2002]

$$\mathcal{A}_{\mathrm{QCD}}^{\mathrm{pert.},(1)} = \bar{v}(p_2) X \bar{u}(p_1) \to \mathrm{Tr}(\Pi_1^{(1)} X)$$

with

$$\Pi_1^{(1)} = \frac{1}{8\sqrt{2}E^2(E+m_c)} (\not{p}_1 + m_c)(\not{P} + 2E)\gamma^5(\not{p}_2 - m_c) \otimes \frac{1_c}{\sqrt{N_c}}$$

and use

$$q^\mu q^\nu \to \frac{\boldsymbol{q}^2}{3}\left(-g^{\mu\nu} + \frac{P^\mu P^\nu}{P^2}\right)$$

- Traces with γ^5 in dim reg: use the t'Hooft-Veltman-Breitenlohner-Maison scheme

- Final result

$$F(^1S_0) = 2\pi e_Q^4 \alpha^2 \left[1 + \frac{C_F \alpha_s}{\pi} \left(\frac{\pi^2}{4} - 5\right)\right]$$

$$G(^1S_0) = -\frac{8\pi e_Q^4 \alpha^2}{3} \left[1 + \frac{C_F \alpha_s}{\pi} \left(\frac{5\pi^2}{16} - \frac{49}{12} - \ln \frac{\mu^2}{4m_c^2}\right)\right]$$

- Let us see how we can reproduce this using **FeynOnium**!