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Microscopic Expansion for Superconformal Indices

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The giant graviton expansion, DG & JHL, 2109.02545

Exact stringy microstates from gauge theories, JHL, 2204.09286

1. Introduction

A central question in AdS/CFT is how the degrees of freedom of gauge theory are organized in the string dual. The purpose of our study is to propose an exact finite N formula that relates the BPS sectors of gauge theories and their string duals.

A way to state AdS/CFT is as an equivalence between the partition functions of gauge and string theories: $\mathcal{Z}_{\text{gauge}} = \mathcal{Z}_{\text{string}}$. The string theory partition function is often evaluated in the supergravity limit at large 't Hooft coupling λ, as a sum over saddle geometries satisfying the boundary condition provided by the CFT:

$$\mathcal{Z}_{\text{sugra}} \simeq \sum_{\text{geometries}} e^{-N^2 S_{\text{on-shell}}}.$$

This is an asymptotic expression valid at large N, with perturbative corrections around each saddle.

We study the relation between gauge/string partition functions at the level of supersymmetric indices, which counts BPS operators up to signs:

$$Z_{CFT} = \text{Tr}_{\mathcal{H}_{\text{BPS}}} (-1)^F e^{-\frac{1}{2}\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} y_1^{\mathcal{C}_1} y_2^{\mathcal{C}_2} \cdots y_s^{\mathcal{C}_s}.$$

The index is topological. Since we can deform the CFT to a free theory and compute the index there, the index is much more tractable than a partition function.

2. Giant graviton expansion

We propose a microscopic expansion for the indices of $U(N)$ gauge theories at finite N. The terms in the expansion admit natural interpretations in string theory as "giant graviton" branes and their open/closed string excitations:

$$Z_N = Z_\infty \sum_{k_1,k_2,\cdots,k_s=0}^{\infty} \left(x_1^{k_1 N} x_2^{k_2 N} \cdots x_s^{k_s N}\right) \hat{Z}_{(k_1,k_2,\cdots,k_s)}.$$

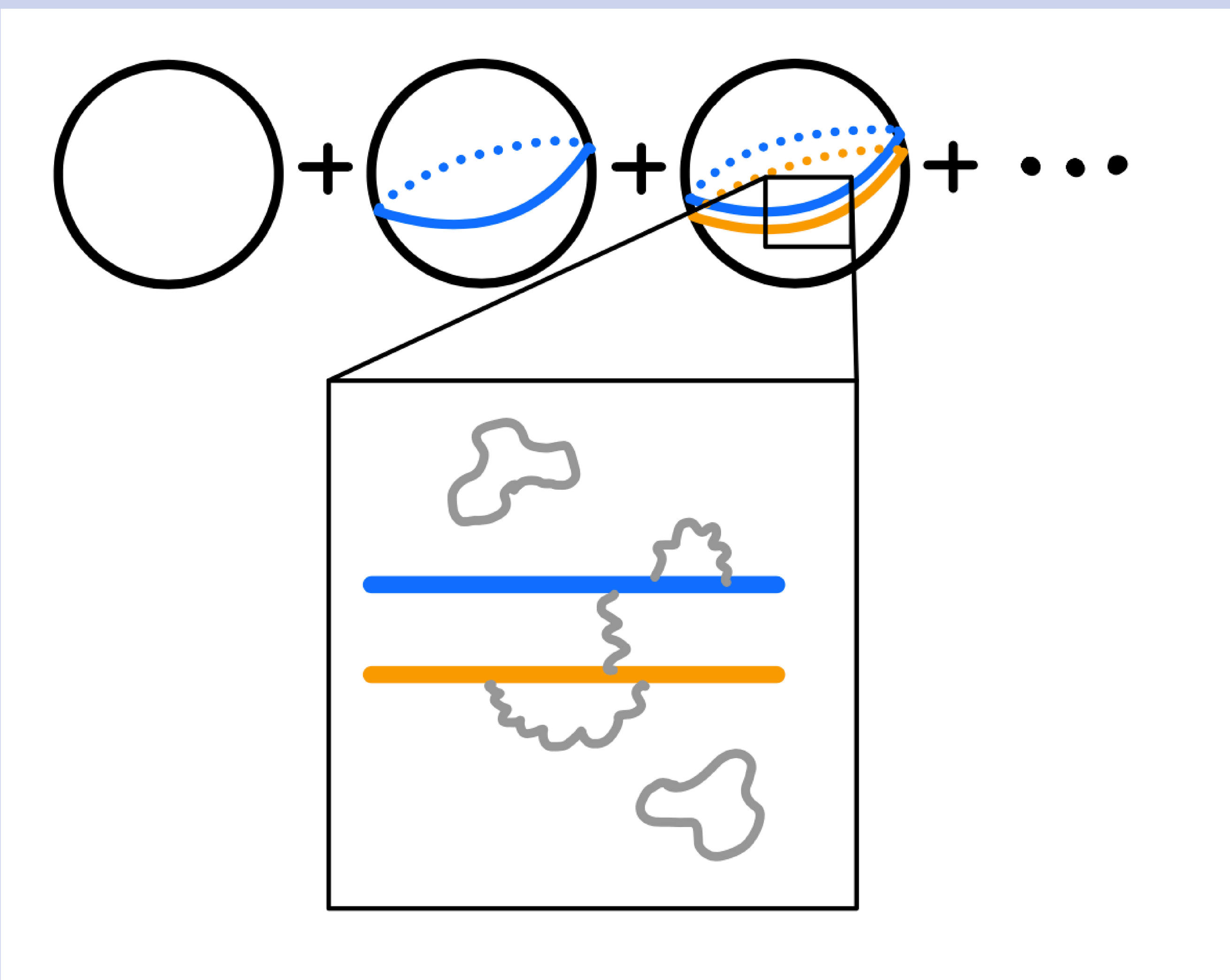

Figure 1: Giant graviton expansion in the half-BPS sector of $\mathcal{N}=4$ SYM. The giants are D3 branes wrapping $S^3 \subset S^5$.

We claim that the sum over strings and branes are *convergent at finite, integer values of* N. We provided a prescription to construct the brane indices $\hat{Z}_{(k_1,k_2,\cdots,k_s)}$ by analyzing the possible modifications of determinant operators in gauge theory. This formula seems to hold for many $U(N)$ gauge theories for which the superconformal index can be defined.

A way to understand the formula is as a *bulk-bulk duality* between (1) coherent states of branes and their excitations in the stringy regime and (2) saddle geometries such as BPS black holes in the supergravity regime. That is, *if the string dual of the $U(N)$ gauge theory admits a supergravity regime with BPS black holes, the string/brane configurations are the bulk microstates of the BPS sector containing such black holes.*

3. Branes and determinants

The branes that appear in the giant graviton expansion are dual to determinant operators in the $U(N)$ gauge theory. Modifying "letters" of large N determinants

$$\det X = \frac{1}{N!}\epsilon_{i_1 i_2 \cdots i_N}\epsilon^{j_1 j_2 \cdots j_N} X^{i_1}_{j_1} X^{i_2}_{j_2} \cdots X^{i_N}_{j_N}$$

with other fields in the theory correspond to open string excitations of branes sitting in a large N string background. We provided a prescription to construct and compute brane indices $\hat{Z}_{(k_1,k_2,\cdots,k_s)}$ by counting determinant modifications in gauge theory.

Let us consider the counting of modifications for a product of determinants

$$(\det X_1)^{k_1} (\det X_2)^{k_2} \cdots (\det X_s)^{k_s}$$

for a set of certain fields $X_1, \cdots, X_s$ in the $U(N)$ gauge theory. Our prescription gives the index of an effective $\prod_i U(k_i)$ quiver gauge theory

$$\hat{Z}_{(k_1,\cdots,k_s)} = \frac{1}{k_1! \cdots k_s!} \oint \prod_{a_1=1}^{k_1} \frac{d\sigma^1_{a_1}}{2\pi i \sigma^1_{a_1}} \cdots \prod_{a_s=1}^{k_s} \frac{d\sigma^s_{a_s}}{2\pi i \sigma^s_{a_s}}$$
$$\prod_{a_1 \neq b_1} \left(1 - \frac{\sigma^1_{a_1}}{\sigma^1_{b_1}}\right) \cdots \prod_{a_s \neq b_s} \left(1 - \frac{\sigma^s_{a_s}}{\sigma^s_{b_s}}\right) \text{PE}\left[\sum_{i,j=1}^{s} \left(\hat{f}^i_j \sum_{a_i, b_j} \frac{\sigma^i_{a_i}}{\sigma^j_{b_j}}\right)\right],$$

whose *holographic interpretation is the worldvolume gauge theory on stacks of branes in the string dual of the $U(N)$ gauge theory*. The brane single letter indices $\hat{f}^i_j$ encode adjoint/bifundamental strings that connect same/different types of branes.

4. BPS Hilbert space in string theory

Consider, at finite N, the half-BPS sector of $\mathcal{N}=4$ SYM and IIB string theory on $\text{AdS}_5 \times S^5$. The half-BPS sector of $\mathcal{N}=4$ SYM is spanned by multitrace operators $\prod_i \text{Tr} X^{m_i}$ subject to trace relations at finite N. Since there are only bosons in gauge theory, the half-BPS index is the half-BPS partition function.

Here, the giant graviton expansion can be written explicitly:

$$\frac{1}{\prod_{n=1}^{N}(1-x^n)} = \frac{1}{\prod_{n=1}^{\infty}(1-x^n)} \sum_{k=0}^{\infty} x^{kN} \frac{1}{\prod_{m=1}^{k}(1-x^{-m})},$$

where stacks of D3 giant gravitons dual to $(\det X)^k$ wrap $S^3 \subset S^5$ in $\text{AdS}_5 \times S^5$. On the RHS, the prefactor is the closed string contribution (KK-modes). The summand is the brane partition function encoding the open string spectra. The inverse power x^{-1} appears because half-BPS open strings take away R-charges from giants. When comparing the brane and gauge theory partition functions, we should analytically continue x from outside to inside the complex unit disk.

An important consequence of the analytic continuation is that $\hat{Z}_k$ gets an overall minus sign for odd k. D3 giants become effective fermions, resulting in an effective $\mathbb{Z}_2$-grading when interpreted in the string BPS Hilbert space. Vast cancellations of coefficients occur on the string side even though the gauge theory only has bosons. *The coefficients of the half-BPS partition function Z_N arise from graded dimensions of the Fock space of giant gravitons and their analytically-continued excitations.*

The lesson from the half-BPS sector is general and is a genuine finite N effect. The effective grading due to analytic continuation shows up in all known examples.

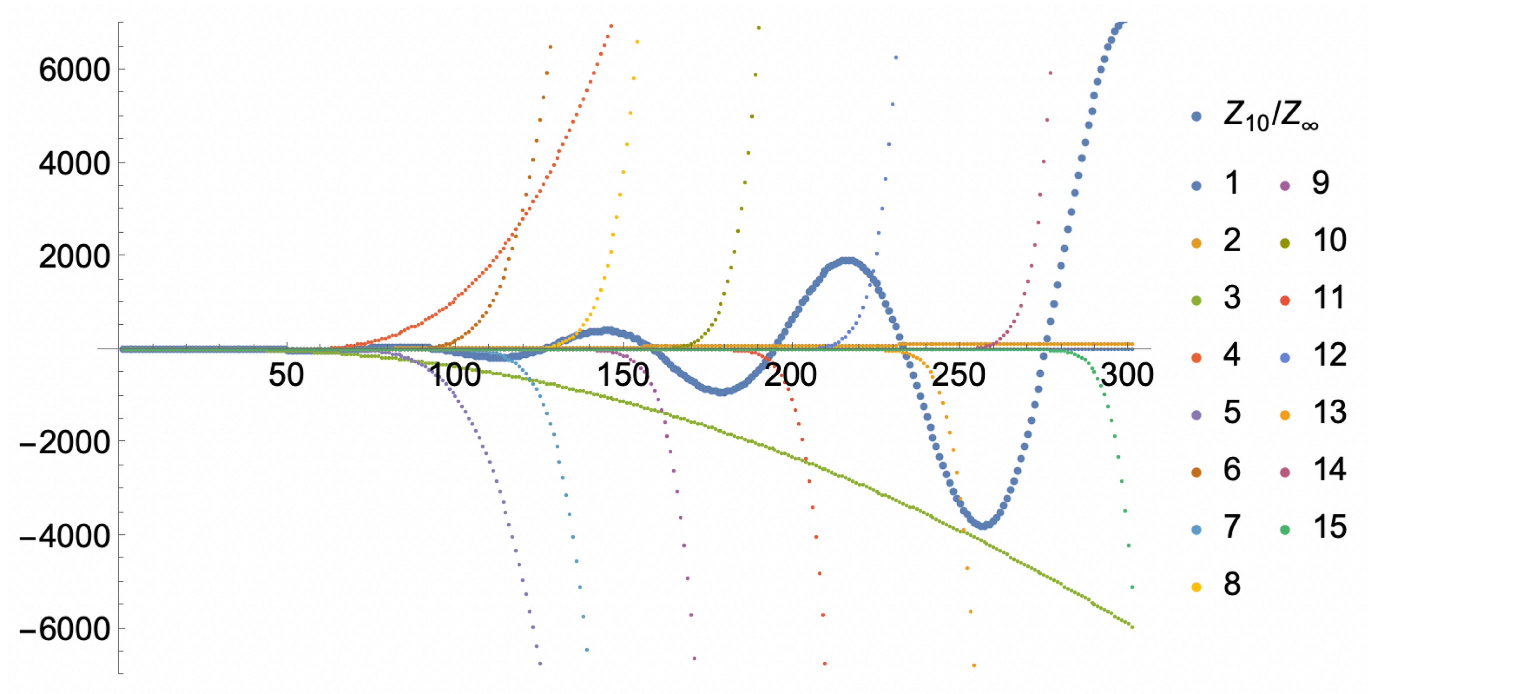

Figure 2: Coefficients of normalized half-BPS partition function Z_N/Z_∞ with $N=10$ (large blue) versus charge n. Large cancellations between brane partition functions $x^{kN}\hat{Z}_k$ (various colors up to $k=15$) induce order N oscillations.