4d physics from 2d chiral correlators

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Based on 2204.05301, 2201.02595 + WIP
A confluence of progress in a few different subfields
points of contact: symmetry, universality

**Twisted holography**

\[ A = A_z d\bar{z} + A_{\bar{w}_1} d\bar{w}_1 + A_{\bar{w}_2} d\bar{w}_2 \]

\[ \int \Omega \wedge CS(A) \]

\[ \eta \in \Omega^{2,1}(M) \]

\[ \frac{1}{2} \int (\partial^{-1} \eta)(\partial \eta) + \frac{1}{6} \int \eta^3 \]

\[ SO^+(1,3) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2 \]

**Bootstrap (CFTs, S-matrix, ...)**

\[ \sum_{\sigma} = \sum_{\sigma} \]

\[ O_1(z, \bar{z}) O_2(0, 0) = \sum_{k=\text{Schar}} \frac{\lambda_{12k}}{2^{h_1+h_2-h_k}} O_k(0) + \{ \mathcal{Q}, \ldots \} \]
In all these arenas we see (hints of):

**Chiral algebras**: asymptotic symmetries in flat space, protected subsectors of (SUSY) CFTs, twisted string theory,…

**Holomorphy**: MHV amplitudes, protected OPEs, worldvolume of B-branes, Kodaira-Spencer theory

One may hope to tie these structures together

(At the level of universal/symmetry-governed/soft/conformal... physics) in a fruitful and clarifying way
Today: we will start to flesh out some of those connections

6d bridge between 4d & 2d massless physics

To any local twistor thy:
1. a 4d (non-unitary) CFT [Penrose transform]
2. a 2d chiral algebra [Koszul duality]

From this perspective:

form factors which reproduce some 4d QCD scattering amplitudes

correlators of a certain 2d chiral algebra
What's nice about 4d theories w/ twistorial uplifts?

\[ \mathbb{P} \mathcal{T} = \mathcal{O}(1) \oplus \mathcal{O}(1) \simeq \mathbb{R}^4 \times \mathbb{C}\mathbb{P}^1 \]

Penrose transform \[ H^{0,1}(\mathbb{P}\mathcal{T}, \mathcal{O}(2h - 2)) \]

hol'c massless fields on \( \mathbb{C}^4 \)

- \( \mathbb{C}\mathbb{P}^1_x, \mathbb{C}\mathbb{P}^1_y \) intersect iff \( |x - y|^2 = 0 \)
- \( \therefore \) correlation fns entire analytic (rational) fns
  - singularities on lightcone
  - no anomalous dimensions
  - integrability (cf. motivations)

entire analytic functions, can pass to any signature

\[ x \in \mathbb{C}^4 \]

See Costello's Strings `21 talk

[Costello: 2111.08879]
The Setting for Today’s Talk

**full (twisted) string embedding?**

- Lift to 10d?

**a local hol’c theory on \( \mathbb{PT} \)**

- "universal defect" on \( \mathbb{CP}^1 \)
- shrink \( \mathbb{CP}^1 \)

- Mathematically: Koszul dual [Costello, Costello-NP, NP-Williams,...]

**A 2d chiral algebra (hol’c “half”of CFT) on celestial sphere**

- inspired by twisted AdS/CFT program [Costello-Gaiotto, Costello-NP]

**non-unitary 4d CFT (SDYM + “axion”)**

- cf. [Witten, Mason-Skinner,...]
  WIP w/ Costello, Sharma
A local hol’c theory on $\mathbb{PT}$

Classically
[Penrose, Ward]

\[ \int_{\mathbb{PT}} \text{Tr}(B F(0,2)(A)) \rightarrow \int_{\mathbb{P}} \text{Tr}(BF(A)_-) \]

\[ B \in \Omega^2_{\mathbb{R}^4, g} \quad B \in \Omega^2_{\mathbb{P}, g} \]

\[ A \in \Omega^0_{\mathbb{P}, g} \]

6d: free “closed string” (BCOV) sector (twist of 6d $\mathcal{N} = (1,0)$ gauge + 1 tensor)

\[ \frac{1}{2} \int (\partial^{-1} \eta)(\bar{\partial} \eta) + k\hat{\lambda}_g \int \eta \text{tr}(\mathcal{A} \partial \mathcal{A}) \]

4d: scalar field w/ quartic kinetic term cf. Fradkin/Tseytlin, Komargodski/Schwimmer, Riegert,…

\[ \frac{1}{2} \int (\Delta \rho)^2 + k'\hat{\lambda}_g \int \rho(F \wedge F) \]

$g = su(2), su(3), so(8), e_{6,7,8}$

$su(N_f) + \text{matter (} N_f = N_c)$
**Self-dual YM**

\[ A \quad B \]

+ \quad -

form factors:

\[ \text{tr}(B^2)(x) \]

L loops, N insertions →

N-L+1 (-) helicity, arbitrary (+) helicity gluons in QCD (integrand)

**4d OPE:**

\[
\text{Tr}B^2(0)\text{Tr}B^2(x_1)\ldots\text{Tr}B^2(x_{n-1}) \sim \\
\sum_i F_i(x_1, \ldots, x_{n-1}) \mathcal{O}^i(0)
\]

rational, constrained by associativity

\[
\text{tr}(B^2)(0) \text{tr}(B^2)(x) \sim \frac{1}{\|x\|^2} B^{a}_{\alpha_1\beta_1} B^{b}_{\alpha_2\beta_2} B^{c}_{\alpha_3\beta_3} f_{abc} \epsilon^{\beta_1\alpha_2} \epsilon^{\beta_2\alpha_3} \epsilon^{\beta_3\alpha_1}.
\]
4d form factors as computed by 2d chiral correlators

\[
p^{\alpha \dot{\alpha}} = \lambda^{\alpha} \tilde{\lambda}^{\dot{\alpha}}
\]
\[
\lambda^{\alpha} \equiv (1, z)
\]

**tree-level**

\[
\langle tr(B^2) \mid \tilde{J}(z_1) \tilde{J}(z_2) J(z_3) \ldots J(z_n) \rangle \leftrightarrow \text{color ordered MHV amps} \quad \text{[Parke-Taylor]}
\]

\[
\frac{1}{|x|^2} \langle tr(B^3) \mid \tilde{J}(z_1) \tilde{J}(z_2) \tilde{J}(z_3) J(z_4) \ldots J(z_n) \rangle \leftrightarrow \text{NMHV amps in CSW form} \quad \text{[Cachazo-Svrcek-Witten]}
\]

**1-loop (axion comes in)**

\[
\langle (\Delta \rho)^2 \mid J_{a_1}(\tilde{\lambda}_1, z_1) \ldots J_{a_n}(\tilde{\lambda}_n, z_n) \rangle \leftrightarrow \text{all-(+)} \text{ one-loop in SDYM/QCD} \quad \text{[Mahlon, Bern et. al.,...]}\]

\[
\langle tr(B^2) \mid \tilde{J}_{a_1}(\tilde{\lambda}_1, z_n) J_{a_2}(\tilde{\lambda}_2, z_2) \ldots J_{a_n}(\tilde{\lambda}_n, z_n) \rangle = \quad \text{[Costello-NP]}
\]
\[
\frac{1}{192\pi^2} \sum_{2 \leq i < j \leq n} \frac{[ij][1i]^2[1j]^2}{\langle ij \rangle \langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \text{Tr}(t_1 \ldots t_n) + \text{perms in } S_{n-1}
\]

cf. [Dixon-Glover-Khoze] for QCD
Whence a chiral algebra?

1. Generators

\[ J[r, s](z_i) \leftrightarrow A = \delta_{z \equiv z_i} (\tilde{\lambda}^1)^r (\tilde{\lambda}^2)^s \]

State in vacuum module = on-shell background field localized on \( \mathbb{C}P^1 \)

Momentum eigenbasis:

\[ J(\tilde{\lambda}, z) = \sum_{r, s} \omega^{r+s} \frac{(\tilde{\lambda}^1)^r (\tilde{\lambda}^2)^s}{r!s!} J[r, s](z) \]

\[ \langle ij \rangle = z_i - z_j \]

### Table 1: The generators of our 2d chiral algebra and their quantum numbers. Dimension refers to the charge under scaling of \( \mathbb{R}^4 \).
Whence a chiral algebra?

2) Conformal blocks

\[ \mathbb{P} T \setminus \mathbb{C}P^1_0 = S^3 \times \mathbb{C}P^1 \times \mathbb{R}_{>0} \]

\[ \mathbb{R}^4 \setminus 0 = \mathbb{R}_{>0} \times S^3 \quad \mathbb{R}_{>0} \times \mathbb{C}P^1_z \]

\( \mathcal{H}(S^3) := \text{space of local operators} \quad \mathcal{H}(\mathbb{C}P^1) := \text{space of conformal blocks} \)

\( \mathcal{O}(0) \leftrightarrow \langle \mathcal{O} | \)

Chiral algebra as 3d bdy algebra:

hol'c-top'l theory (twist of 3d \( \mathcal{N} = 2 \))

- chiral algebra a boundary condition at \( \infty \)
- conformal block a state at 0
- correlator in 3d bulk/boundary system
Whence a chiral algebra? (3) OPEs

\[ \sum_{r,s \geq 0} \int_{\mathbb{CP}^1_z} (\partial^r \bar{\lambda} \cdot \bar{\partial}^s \mathcal{B}^a \bar{z}) \tilde{J}_a[r, s](z) \]

\[ \sum_{r,s \geq 0} \int_{\mathbb{CP}^1_z} (\partial^r \bar{\lambda} \cdot \bar{\partial}^s \mathcal{A}^a \bar{z}) J_a[r, s](z) \]

Gauge inv't couplings to arbitrary defect
\[ \rightarrow \text{Hom from Koszul dual algebra into defect algebra} \]

Koszul duality:
In twisted holography [Costello, Costello-NP] + WIP
In field theory (includes math review for physicists!) [NP-Williams]

OPEs among currents on defect by imposing gauge (BV-BRST) invariance
\[ \rightarrow \text{universal or "Koszul dual" algebra} \]
Tree level: recover current algebra for gauge symmetry

Quantum deformations:

cf. e.g. integrating out d.o.f. to obtain Wilson lines [Gomis-Passerini]
\[
J^a[r, s](0)J^b[t, u](z) \sim \frac{1}{z} f^a_{cd} J^c[r + t, s + u](0)
\]
\[
J^a[r, s](0)\mathcal{J}^b[t, u](z) \sim \frac{1}{z} f^a_{cd} \mathcal{J}^c[r + t, s + u](0)
\]

Tree-level

\[
J^a[r, s](0)E[t, u](z) \sim \frac{1}{z} \frac{(ts - ur)}{t + u} \mathcal{J}^a[t + r - 1, s + u - 1](0)
\]
\[
J^a[r, s](0)F[t, u](z) \sim -\frac{1}{z} \partial_z \mathcal{J}^a[r + t, s + u](0) - \frac{1}{z^2} \frac{r + s}{t + u + 2} \mathcal{J}^a[r + t, s + u](0)
\]
\[
J^a[r, s](0)J^b[t, u](z) \sim \frac{1}{z} K^{ab}(ru - st)E[r + t - 1, s + u - 1](0)
\]
\[
- \frac{1}{z} K^{ab}(t + u)\partial_z E[r + t, s + u](0) - \frac{1}{z^2} K^{ab}(r + s + t + u)E[r + t, s + u](0).
\]

Failure of associativity in pure SDYM theory in one-loop
Axion field necessary for its restoration

\[
\text{Split}^{[1]}_+(a^+, b^+) = -\frac{N_c}{96\pi^2} \left[ \frac{[ab]}{\langle ab \rangle^2} \right]
\]

Quantum deformation

\[
C, D \text{ fixed by anomaly coefficient in 6d}
\]
## Main Theorems [Costello-Paquette: 2201.02595]

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<th>2d chiral algebra</th>
<th>4d theory</th>
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- Hol’c defect on $\mathbb{CP}_1^0$ in 6d, integrate out 2d fields → state/operator at $0 \in \mathbb{R}^4$
- **Exchange of 6d fields doesn’t contribute to amplitude:** *localization to 2d physics*
- 6d conformal primaries at different points on sphere *talk to each other by exchange of 2d defect fields only*
Final thoughts & future directions

- Top-down topological string model of celestial holography (WIP w/ Costello & Sharma)
- Magnetic monopoles/spectral flow in chiral algebra (WIP w/ Garner)
- Self-dual gravity! (Anomaly cancellation: Bittleston-Sharma-Skinner)
- Compute more QCD (+ matter?) amplitudes: via multi-point axion exchange? via anomaly-cancelling theories for special choices of matter?
Thank you!
Example: the Parke-Taylor Formula

\[ \langle tr(B^2) | \tilde{J}(z_1)\tilde{J}(z_2) J(z_3) \ldots J(z_n) \rangle \leftrightarrow \text{color ordered MHV amps} \]

1. \[ \langle tr(B^2) | \tilde{J}^a(z_1)\tilde{J}^b(z_2) \rangle = K^{ab}(z_1 - z_2)^2 \]  
   direct computation from 6d, or fixed by symmetry arguments

2. \[ \langle tr(B^2) | \tilde{J}^a(z_1)\tilde{J}^b(z_2) J^c(z_3) \rangle = \frac{f_c^{ab}}{z_{23}} \langle tr(B^2) | \tilde{J}^a(z_1)\tilde{J}^d(z_2) \rangle + \frac{f_d^{ca}}{z_{13}} \langle tr(B^2) | \tilde{J}^d(z_1)\tilde{J}^b(z_2) \rangle \]  
   use OPE

\[ = \frac{z_1^3}{z_1 z_2 z_3} f_{abc} \]  
   identity 1.

+ elementary induction:

\[ \langle tr(B^2) | J^{a_1}(z_1) \ldots \tilde{J}^{a_i}(z_i) \ldots \tilde{J}^{a_j}(z_j) \ldots J^{a_n}(z_n) \rangle = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} tr(t^{a_1} \ldots t^{a_n}) + \text{permutations} \]