Fluxes, holography and the uses of exceptional generalised geometry

Daniel Waldram, Imperial College London
20 July 2022

Review talk, *Strings 2022, Vienna*
Overview and motivation

Geometrical backgrounds are ubiquitous in string theory
phenomenology, swampland, holography, . . .

and we have many tools for case without non-trivial fluxes

- Lie groups, cosets $G/H$, special holonomy (Calabi–Yau, $G_2$, Sasaki–Einstein etc), . . .
- $\leadsto$ moduli, spectra, existence of solutions, . . .

What about when there are (large) non-trivial fluxes?

exceptional generalised geometry is a framework to extend standard geometrical constructions to include fluxes

building on history of using $G$-structures and generalised complex geometry
Thank my collaborators

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Exceptional generalised geometry

Supersymmetry and generalised $G$-structures

Consistent truncations

Holography
Exceptional generalised geometry
Set up: compactification geometry $X_D \times M_d$

- **on-shell:** $X$ is Minkowski or AdS warped product

$$ds^2 = e^{2\Delta} ds^2(X) + ds^2(M) + \text{flux on } M$$

no-go theorems for Minkowski $\Rightarrow$ need sources for flux \cite{Maldacena, Nunez 00; Ivanov, Papadopoulos 00; . . .}

- **off-shell:** repackage full $(D+d)$-dim (or truncated) theory as theory on $X$

  scalars: $g_{mn}(y, x)$, $A_{m_1 \ldots m_p}(y, x)$, etc.

  vectors: $g_{\mu m}(y, x)$, $A_{\mu m_1 \ldots m_{p-1}}(y, x)$, etc.

\cite{de Wit, H. Nicolai 86; . . .}
• symmetries of the NSNS fields are diffeos $\xi^\mu$ and gauge transf $\lambda_\mu$

$$\delta g_{mn} = (\mathcal{L}_\xi g)_{mn}, \quad \delta B_{mn} = (\mathcal{L}_\xi B)_{mn} + (d\lambda)_{mn}, \quad \delta \phi = \mathcal{L}_\xi \phi$$

with the algebra $\xi'' = [\xi, \xi']$ and $d\lambda'' = \mathcal{L}_\xi d\lambda' - \mathcal{L}_{\xi'} d\lambda$

• package into generalised tangent space $E \simeq TM \oplus T^*M$

$$V^M = \begin{pmatrix} \xi^m \\ \lambda_m \end{pmatrix} \in \Gamma(E) \quad \text{generalised vector, } M = 1 \ldots, 2d$$

and choose integration to give generalised Lie derivative

$$V'' = \mathcal{L}_V V' = \begin{pmatrix} [\xi, \xi'] \\ \mathcal{L}_\xi \lambda' - \iota_{\xi'} d\lambda \end{pmatrix} \quad \text{why?}$$

[Liu, Weinstein, Xu 97; Hitchin 02; Gualtieri 04]
Generalised geometry II

• preserves the natural $O(d, d)$ metric on $E$

$$\eta_{MN} V^M V^N = V^T \begin{pmatrix} 0 & 1 \frac{1}{2} \mathbb{1} \\ \frac{1}{2} \mathbb{1} & 0 \end{pmatrix} V = \frac{1}{2} \xi^m \lambda_m, \quad L_V \eta = 0$$

• so can extend generalised Lie derivative $L_V$ to

generalized tensor = rep of $O(d, d) \times \mathbb{R}^+ \supset GL(d, \mathbb{R})$

where $\mathbb{R}^+$ weight $p$ counts powers of $(\det T^* M)^p$

Basic idea is to “geometrize the flux”

reformulate supergravity and supersymmetric background geometries in terms of generalised tensors
Generalised Riemannian geometry

- **generalised metric** $G \in \Gamma(S^2E^* \otimes \det T^*M)$
  
  $$G_{MN} = e^{-2\phi} \sqrt{g} \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}_{MN}$$

  invariant under $O(d) \times O(d)$ subgroup

- **family of generalised Levi–Civita connections** $DG = 0$ and $T(D) = 0$
  
  $$(D_V W)^M = \xi^\mu \left( \partial_\mu W^M + \Omega^M_{\mu N} W^N \right) + \lambda_\mu (\tilde{\Omega}^M_{\mu N} W^N)$$

  where $T(D) \in \Gamma(\Lambda^3E \oplus E)$

- **Ricci tensor is unique and gives NSNS equations of motion, $R_{MN} = 0$**
  
  $$\int_M \text{vol}_G R = \int_M \sqrt{g} e^{-2\phi} \left( \mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right),$$

  other field $RR = O(d, d) \times \mathbb{R}^+$ spinors; fermions $= \text{Spin}(d) \times \text{Spin}(d)$ spinors

[Siegel 93; Hohm, Kwak 10; Jeon, Lee, Park 11; Coimbra, Strick.-Const. DW 13]
Exceptional generalised geometry

[ Hull 07; Pacheco, DW 08; Berman, Perry 10; Coimbra, Strick.-Const. DW 13]

How extend to RR sector? M-theory? \( F = dA, \tilde{F} = *F = d\tilde{A} - \frac{1}{2} A \wedge F \)

\[
\delta g = \mathcal{L}_{\xi} g, \quad \delta A = \mathcal{L}_{\xi} A + d\omega, \quad \delta \tilde{A} = \mathcal{L}_{\xi} \tilde{A} + d\sigma - \frac{1}{2} d\omega \wedge A
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- for example $d = 6$: $E \simeq TM \oplus \Lambda^2 T^* M \oplus \Lambda^5 T^* M$

$\mathcal{L}_\nu V' = [\xi, \xi'] + (\mathcal{L}_\xi \omega' - \iota_{\xi'} d\omega) + (\mathcal{L}_\xi \sigma' - \iota_{\xi'} d\sigma - \omega' \wedge d\omega)$

preserves $E_{6(6)} \times \mathbb{R}^+$ cubic invariant $c_{MNP} V^M V^N V^P$ and $E \sim 27_{1/2}$
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- gen metric $G_{MN}$ invariant under $\text{USp}(8) \subset E_{6(6)} \times \mathbb{R}^+$

bosonic supergravity on $M = \text{generalised Einstein gravity}$

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- Extend to $E_{d(d)}$ ($d \leq 7$) and IIB by different $\text{GL}(d - 1, \mathbb{R}) \subset E_{d(d)} \times \mathbb{R}^+$

$$E \simeq TM \oplus 2 T^* M \oplus \Lambda^3 T^* M \oplus 2 \Lambda^5 T^* M$$
Formalism

- reformulation of full 11d M-theory on $X \times M$ [Hohm, Samtleben 13]
  
  scalars: $G_{MN}(x, y) \in \Gamma(S^2 E^* \otimes \det T^* M)$
  
  vectors: $A^{\mu M}(x, y) = (g^{\mu n}, A_{\mu mn}, \tilde{A}_{\mu m_1...m_5}) \in \Gamma(T^* X \otimes E)$ etc

- $O(d, d + n) \times \mathbb{R}^+$ description of heterotic

- DFT/ExFT: extend spacetime ($T_p M = T_p X \oplus E_p$), locally same formalism [Hull, Zwiebach 09] [Hohm, Samtleben 13; ...]

- $d > 7?$ [Hohm, Samtleben 14; Bossard, Ciceri, Inverso, Kleinschmidt, Samtleben 19, 21]
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Higher-derivative corrections? difficult: need to modify $L_V$ for $\alpha'$ and M-theory

\cite{Hohm, Zweibach 14; Marques, Nuñez 15, ...} \cite{Coimbra, Minasian 17; Coimbra 19} \cite{Bossard, Kleinschmidt 15}
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Why this structure? "$G$-algebroid", descends from $L_\infty$ symmetry of closed SFT [Bugden, Hulik, Valach, DW 21] [Sen 16; Arvanitakis, Hohm, Hull, Lekeu 20,21;…]
Supersymmetry and generalised $G$-structures
Conventional $G$-structures

Supersymmetric bkgrd $\Rightarrow$ new geometric structure on $M$ eg cplx structure

- **topological**: “almost complex structure”

  \[ T_C M = T^{1,0} \oplus T^{0,1} \iff \text{global tensor } I^{m^p} \]

  reduction of structure group of $TM$ to $\text{GL}(n, \mathbb{C}) \subset \text{GL}(2n, \mathbb{R})$

- **differential**: “integrable complex structure”

  \[ [T^{1,0}, T^{1,0}] \subset T^{1,0} \text{ involutive} \]

  \[ \iff N_{mn^p} = I^p_q \partial_m I^q_n - I^p_q \partial_n I^q_m - I^q_m \partial_q I^p_n + I^q_n \partial_q I^p_m = 0 \]

  or, there exists a connection $\nabla$ such that

  \[ \nabla I = 0 \text{ \ "compatible" } \quad T(\nabla) = 0 \text{ \ "torsion-free"} \]
Supersymmetric backgrounds

\[ \delta(\text{fermion}) = (\nabla^L + \text{flux}) \epsilon = 0 \quad \text{Killing spinor eqns} \]
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\[ \delta(\text{fermion}) = (\nabla^{\text{LC}} + \text{flux})\epsilon = 0 \]  

Killing spinor eqns

- no flux ⇒ special holonomy ⇒ integrable \( G \subset SO(d) \) structure e.g.

  - \( SU(n) \subset SO(2n) \) \hspace{1cm} Calabi–Yau \hspace{1cm} \( d\omega = d\Omega = 0 \)
  - \( G_2 \subset SO(7) \) \hspace{1cm} Joyce \hspace{1cm} \( d\phi = d*\phi = 0 \) etc.

- flux ⇒ non-integrable, local \( G \subset SO(d) \) structure

  \[ d(\text{structure form}) = \text{flux} \text{ "intrinsic torsion" } \]

  \( \Rightarrow \) classification, new solutions, but e.g. moduli hard

  [Gauntlett, Martelli, Pakis, DW 02; Gutowski, Hull, Pakis, Reall 02, . . . ]

- for type II: \( O(d, d) \times \mathbb{R}^+ \) geometrizes NSNS flux

  \[ d\Phi^+ = 0, \hspace{0.5cm} d\Phi^- = \text{RR flux} \]

[Hitchin 02; Gualtieri 04; Graña, Minasian, Petrini, Tomasiello 04,05, . . . ]
Supersymmetric backgrounds

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[Hitchin 02; Gualtieri 04; Graña, Minasian, Petrini, Tomasiello 04,05; ...]
By analogy, **generalised $G$-structure** on $E$ for $G \subset E_{d(d)} \times \mathbb{R}^+$

**Theorem:** Generic supersymmetric flux backgrounds (M-theory, type II, $d \leq 7$) are equivalent to

- **Minkowski** $\Rightarrow G \subset H_d$ integrable structure
- **AdS** $\Rightarrow G \subset H_d$ structure with singlet intrinsic torsion

where $G$ is the stabiliser group of the Killing spinor(s)
Generalised $G$-structures and supersymmetry

[Coimbra, Strick.-Const., DW 14; Coimbra, Strick.-Const. 16, 17]

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for example $D = 4$: susy parameter $\epsilon$ in 8 of $H_7 = SU(8) \subset E_{7(7)} \times \mathbb{R}^+$

- $\mathcal{N} = 1$ $\quad G = \text{Stab}(\epsilon_1) = SU(7)$
- $\mathcal{N} = 2$ $\quad G = \text{Stab}(\epsilon_1, \epsilon_2) = SU(6)$ etc
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Analogue of special holonomy $\leadsto$ classification, new solutions, moduli
Example: “generalising $G_2$” $\mathcal{N} = 1$, $D = 4$ in M-theory

[Ashmore, Strick.-Const., Tennyson, DW 19] [c.f. Lukas, Saffin 04]

For SU(7), generalised invariant tensor in $912_{3/2}$,

$$\psi^{MNP} \in \Gamma(W_C) \quad W \cong \mathbb{R} \oplus \Lambda^3 T^* M \oplus (T^* M \otimes \Lambda^5 T^* M) \oplus \ldots$$

viewing 11d M-theory as 4d $\mathcal{N} = 1$ on $X$ with chiral matter $\psi$

$$\mathcal{Z} = \{\text{SU}(7) \text{ structures } \psi\} \quad \infty\text{-dim Kähler manifold}$$
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- F-terms: from superpotential $\Leftrightarrow$ involutive SU(7)-inv sub-bundle

$$L_{C_3} C_3 \subset C_3 \quad E_{\mathbb{C}} \cong C_3 \oplus C_{-1} \oplus C_{-3} \oplus C_1$$

$$56 = 7 \oplus 21 \oplus 7 \oplus 2\bar{1}$$
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- **D-terms**: moment map for GDiff symmetry acting on $\mathcal{Z}$ by $\delta \psi = L_V \psi$

  $$\mu(V) = 0, \quad \forall V \in \Gamma(E) \simeq \text{gdiff}$$
Typical of supersymmetry conditions: first solve F-terms (holomorphic)

- $\mathcal{Z}$ is Kähler (infinite-dimensional) with group action $G$
- orbits of $G_{\mathbb{C}}$ intersect $\mu = 0$ if “stable” – algebraic condition

Kähler–Einstein, Sasaki–Einstein, Hermitian Yang-Mills, ...

[Yau; Tian; Donaldson, ...]

[9x256]Symplectic quotient/GIT

[28x88]• $Z$ is Kähler (infinite-dimensional) with group action $G$

[28x73]• orbits of $G_{\mathbb{C}}$ intersect $\mu = 0$ if “stable” – algebraic condition

[28x53]Kähler–Einstein, Sasaki–Einstein, Hermitian Yang-Mills, ...

[31x36]Yau; Tian; Donaldson, ...]
Involutive structure defines a complex

\[ \cdots \xrightarrow{\text{d}_C} \Lambda^p C^*_3 \xrightarrow{\text{d}_C} \Lambda^{p+1} C^*_3 \xrightarrow{\text{d}_C} \cdots \]

- deforming \( \psi \) but not flux sources gives

\[
\text{local moduli space} \simeq H^3(\Lambda^* C^*_3, d_C) \oplus H^6(\Lambda^* C^*_3, d_C) \\
\simeq H^3_{dR}(M, \mathbb{C}) \oplus H^6_{dR}(M, \mathbb{C})
\]

- same as \( G_2 \)! however currently no good flux examples ...
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Can extend to type II (e.g. GMTP backgrounds)

- other new results e.g. moduli of Graña–Polchinksi background (matches naive superpotential expectation) [Ashmore, Stric.-Const., Tennyson, DW 19; Smith, Tennyson, DW w.i.p.]
Further directions and extensions

Other dimensions and amounts of supersymmetry

- $\frac{1}{4}$-susy: “exceptional Calabi–Yau” including moduli [Ashmore, DW 15];
- $\frac{1}{2}$-susy: (Mink. and AdS) [Malek 17]

- heterotic Hull–Strominger system including moduli [Ashmore, Strick.-Const., Tennyson, DW 19] [cf de la Ossa, Svanes 14; Garcia-Fernandez, Rubio, Tipler 19]
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Kähler potential on $\mathcal{Z}$ gives “exceptional Hitchin functionals”

- SU(7) structure extends $G_2$, ECY extends cplx-struct functional
- quantisation and topological theories? (heterotic [Svanes, Tennyson w.i.p.])
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Existence of solutions from stability?

- for extended $G_2$: $d\phi = 0$ then vary in $H^{3}_{dR}(M)$ for $d \ast \phi = 0$
- toric backgrounds for type II and ECY?
Consistent truncations
Basic idea

Solutions of truncated theory are solutions of full theory

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} M^2 \phi^2 + \frac{1}{2} (\partial \pi)^2 - \frac{1}{2} m^2 \pi^2 - \lambda \phi \pi^2 \]

\[ (\partial^2 + M^2)\phi = -\lambda \pi^2 \quad (\partial^2 + m^2)\pi = -2\lambda \phi \pi \]

- truncate to \(\phi \surd\), truncate to \(\pi \times\), even if \(M \gg m\)
- symmetry: keep singlets under \(\mathbb{Z}_2\) where \(\phi \rightarrow \phi, \pi \rightarrow -\pi\)

Usually fields come from Kaluza–Klein modes in compactification

- gives consistent “uplift” of dimensionally reduced theory
- closed sector at large \(N\) in holography
- (partial) check of stability (AdS swampland conjecture)
Long history searching for suitable ansätze

- **Scherk–Schwarz**: \( M = \mathcal{G} \), expand in \((\text{left-})\text{invariant objects}\) on \( M \)

\[
g^{\mu \nu} = \phi^{a b}(y) \hat{e}^a(\mathbf{x}) \hat{e}^b(\mathbf{x}), \quad \text{etc} \quad [\hat{e}_a, \hat{e}_b] = f_{a b}^\ c \hat{e}_c
\]

giving theory with maximal susy \cite{Scherk, Schwarz 79}

- “mysterious spheres”: \( S^4 \) and \( S^7 \) in M-theory, \( S^5 \) in IIB, complicated ansatz, maximal susy \cite{de Wit, Nicolai 87; Natase, Vaman, van Nieuwenhuizen 99}

- conv. \( G \)-structure with constant singlet torsion \cite{Gauntlett, Kim, Varela, DW 09; Cassani, Dall’Agata, Faedo 10; Gauntlett, Varela 10; \ldots}
General framework

Long history searching for suitable ansätze

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$$g^{\mu\nu} = \phi^{ab}(y) \hat{e}_a^\mu(x) \hat{e}_b^\nu(x), \text{ etc } \quad [\hat{e}_a, \hat{e}_b] = f_{ab}^c \hat{e}_c$$

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**General picture?**

**Theorem**: Given a generalised $G$-structure on $M$ with constant, singlet intrinsic torsion, keeping all $G$-invariant fields gives consistent truncation of M-theory or type II on $M$ [Cassani, Josse, Petrini, DW 19]
Maximal susy: “generalised Scherk–Schwarz”, \( G = \mathbb{1} \) “trivial structure”

\( E \) is parallelisable, admits global frame, \( M = G_E/H_E \) where \( g_E = a/i \)

invariant gen. tensors: \( \hat{E}_A \in \Gamma(E) \) basis for \( E \)

singlet torsion: \( L_{\hat{E}_A} \hat{E}_B = X_{AB}^C \hat{E}_C \) Leibniz alg. \( a \)

scalars \( G^{MN} = \phi^{AB}(y) \hat{E}_A^M \hat{E}_B^N \), vectors \( A_{\hat{\mu}}^M = a_{\hat{\mu}}^A(y) \hat{E}_A^M \) etc.
Generalised Scherk–Schwarz

Maximal susy: “generalised Scherk–Schwarz”, $G = 1$ “trivial structure”

$E$ is parallelisable, admits global frame, $M = G_E/H_E$ where $g_E = a/i$

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scalars $G^{MN} = \phi^{AB}(y)\hat{E}_A^M \hat{E}_B^N$, vectors $A^M_{\mu} = a^A_{\mu}(y)\hat{E}_A^M$ etc.

• “mysterious spheres” are generalised parallelisable [Lee, Strick.-Const., DW 14]

• $\sim$ gauged maximal supergravity, embedding tensor $X_{AB}^C$

[de Wit, Nicolai 87; Hull, Reid-Edwards 05; Geissbuhler 11; Graña, Marquéz 12; Berman, Musaev, Thompson 12; Godazgar, Godazgar, Nicolai 13; ...]
Developments

For generalised Scherk–Schwarz

- **full consistency** of IIB $S^5$ truncation (and massive IIA $S^6$) [Baguet, Hohm, Samtleben 15; ...] [also Ciceri, de Wit, Varela 14]

- **reduction to algebraic problem** [Inverso 17; Bugden, Hulik, Valach, DW 21]
  → classification of all compact simple gaugings [Valach, DW w.i.p.]

- **Poisson–Lie U-duality**: $\{\hat{E}_A\}$ and $\{\hat{E}'_A\}$ give same algebra $\alpha$ [Sakatani 20; Malek, Thompson 20; Bugden, Hulik, Valach, DW 21]

- **no dyonic** $N=8$ SO(8) gaugings, but other dyonic gaugings possible [Lee, Strick.-Const., DW 15; Guarino, Jafferis, Varela 15; Inverso, Samtleben, Trigiante 16]

- **all** $\frac{1}{2}$-susy $D=5$, 6, 7 gaugings [Malek, Samtleben 19; Malek, Vall Camell 20]

- **all** $\frac{1}{4}$-susy $D=5$ gaugings [Josse, Malek, Petrini, DW 21]

- **proof of “pure supergravity” conjecture** of [Gauntlett, Varela 19]
  also $d>7$ and many other cases (Maldacena–Nunez, $\beta$-deformed, etc ... )
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- Poisson–Lie U-duality: $\{\hat{E}_A\}$ and $\{\hat{E}'_A\}$ give same algebra $\mathfrak{a}$ [Sakatani 20; Malek, Thompson 20; Bugden, Hulik, Valach, DW 21]

Mapping the landscape

- no dyonic $\mathcal{N} = 8$ SO(8) gaugings, but other dyonic gaugings possible [Lee, Strick.-Const., DW 15; Guarino, Jafferis, Varela 15; Inverso, Samtleben, Trigiante 16]

- all $\frac{1}{2}$-susy $D = 5, 6, 7$ gaugings [Malek, Samtleben 19; Malek, Vall Camell 20];
  all $\frac{1}{4}$-susy $D = 5$ gaugings [Josse, Malek, Petrini, DW 21]

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Kaluza–Klein spectroscopy

[Malek, Samtleben 19,20]

Can you do more? Complete spectrum?? Still determined by $a$??

- potential $V(\phi)$ for truncation scalars $\phi^{AB}$ has several AdS extrema
- $M = \text{SO}(d+1)/\text{SO}(d)$ $\Rightarrow$ expand fluctuations in $\text{SO}(d+1)$ reps $r_i$; only mass eigenstates for round sphere
- however mass matrix only depends on $r_i$, $\phi^{AB}$ and $X_{AB}^C$
- gives full spectrum at any extremum (in BPS multiplets if supersymmetric)
- includes example with no isometries [Cesaro, Larios, Varela 21]
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AdS swampland conjecture: no stable, non-susy AdS bkgds [Ooguri, Vafa 16]

- $S^7$ in M-theory $\text{SO}(3) \times \text{SO}(3)$ extremum: unstable to higher KK modes
  [Malek, Nicolai, Samtleben 20]
- $S^6$ in massive IIA $G_2$ extremum: perturbatively stable!
  [Guarino, Malek, Samtleben 20,21]
- IIB S-fold gauging: conformal manifold of non-susy perturb. stable vacua!
  [Giambrone, Guarino, Malek, Samtleben, Sterckx, Trigiante 21]
Implications and new directions

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Full spectrum of conformal dimensions in holographic dual $\rightsquigarrow$ topology of conformal manifold [Giambrone, Malek, Samtleben, Trigiante 21; GGMSST 21]
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Full spectrum of conformal dimensions in holographic dual $\leadsto$ topology of conformal manifold [Giambrone, Malek, Samtleben, Trigiante 21; GGMSST 21]

Cubic interactions? Consistent truncations with less susy? susy breaking deformations? . . . many new possibilities
Holography
Old problem: $\mathcal{N} = 1$ marginal deformations for $\mathcal{N} = 4$

Superpotential deformation \cite{Leigh, Strassler 95}

$$\Delta \mathcal{W} = \frac{1}{2} \lambda_1 \text{tr} \Phi^1 \Phi^2 \Phi^3 + \frac{1}{6} \lambda_2 \text{tr} \left[ (\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3 \right]$$

Gravity dual? deforming $S^5$ and adding fluxes

- $\lambda_2 = 0$: "$\beta$-deformation", $U(1)^3$ isometry, exact dual \cite{Lunin, Maldacena 05}
- $\lambda_1 \neq 0, \lambda_2 \neq 0$: tour de force to 2nd/3rd order \cite{Aharony, Kol, Yankielowicz 02}
  only $U(1)_R$ isometry, as hard as finding explicit Calabi–Yau metrics
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Superpotential deformation [Leigh, Strassler 95]

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But ... much of field theory quite simple, since only depends on holomorphic structure. Is there a generic supergravity analogue?

Can we understand the dual geometry beyond classic Sasaki–Einstein examples?
Inv. tensors define $USp(6) \subset E_6(6) \times \mathbb{R}^+$ structure $\Rightarrow g_{mn}, B^i_{mn}, C_{mnpq}, \phi, \chi, \Delta$

$V$ structure, $K$

\[ E \simeq TM \oplus 2 T^* M \oplus \Lambda^3 T^* M \oplus 2\Lambda^5 T^* M \]

$H$ structure, $X$

\[ \text{ad} \tilde{F}_C \otimes \text{det} T^* M \simeq T^* M \oplus 2\Lambda^3 T^* M \oplus \ldots \]

Differential conditions: singlet intrinsic torsion

- **F-terms**: $X$ defines involutive sub-bundle

\[ E_C \simeq C_+ \oplus C_- \oplus C_0, \quad L_{C+} C_+ \subset C_+ \]

- **D-terms**: moment map for $GDiff$, generated by $L_V$

\[ \mu(V) = -\frac{i}{4} \int_M \text{tr} X(L_V \bar{X}) = \int_M c(K, K, V) \quad \forall V \in \mathfrak{gdiff} \]

- **R-charges**: $L_K X = 3iX$ and $L_K K = 0$
For Sasaki–Einstein writing $\tau = \chi + i e^{-2\phi}$ and $u^i = \tau_2^{-2}(\tau, 1)^i$

\[
\begin{align*}
X &= -\frac{1}{2}i u^i e^{\frac{1}{4}i\Omega \wedge \bar{\Omega}} \cdot \sigma \wedge \Omega, \\
K &= e^{C} \cdot (\xi - \sigma \wedge \omega),
\end{align*}
\]

"Cauchy–Riemann structure"

"contact structure"

Universal form of central charge

\[
a^{-1} \propto \int_M c(K, K, K)
\]

SCFT result [Kol 02,03; Green, Komargodski, Seiberg, Tachikawa, Wecht 10]

all marginal deformations are in the superpotential and are all exactly marginal unless there is a global symmetry

follows directly from moment map structure of ESE [Ashmore, Gabella, Graña, Petrini, DW 16]

What about the missing deformed solutions? Analogue with CY theorem …
Exceptional Sasaki geometry and the superpotential dual

[Ashmore, Petrini, Tasker, DW 21]

Using the GIT picture
Exceptional Sasaki geometry and the superpotential dual

[Ashmore, Petrini, Tasker, DW 21]

Using the GIT picture

- “Exceptional Sasaki” = relax D-term (n.b. Sasaki ⊂ ES)
- GDiff\_C orbit generated by $\delta X = L_V X$ with $V \in \Gamma(E_C) \simeq \text{gdiff}_C$ and intersects moment map condition on ESE background

Physically

$$\text{orbit } [X] = \{ X' : X' = \text{GDiff}_C \cdot X \}$$ encodes superpotential $\mathcal{W}$

- $\delta X = L_V X$ part of long vector deforming Kähler potential
- orbit is renormalisation flow of Kähler potential; class $[X]$ does not change for domain wall flow – non-renormalization of $\mathcal{W}$

$$X' = -\frac{1}{3} i L_K X, \quad K^* = \mu \quad \text{where } K^*_M = c_{MNP} K^N K^P$$
We find new family of Exceptional Sasaki solutions with $\mathcal{L}_\xi f = 3i f$

$$K = e^c \cdot (\xi - \sigma \wedge \omega)$$

$$X = e^{b^i(\tau, f) + c} \cdot (df + \nu^i(\tau, f) \sigma \wedge \Omega)$$

with $b^i \in \Gamma(\wedge^2 T^*_C M)$ linear in $f$ and $\nu^i$ quadratic in $f$

- complicated deformed metric $g$, axion-dilaton and fluxes
- valid for marginal deformation of any Sasaki–Einstein
- for $S^5$ matches to 2nd order [Aharony et al 02] with

$$f = \frac{1}{6} d_{ijk} z^i z^j z^k$$

on CY cone over SE

and same discrete symmetries as $\Delta W$ [cf Baggio, Bobev, Chester, Lauria, Pufu 17]

- for $f = z^1 z^2 z^3$ on $S^5$ gives $GDiff_C$ of LM solution
ESE solution exists in open neighbourhood of Sasaki–Einstein point

- moment map $\tilde{\mu} = \mu - K^*$ for $\text{GDiff}_K$ (preserves $K$)
- stable points form open set in $\mathcal{Z}$ (Kempf–Ness + no additional sym)
- Monge–Ampère-type equation?
Spectrum of single trace chiral operators

Count single-trace mesonic operators $\text{tr } \Phi^{i_1} \cdots \Phi^{i_n}$ of R-charge $\frac{2}{3} k$

- deformations of $C_+$, counted by cohomology of

  \[ \cdots \xrightarrow{dC} \Lambda^p C^*_+ \xrightarrow{dC} \Lambda^{p+1} C^*_+ \xrightarrow{dC} \cdots \]

  independent of choice of $X$ in class $[X]$

  i.e. holomorphic data and can calculate at ES point

- for SE gives “transverse Dolbeault cohomology” [Eager, Schmude, Tachikawa 12]

- if $\eta = df$ nowhere vanishing (not $\beta$-def, not $Y^{p,q}$) gives “$\eta$-cohomology”

  \[ \cdots \xrightarrow{d} \eta \wedge \Lambda^p T^*_C M \xrightarrow{d} \eta \wedge \Lambda^{p+1} T^*_C M \xrightarrow{d} \cdots \]

  can calculate using transverse Dolbeault [Tasker 21]
New results

- **universal result** for Hilbert series, counting chirals with R-charge \( \frac{2}{3}k \)

\[
\tilde{H}(t) = \sum_k (\text{# of chiral ops.}) t^{2k} = 1 + I_{s.t.}(t) - \frac{t^6}{1 - t^6}
\]

where \( I_{s.t.}(t) \) is single trace index

- for regular Sasaki–Einstein

  \( S^5 \) : \( \tilde{H}(t^{1/2}) = \frac{(1 + t)^3}{1 - t^3} \) matches math \( HC_0(k) \) [van den Bergh]

  \( T^{1,1} \) : \( \tilde{H}(t^{1/3}) = \frac{1 + 4t + 2t^2}{1 - t^2} \) prediction (checked to level 7)

  \( dP_6 \) : \( \tilde{H}(t^{1/6}) = \frac{1 + 7t}{1 - t} \) prediction
Future directions

Key point is that $[X]$ captures holomorphic information:

- **same formalism for M-theory** (MN, BBBW, ...): count chirals?
- **3d $\mathcal{N} = 2$ theories and SE$_7$ deformations; $d > 7$ and relation to spindles?**
- **superconformal index** from $(\Lambda^p C^*_+, d_C)$ complex via localisation on $M$

  [Ashmore, Smith, Tasker, Tennyson, DW w.i.p.]

should reduce to holomorphic structure of probe geometry

From moment map/GIT:

- **general picture of dual of $a$-max$/\mathcal{F}$-max** [Ashmore, Petrini, DW w.i.p.]
- **extension of “K-stability” of SE and existence of solutions; relation to flow in QFT?** [cf Collins, Xie, Yau 16; Fazzi, Tomasiello 19]
Exceptional generalised geometry $\rightsquigarrow$ new results for flux backgrounds

- moduli for flux compactifications, but need more examples
- mapping out landscape of consistent truncations
- powerful new Kaluza–Klein spectroscopy
- new holomorphic structures in holography
Exceptional generalised geometry $\rightsquigarrow$ new results for flux backgrounds

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*Thank you!*