Symmetries from string theory

Iñaki García Etxebarria

Based on

- 1908.08027 with B. Heidenreich and D. Regalado,
- 2112.02092 with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki

Durham University

Department of Mathematical Sciences

Simons Collaboration on Global Categorical Symmetries

QFTs from geometry

This is a talk about "geometric engineering": I will view string theory as a tool for associating Quantum Field Theories (QFTs) $\mathcal{T}[X]$ to singular manifolds X, and I will explain how the generalised symmetries of $\mathcal{T}[X]$ can be read from X.

QFTs from geometry

This is a talk about "geometric engineering": I will view string theory as a tool for associating Quantum Field Theories (QFTs) $\mathcal{T}[X]$ to singular manifolds X, and I will explain how the generalised symmetries of $\mathcal{T}[X]$ can be read from X.

More precisely: to any given theory $\mathcal{T}[X]$ we can associate a "symmetry TFT" $\mathrm{Symm}[\mathcal{T}[X]]$, a TFT in one dimension higher encoding symmetries and anomalies of the theory, and all its gaugings.

QFTs from geometry

This is a talk about "geometric engineering": I will view string theory as a tool for associating Quantum Field Theories (QFTs) $\mathcal{T}[X]$ to singular manifolds X, and I will explain how the generalised symmetries of $\mathcal{T}[X]$ can be read from X.

More precisely: to any given theory $\mathcal{T}[X]$ we can associate a "symmetry TFT" $\mathrm{Symm}[\mathcal{T}[X]]$, a TFT in one dimension higher encoding symmetries and anomalies of the theory, and all its gaugings.

It turns out that $\mathrm{Symm}[\mathcal{T}[X]]$ is significantly easier to understand than $\mathcal{T}[X]$ itself, so our goal will be to construct $\mathrm{Symm}[X] \coloneqq \mathrm{Symm}[\mathcal{T}[X]]$ directly from the geometry.

Why

Given a Lagrangian description of $\mathcal{T}[X]$ it is in principle possible (but subtle) to find its generalised symmetry structure.

Why

Given a Lagrangian description of $\mathcal{T}[X]$ it is in principle possible (but subtle) to find its generalised symmetry structure. (See Rudelius' and Ohmori's talks for nice reviews of generalised symmetries, but in a nutshell: symmetry $=$ topological operator.)

Why

Given a Lagrangian description of $\mathcal{T}[X]$ it is in principle possible (but subtle) to find its generalised symmetry structure. (See Rudelius' and Ohmori's talks for nice reviews of generalised symmetries, but in a nutshell: symmetry $=$ topological operator.)

Lots of progress on this in the last few years, see the talks by Rudelius, Ünsal, Komargodski, Nardoni and Ohmori in this conference for beautiful examples of symmetries and their applications.

Why

Given a Lagrangian description of $\mathcal{T}[X]$ it is in principle possible (but subtle) to find its generalised symmetry structure. (See Rudelius' and Ohmori's talks for nice reviews of generalised symmetries, but in a nutshell: symmetry $=$ topological operator.)

Lots of progress on this in the last few years, see the talks by Rudelius, Ünsal, Komargodski, Nardoni and Ohmori in this conference for beautiful examples of symmetries and their applications.

Nevertheless, in the context of geometric engineering having a Lagrangian description of $\mathcal{T}[X]$ is more the exception than the rule: what we know is the topology (and sometimes metric) of X.

Why

Given a Lagrangian description of $\mathcal{T}[X]$ it is in principle possible (but subtle) to find its generalised symmetry structure. (See Rudelius' and Ohmori's talks for nice reviews of generalised symmetries, but in a nutshell: symmetry $=$ topological operator.)

Lots of progress on this in the last few years, see the talks by Rudelius, Ünsal, Komargodski, Nardoni and Ohmori in this conference for beautiful examples of symmetries and their applications.

Nevertheless, in the context of geometric engineering having a Lagrangian description of $\mathcal{T}[X]$ is more the exception than the rule: what we know is the topology (and sometimes metric) of X.

It is precisely in the cases where we don't know a Lagrangian that the information about symmetries and anomalies is most valuable, for example to suggest/test dualities.

Why

A more formal reason to care about this problem is that it hints towards a geometric version of the Landau paradigm: as we will see the map $X \to \mathrm{Symm}[X]$ is very sensitive to the details of X.

Why

A more formal reason to care about this problem is that it hints towards a geometric version of the Landau paradigm: as we will see the map $X \to \mathrm{Symm}[X]$ is very sensitive to the details of X.

Geometric Landau question

Can we reconstruct X (modulo string dualities and deformations) given $\mathrm{Symm}[X]$?

Why

A more formal reason to care about this problem is that it hints towards a geometric version of the Landau paradigm: as we will see the map $X \to \mathrm{Symm}[X]$ is very sensitive to the details of X.

Geometric Landau question

Can we reconstruct X (modulo string dualities and deformations) given $\mathrm{Symm}[X]$?

(In terms of Shlomo's analogies on Tuesday: which pieces of the skeleton do you need to recognise which animal it is?)

Why

A more formal reason to care about this problem is that it hints towards a geometric version of the Landau paradigm: as we will see the map $X \to \mathrm{Symm}[X]$ is very sensitive to the details of X.

Geometric Landau question

Can we reconstruct X (modulo string dualities and deformations) given $\mathrm{Symm}[X]$?

(In terms of Shlomo's analogies on Tuesday: which pieces of the skeleton do you need to recognise which animal it is?)

There is a categorical version of this question, where we ask about some category associated to X instead. For instance, in some cases we can associate a cluster category to X. The Grothendieck group of this cluster category is easy to read from $\mathrm{Symm}[X]$. [Caorsi, Cecotti '17], [Del Zotto, IGE, Hosseini '20], [Del Zotto, IGE '22].

Geometric engineering

For reasons of analytic control we want to impose restrictions on the manifolds X that we consider. These are:

- X is non-compact, to decouple gravity. To make our life simpler I'll assume that X is a real cone over some base B.

Geometric engineering

For reasons of analytic control we want to impose restrictions on the manifolds X that we consider. These are:

- X is non-compact, to decouple gravity. To make our life simpler I'll assume that X is a real cone over some base B.
- In order for the field theory to be supersymmetric, we assume that X has reduced holonomy (Calabi-Yau, for instance).

Geometric engineering

For reasons of analytic control we want to impose restrictions on the manifolds X that we consider. These are:

- X is non-compact, to decouple gravity. To make our life simpler I'll assume that X is a real cone over some base B.
- In order for the field theory to be supersymmetric, we assume that X has reduced holonomy (Calabi-Yau, for instance).

For instance, if X is a complex two-fold, these assumptions restrict it to be an ALE space of the form $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$, with $\Gamma_{\mathfrak{g}} \subset SU(2)$. This is a cone over $S^3/\Gamma_{\mathfrak{g}}$, with $\Gamma_{\mathfrak{g}}$ acting freely on S^3. On $\mathbb{C}^2$ the origin is fixed by all elements of $\Gamma_{\mathfrak{g}}$, so we have an orbifold singularity there.

Geometric engineering

For reasons of analytic control we want to impose restrictions on the manifolds X that we consider. These are:

- X is non-compact, to decouple gravity. To make our life simpler I'll assume that X is a real cone over some base B.
- In order for the field theory to be supersymmetric, we assume that X has reduced holonomy (Calabi-Yau, for instance).

For instance, if X is a complex two-fold, these assumptions restrict it to be an ALE space of the form $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$, with $\Gamma_{\mathfrak{g}} \subset SU(2)$. This is a cone over $S^3/\Gamma_{\mathfrak{g}}$, with $\Gamma_{\mathfrak{g}}$ acting freely on S^3. On $\mathbb{C}^2$ the origin is fixed by all elements of $\Gamma_{\mathfrak{g}}$, so we have an orbifold singularity there.

If we place IIB string theory on this geometry we obtain a $(2,0)$ SCFT $\mathfrak{g}_{(2,0)}$ in six dimensions, arising from modes at the singularity. These theories are believed to be indexed by $\Gamma_{\mathfrak{g}}$, or equivalently by an algebra $\mathfrak{g}$ of type $\mathfrak{a}_n$, $\mathfrak{d}_n$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ or $\mathfrak{e}_8$.

Local vs global

One important property of the $(2,0)$ theory with algebra $\mathfrak{g}$ is that upon reduction on T^2 with complex structure τ it gives rise to 4d $\mathcal{N}=4$ SYM with algebra $\mathfrak{g}$ and complexified gauge coupling τ. Let me call this object $\mathfrak{g}_4$.

Local vs global

One important property of the $(2,0)$ theory with algebra $\mathfrak{g}$ is that upon reduction on T^2 with complex structure τ it gives rise to 4d $\mathcal{N}=4$ SYM with algebra $\mathfrak{g}$ and complexified gauge coupling τ. Let me call this object $\mathfrak{g}_4$.

What I have just described fully specifies the behaviour of local operators in $\mathfrak{g}_4$, but it does not fully fix the theory. For example it does not fully fix the partition function on compact manifolds.

Local vs global

One important property of the $(2,0)$ theory with algebra $\mathfrak{g}$ is that upon reduction on T^2 with complex structure τ it gives rise to 4d $\mathcal{N}=4$ SYM with algebra $\mathfrak{g}$ and complexified gauge coupling τ. Let me call this object $\mathfrak{g}_4$.

What I have just described fully specifies the behaviour of local operators in $\mathfrak{g}_4$, but it does not fully fix the theory. For example it does not fully fix the partition function on compact manifolds.

For instance, for $\mathfrak{g}=\mathfrak{su}(2)$ it does not tell us whether in the path integral we should sum over $SU(2)$ bundles or all $SO(3)$ bundles. All matter is in the adjoint of $SU(2)$, which is a representation of $SO(3)$, so both choices are consistent.

Local vs global

One important property of the $(2,0)$ theory with algebra $\mathfrak{g}$ is that upon reduction on T^2 with complex structure τ it gives rise to 4d $\mathcal{N}=4$ SYM with algebra $\mathfrak{g}$ and complexified gauge coupling τ. Let me call this object $\mathfrak{g}_4$.

What I have just described fully specifies the behaviour of local operators in $\mathfrak{g}_4$, but it does not fully fix the theory. For example it does not fully fix the partition function on compact manifolds.

For instance, for $\mathfrak{g}=\mathfrak{su}(2)$ it does not tell us whether in the path integral we should sum over $SU(2)$ bundles or all $SO(3)$ bundles. All matter is in the adjoint of $SU(2)$, which is a representation of $SO(3)$, so both choices are consistent.

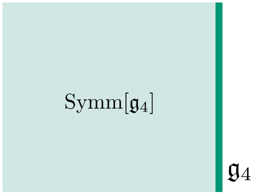
The standard prescription is to decorate $\mathfrak{g}_4$ with some extra structure (a choice of global form for the gauge group) to define a proper 4d theory.

$\mathfrak{g}_4$ as a relative theory

The way string theory "sees" this is a bit different.

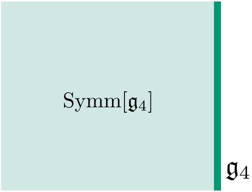
$\mathfrak{g}_4$ as a relative theory

The way string theory "sees" this is a bit different. We can think of $\mathfrak{g}_4$ itself as a "relative theory" [Freed, Teleman '12]: in physical terms it is a set of boundary gapless modes for a TFT in one dimension higher ($4+1=5$ here). This 5d TFT includes information about the symmetries, anomalies and gaugings of all theories with local dynamics given by $\mathfrak{g}_4$. We refer to this TFT as the "symmetry theory" $\mathrm{Symm}[\mathfrak{g}_4]$.



$\mathfrak{g}_4$ as a relative theory

The way string theory "sees" this is a bit different. We can think of $\mathfrak{g}_4$ itself as a "relative theory" [Freed, Teleman '12]: in physical terms it is a set of boundary gapless modes for a TFT in one dimension higher $(4+1=5$ here). This 5d TFT includes information about the symmetries, anomalies and gaugings of all theories with local dynamics given by $\mathfrak{g}_4$. We refer to this TFT as the "symmetry theory" $\mathrm{Symm}[\mathfrak{g}_4]$.

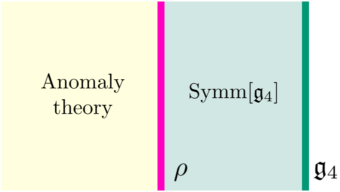


$\mathrm{Symm}[\mathfrak{g}_4]$ is often simple. For instance, for $\mathfrak{g}=\mathfrak{su}(N)$ its most interesting part is a $\mathbb{Z}_N$ gauge theory with action

$$S_{\mathrm{Symm}} = 2\pi i \cdot N \int B_2 \wedge dC_2\,.$$

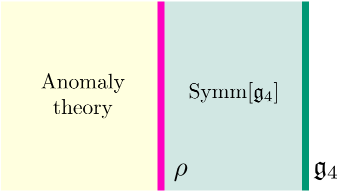
"Absolute" theories

We can obtain more familiar 4d theories by introducing a gapped interface ρ between $\mathrm{Symm}[\mathfrak{g}_4]$ and an invertible TFT, the anomaly theory (in case anomalies remain, otherwise ρ is a boundary).



"Absolute" theories

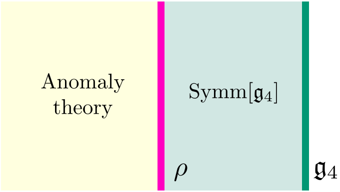
We can obtain more familiar 4d theories by introducing a gapped interface ρ between $\mathrm{Symm}[\mathfrak{g}_4]$ and an invertible TFT, the anomaly theory (in case anomalies remain, otherwise ρ is a boundary).



Colliding ρ and $\mathfrak{g}_4$ we obtain what we usually think of as SYM theories in $d = 4$ with a choice of global form.

"Absolute" theories

We can obtain more familiar 4d theories by introducing a gapped interface ρ between $\mathrm{Symm}[\mathfrak{g}_4]$ and an invertible TFT, the anomaly theory (in case anomalies remain, otherwise ρ is a boundary).



Colliding ρ and $\mathfrak{g}_4$ we obtain what we usually think of as SYM theories in $d = 4$ with a choice of global form. The possible choices of ρ were classified by [Aharony, Seiberg, Tachikawa '13] from a different viewpoint. The connection with the picture above was essentially done (for $SU(N)$, holographically) in [Witten '98], and extended to the $\mathfrak{d}_i$, $\mathfrak{e}_i$ cases in [IGE, Heidenreich, Regalado '19].

Back to 10d

Our starting point is not the 4d theory $\mathfrak{g}_4$ on $\mathcal{M}_4$, but rather IIB on $\mathcal{M}_4 \times \mathbb{C}^2/\Gamma_{\mathfrak{g}} \times T^2$.

Back to 10d

Our starting point is not the 4d theory $\mathfrak{g}_4$ on $\mathcal{M}_4$, but rather IIB on $\mathcal{M}_4 \times \mathbb{C}^2/\Gamma_{\mathfrak{g}} \times T^2$. How do we reproduce the previous discussion from the string theory perspective? That is, where is $\mathrm{Symm}[\mathfrak{g}_4]$ in the string construction?

Back to 10d

Our starting point is not the 4d theory $\mathfrak{g}_4$ on $\mathcal{M}_4$, but rather IIB on $\mathcal{M}_4 \times \mathbb{C}^2/\Gamma_{\mathfrak{g}} \times T^2$. How do we reproduce the previous discussion from the string theory perspective? That is, where is $\mathrm{Symm}[\mathfrak{g}_4]$ in the string construction?

My goal will be to derive $\mathrm{Symm}[\mathfrak{g}_4]$ without using any knowledge about the Lagrangian of the theory.

Heavy branes

We are interested in understanding generalised symmetries. The objects charged under generalised symmetries are generically extended operators. Where are these in our geometric setup?

Heavy branes

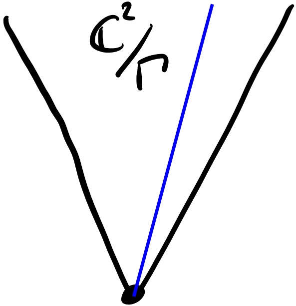
We are interested in understanding generalised symmetries. The objects charged under generalised symmetries are generically extended operators. Where are these in our geometric setup?

These are infinitely heavy branes inserted into our configuration.

Heavy branes

We are interested in understanding generalised symmetries. The objects charged under generalised symmetries are generically extended operators. Where are these in our geometric setup?

These are infinitely heavy branes inserted into our configuration. The mass of a wrapped brane is proportional to the volume wrapped in X. So defects will arise from branes wrapping non-compact cycles ending on the singular point.



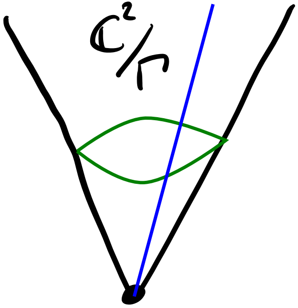
Charge operators

So defect operators (generalised Wilson/'t Hooft lines) in the field theory are branes wrapping non-compact cycles. They are in general *not* topological, so they are not symmetries.

Charge operators

So defect operators (generalised Wilson/'t Hooft lines) in the field theory are branes wrapping non-compact cycles. They are in general *not* topological, so they are not symmetries.

The symmetry operators are rather the flux operators measuring which non-compact lines we have in our configuration:



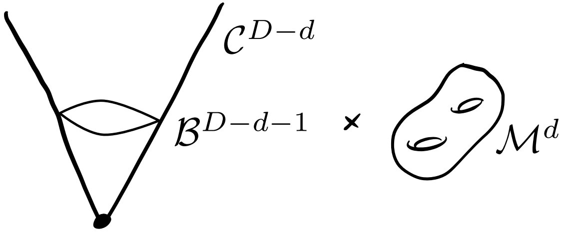
Behaviour at infinity

To fully specify the string background we need to specify the expectation value of these flux operators at infinity.

Behaviour at infinity

To fully specify the string background we need to specify the expectation value of these flux operators at infinity.

Consider our D-dimensional spacetime $\mathcal{M}^D$, which we take to be a d-dimensional manifold $\mathcal{M}^d$ where the QFT lives times a $(D-d)$-dimensional cone $\mathcal{C}^{D-d}$ over a $D-d-1$ base $\mathcal{B}^{D-d-1}$. In order to determine the behaviour at infinity, we'll quantise the theory taking the cone radial direction as "time", and $\mathcal{M}^{D-1} := \mathcal{M}^d \times \mathcal{B}^{D-d-1}$.



Behaviour at infinity

Quantising string theory is of course very difficult, but we can understand the basic physics by studying (generalised) Maxwell theory for a p-form C_p, with action

$$S_{\mathsf{gM}} = \int_{\mathcal{M}^D} F_{p+1} \wedge \star F_{p+1}$$

with $F_{p+1} = dC_p$.

Flux non-commutativity

In generalised Maxwell theory we have flux measuring operators $\Phi^e(\eta_e)$, $\Phi^m(\eta_m)$ with $\eta_e \in H^p(\mathcal{M}^{D-1};\mathbb{R}/\mathbb{Z})$ and $\eta_m \in H^{D-p-2}(\mathcal{M}^{D-1};\mathbb{R}/\mathbb{Z})$.

Flux non-commutativity

In generalised Maxwell theory we have flux measuring operators $\Phi^e(\eta_e)$, $\Phi^m(\eta_m)$ with $\eta_e \in H^p(\mathcal{M}^{D-1}; \mathbb{R}/\mathbb{Z})$ and $\eta_m \in H^{D-p-2}(\mathcal{M}^{D-1}; \mathbb{R}/\mathbb{Z})$.

If there is no torsion we have

$$\begin{aligned} H^k(\mathcal{M}^{D-1}; \mathbb{R}/\mathbb{Z}) &= H^k(\mathcal{M}^{D-1}; \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \\ &= H_{D-k-1}(\mathcal{M}^{D-1}; \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \end{aligned}$$

and we can write ($k = D - p - 2$)

$$\Phi^m(\eta_m) = \exp(2\pi i \alpha \int_{\tilde{\eta}^m} F_{p+1})$$

with $\tilde{\eta}^m \in H_{p+1}(\mathcal{M}^{D-1}; \mathbb{Z})$ and $\alpha \in \mathbb{R}/\mathbb{Z}$, which is a familiar expression for the operator measuring magnetic flux. [Gukov, Witten '08], [Gaiotto, Kapustin, Seiberg, Willett '08]

Flux non-commutativity

If there is torsion the story is more subtle (see [Freed, Moore, Segal '06]), but working in cohomology with $\mathbb{R}/\mathbb{Z}$ coefficients is the right prescription valid in all cases.

Flux non-commutativity

If there is torsion the story is more subtle (see [Freed, Moore, Segal '06]), but working in cohomology with $\mathbb{R}/\mathbb{Z}$ coefficients is the right prescription valid in all cases.

As shown in [Moore '04], [Freed, Moore, Segal '06] we have

$$\Phi^e(\eta_e)\Phi^m(\eta_m) = e^{2\pi i\,\mathsf{L}(\beta(\eta_e),\beta(\eta_m))}\Phi^m(\eta_m)\Phi^e(\eta_e)$$

with

$$\beta\colon H^{k-1}(\mathcal{M}^{D-1};\mathbb{R}/\mathbb{Z}) \to \operatorname{Tor} H^k(\mathcal{M}^D;\mathbb{Z})$$

a Bockstein map and

$$\mathsf{L}\colon \operatorname{Tor} H^p(\mathcal{M}^{D-1}) \times \operatorname{Tor} H^{D-p-2}(\mathcal{M}^{D-1}) \to \mathbb{R}/\mathbb{Z}$$

the “linking pairing”.

Flux non-commutativity

The details are really not that important now. The important thing is that the operators measuring flux at infinity do not necessarily commute when there is torsion.

Flux non-commutativity

The details are really not that important now. The important thing is that the operators measuring flux at infinity do not necessarily commute when there is torsion. So in this case it is not possible to simply turn off all fluxes at infinity. This would correspond to setting all $\Phi^{e,m}(\eta_{e,m}) = 1$, but if for instance

$$\Phi^e(\eta_e)\Phi^m(\eta_m) = -\Phi^m(\eta_m)\Phi^e(\eta_e)$$

this is inconsistent. One has to make choices!

Flux non-commutativity

The details are really not that important now. The important thing is that the operators measuring flux at infinity do not necessarily commute when there is torsion. So in this case it is not possible to simply turn off all fluxes at infinity. This would correspond to setting all $\Phi^{e,m}(\eta_{e,m}) = 1$, but if for instance

$$\Phi^e(\eta_e)\Phi^m(\eta_m) = -\Phi^m(\eta_m)\Phi^e(\eta_e)$$

this is inconsistent. One has to make choices!

In [IGE, Heidenreich, Regalado '19] we performed an analysis of the choices at infinity taking this phenomenon into account, and in this way reproduced the rules in [Aharony, Seiberg, Tachikawa '15].

Flux non-commutativity

The details are really not that important now. The important thing is that the operators measuring flux at infinity do not necessarily commute when there is torsion. So in this case it is not possible to simply turn off all fluxes at infinity. This would correspond to setting all $\Phi^{e,m}(\eta_{e,m}) = 1$, but if for instance

$$\Phi^e(\eta_e)\Phi^m(\eta_m) = -\Phi^m(\eta_m)\Phi^e(\eta_e)$$

this is inconsistent. One has to make choices!

In [IGE, Heidenreich, Regalado '19] we performed an analysis of the choices at infinity taking this phenomenon into account, and in this way reproduced the rules in [Aharony, Seiberg, Tachikawa '15].

A virtue of the boundary perspective is that it straightforwardly extends to theories without a Lagrangian formulation.

Other cases

This philosophy is very general, and it explains (and predicts) the higher form symmetries of geometrically engineered QFTs in a multitude of settings. See also [Tachikawa '13] for a derivation of the higher form symmetries from class-$\mathcal{S}$ (without punctures) and [Del Zotto, Heckman, Park, Rudelius '15] for a more direct translation of [Aharony, Seiberg, Tachikawa '15].

Other cases

This philosophy is very general, and it explains (and predicts) the higher form symmetries of geometrically engineered QFTs in a multitude of settings. See also [Tachikawa '13] for a derivation of the higher form symmetries from class-$\mathcal{S}$ (without punctures) and [Del Zotto, Heckman, Park, Rudelius '15] for a more direct translation of [Aharony, Seiberg, Tachikawa '15].

By now there is a very good understanding of how to determine higher form symmetries for a multitude of ways of engineering field theories geometrically:
[Morrison, Schäfer-Nameki, Willett '20], [Albertini, Del Zotto, IGE, Hosseini '20], [Bah, Bonetti, Minasian '20], [Closset, Schäfer-Nameki, Wang '20], [Del Zotto, IGE, Hosseini '20], [Apruzzi, Dierigl, Lin '20], [Bhardwaj, Schäfer-Nameki '20], [Cvetič, Dierigl, Lin, Zhang '20], [Gukov, Hsin, Pei '20], [Bhardwaj, Hübner, Schäfer-Nameki '21], [Hosseini, Moscrop '21], [Cvetič, Dierigl, Lin, Zhang '21], [. . .]

Other cases

This philosophy is very general, and it explains (and predicts) the higher form symmetries of geometrically engineered QFTs in a multitude of settings. See also [Tachikawa '13] for a derivation of the higher form symmetries from class-$\mathcal{S}$ (without punctures) and [Del Zotto, Heckman, Park, Rudelius '15] for a more direct translation of [Aharony, Seiberg, Tachikawa '15].

By now there is a very good understanding of how to determine higher form symmetries for a multitude of ways of engineering field theories geometrically:
[Morrison, Schäfer-Nameki, Willett '20], [Albertini, Del Zotto, IGE, Hosseini '20], [Bah, Bonetti, Minasian '20], [Closset, Schäfer-Nameki, Wang '20], [Del Zotto, IGE, Hosseini '20], [Apruzzi, Dierigl, Lin '20], [Bhardwaj, Schäfer-Nameki '20], [Cvetič, Dierigl, Lin, Zhang '20], [Gukov, Hsin, Pei '20], [Bhardwaj, Hübner, Schäfer-Nameki '21], [Hosseini, Moscrop '21], [Cvetič, Dierigl, Lin, Zhang '21], [. . .]
(Not all cases are understood, for instance $\mathcal{N} = 3$ S-folds aren't.)

Other cases

This philosophy is very general, and it explains (and predicts) the higher form symmetries of geometrically engineered QFTs in a multitude of settings. See also [Tachikawa '13] for a derivation of the higher form symmetries from class-$\mathcal{S}$ (without punctures) and [Del Zotto, Heckman, Park, Rudelius '15] for a more direct translation of [Aharony, Seiberg, Tachikawa '15].

By now there is a very good understanding of how to determine higher form symmetries for a multitude of ways of engineering field theories geometrically:
[Morrison, Schäfer-Nameki, Willett '20], [Albertini, Del Zotto, IGE, Hosseini '20], [Bah, Bonetti, Minasian '20], [Closset, Schäfer-Nameki, Wang '20], [Del Zotto, IGE, Hosseini '20], [Apruzzi, Dierigl, Lin '20], [Bhardwaj, Schäfer-Nameki '20], [Cvetič, Dierigl, Lin, Zhang '20], [Gukov, Hsin, Pei '20], [Bhardwaj, Hübner, Schäfer-Nameki '21], [Hosseini, Moscrop '21], [Cvetič, Dierigl, Lin, Zhang '21], [. . .]
(Not all cases are understood, for instance $\mathcal{N} = 3$ S-folds aren't.)

All this approaches can be related, but the non-commuting flux viewpoint connects well with the "symmetry theory" approach.

Relative theories

So in geometric engineering we have something like a "QFT on a singularity relative to the string theory bulk":

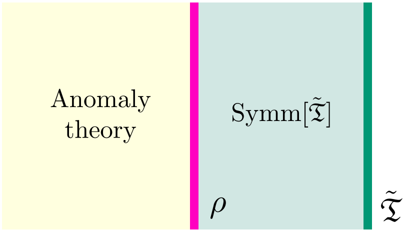
Relative theories

So in geometric engineering we have something like a "QFT on a singularity relative to the string theory bulk": the full QFT is only defined only after specifying boundary values for the supergravity fields, even in the deep IR limit where dynamical excitations for the bulk decouple. There's always a non-trivial topological sector due to the non-commuting flux operators that doesn't decouple.

Relative theories

So in geometric engineering we have something like a "QFT on a singularity relative to the string theory bulk": the full QFT is only defined only after specifying boundary values for the supergravity fields, even in the deep IR limit where dynamical excitations for the bulk decouple. There's always a non-trivial topological sector due to the non-commuting flux operators that doesn't decouple.

This relates a D-dimensional field theory to a $(D+n)$-dimensional topological bulk, with $n > 1$. I will now reduce this picture to the better understood relative QFTs of Freed and Teleman, with $n = 1$:



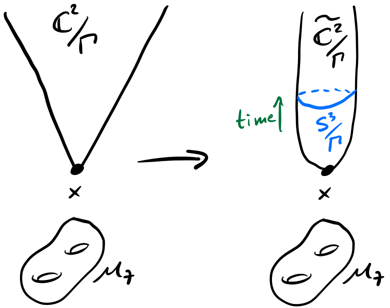
How symmetry theories appear in string theory

Consider, for concreteness, M-theory on $\mathcal{M}^7 \times \mathbb{C}^2/\Gamma$.

How symmetry theories appear in string theory

Consider, for concreteness, M-theory on $\mathcal{M}^7 \times \mathbb{C}^2/\Gamma$.

The approach in [IGE, Heidenreich, Regalado '19] uses a modified asymptotic structure.

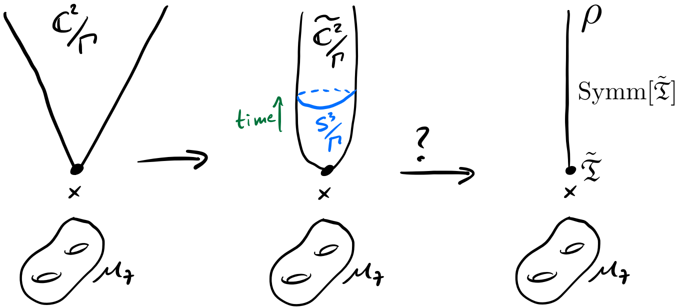


This suggests a strategy for deriving the symmetry theory associated to the field theory: dimensional reduction on the link of the singularity: [Apruzzi, Bonetti, IGE, Hosseini, S. Schäfer-Nameki '21]

How symmetry theories appear in string theory

Consider, for concreteness, M-theory on $\mathcal{M}^7 \times \mathbb{C}^2/\Gamma$.

The approach in [IGE, Heidenreich, Regalado '19] uses a modified asymptotic structure.

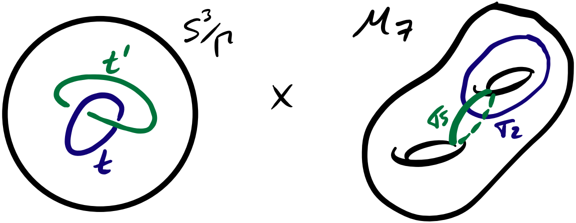


In this picture the boundary conditions at infinity that we need to specify in string theory correspond to ρ, so the object that arises from reduction is the symmetry theory. ("Symmetry inflow" instead of "anomaly inflow".)

The BF theory

In the full theory on $S^3/\Gamma \times X^8$ there are non-commuting flux operators wrapping[1] $t \times \sigma_2$ and $t' \times \sigma_5$, with $t, t' \in H_1(S^3/\Gamma) = \Gamma^{\mathsf{ab}}$ and $\sigma_i \in H_i(X^8)$. Their commutation relations (on a spatial slice $\mathcal{M}_7$ of X^8) are

$$\Phi(t \times \sigma_2)\Phi(t' \times \sigma_5) = e^{2\pi i \mathsf{L}(t,t')\sigma_2 \cdot \sigma_5}\Phi(t' \times \sigma_5)\Phi(t \times \sigma_2)\,.$$



[1] Going to homology so I can draw pictures.

The BF theory (continued)

Fix $\Gamma = \mathbb{Z}_N$ for concreteness. Then $\mathsf{L}(t,t) = 1/N$ for the generator t of $H_1(S^3/\mathbb{Z}_N) = \mathbb{Z}_N$. From the point of view of the effective theory on X_8 we have $\mathbb{Z}_N$ 2-surface operators and 5-surface operators whose relative phase goes with the intersection number divided by N:

$$\Phi(t \times \sigma_2)\Phi(t \times \sigma_5) = e^{2\pi i \sigma_2 \cdot \sigma_5 / N} \Phi(t \times \sigma_5)\Phi(t \times \sigma_2)\,.$$

The BF theory (continued)

Fix $\Gamma = \mathbb{Z}_N$ for concreteness. Then $\mathsf{L}(t,t) = 1/N$ for the generator t of $H_1(S^3/\mathbb{Z}_N) = \mathbb{Z}_N$. From the point of view of the effective theory on X_8 we have $\mathbb{Z}_N$ 2-surface operators and 5-surface operators whose relative phase goes with the intersection number divided by N:

$$\Phi(t \times \sigma_2)\Phi(t \times \sigma_5) = e^{2\pi i \sigma_2 \cdot \sigma_5 / N}\Phi(t \times \sigma_5)\Phi(t \times \sigma_2)\,.$$

This is the 8d $\mathbb{Z}_N$ theory with topological action

$$S_{\mathsf{top}} = 2\pi i\, N \int_{X_8} B_2 \wedge dC_5\,.$$

The BF theory (continued)

Fix $\Gamma = \mathbb{Z}_N$ for concreteness. Then $\mathsf{L}(t,t) = 1/N$ for the generator t of $H_1(S^3/\mathbb{Z}_N) = \mathbb{Z}_N$. From the point of view of the effective theory on X_8 we have $\mathbb{Z}_N$ 2-surface operators and 5-surface operators whose relative phase goes with the intersection number divided by N:

$$\Phi(t \times \sigma_2)\Phi(t \times \sigma_5) = e^{2\pi i \sigma_2 \cdot \sigma_5 / N}\Phi(t \times \sigma_5)\Phi(t \times \sigma_2)\,.$$

This is the 8d $\mathbb{Z}_N$ theory with topological action

$$S_{\text{top}} = 2\pi i\, N \int_{X_8} B_2 \wedge dC_5\,.$$

(In upcoming work with S. Hosseini we derive this more directly from a reduction on S^3/Γ, following [Belov, Moore '06].)

Mixed anomalies

(2112.02092, with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki)

The 7d theory, in addition to the 1-form and/or 4-form symmetries acting on Wilson lines / 't Hooft surfaces, has a $U(1)_I$ continuous 2-form symmetry acting on instanton surfaces.

Mixed anomalies

(2112.02092, with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki)

The 7d theory, in addition to the 1-form and/or 4-form symmetries acting on Wilson lines / 't Hooft surfaces, has a $U(1)_I$ continuous 2-form symmetry acting on instanton surfaces.

There is a mixed 't Hooft anomaly between the $U(1)_I$ symmetry and the 1-form symmetry, of the form

$$S_{\text{anomaly}} = \frac{r_{\mathfrak{g}}}{2} \int_{X_8} F_I^{(4)} \cup \mathcal{P}(B_2)$$

with $r_{\mathfrak{g}}\mathcal{P}(B_2)/2$ the fractional instanton number in the presence of a background for the 1-form symmetry, $F_I^{(4)} = dC^{(3)}$ and $C_I^{(3)}$ the background for the instanton 2-form symmetry.

Mixed anomalies

(2112.02092, with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki)

The 7d theory, in addition to the 1-form and/or 4-form symmetries acting on Wilson lines / 't Hooft surfaces, has a $U(1)_I$ continuous 2-form symmetry acting on instanton surfaces.

There is a mixed 't Hooft anomaly between the $U(1)_I$ symmetry and the 1-form symmetry, of the form

$$S_{\text{anomaly}} = \frac{r_{\mathfrak{g}}}{2} \int_{X_8} F_I^{(4)} \cup \mathcal{P}(B_2)$$

with $r_{\mathfrak{g}}\mathcal{P}(B_2)/2$ the fractional instanton number in the presence of a background for the 1-form symmetry, $F_I^{(4)} = dC^{(3)}$ and $C_I^{(3)}$ the background for the instanton 2-form symmetry.

This anomaly theory can be derived by "reducing" $\int_{\mathcal{M}_{11}} C_3 G_4 G_4 + C_3 X_8$ on S^3/Γ, keeping track of the torsion sector. (See also recent work by [Cvetič, Dierigl, Lin, Zhang '21].)

Differential cohomology

KK reductions beyond de Rham

Mathematically, we want to extract a (discrete) cohomology invariant on $d+1$ dimensions from the Chern-Simons coupling "$\int_{\mathsf{Link}^{10-d}} (C_3 \wedge G_4 \wedge G_4 + C_3 \wedge X_8)$".

Differential cohomology

KK reductions beyond de Rham

Mathematically, we want to extract a (discrete) cohomology invariant on $d+1$ dimensions from the Chern-Simons coupling "$\int_{\mathsf{Link}^{10-d}}(C_3 \wedge G_4 \wedge G_4 + C_3 \wedge X_8)$". Tricky:

- C_3 is not globally well defined

Differential cohomology

KK reductions beyond de Rham

Mathematically, we want to extract a (discrete) cohomology invariant on $d+1$ dimensions from the Chern-Simons coupling "$\int_{\mathsf{Link}^{10-d}}(C_3 \wedge G_4 \wedge G_4 + C_3 \wedge X_8)$". Tricky:

- C_3 is not globally well defined
- and $G_4 = 0$.

Differential cohomology

KK reductions beyond de Rham

Mathematically, we want to extract a (discrete) cohomology invariant on $d+1$ dimensions from the Chern-Simons coupling "$\int_{\mathsf{Link}^{10-d}}(C_3 \wedge G_4 \wedge G_4 + C_3 \wedge X_8)$". Tricky:

- C_3 is not globally well defined
- and $G_4 = 0$.

Luckily these problems essentially cancel each other: we can make sense of this by using **differential cohomology** (aka Cheeger-Simons cohomology or Deligne cohomology), a way of packing differential forms and cohomology classes together, and then the answer is nonzero.

Results in 7d

$$S_{\text{symm}} = \ldots + \left(-\frac{1}{2}\int_{S^3/\Gamma} \breve{t} \star \breve{t}\right) \int_{\mathcal{M}^8} \breve{\gamma}_4 \breve{B}_2^2 \,.$$

We can identify the term in brackets (times $\breve{B}_2^2$), with the fractional instanton number n_{inst}. In particular $r_{\mathfrak{g}}/2$ is given by the classical level $-\frac{1}{2}$ spin-Chern-Simons invariant of S^3/Γ evaluated on a flat connection:

$$\frac{r_{\mathfrak{g}}}{2} = -\frac{1}{2}\int_{S^3/\Gamma} \breve{t} \star \breve{t} \,.$$

Results in 7d

$$S_{\mathsf{symm}} = \ldots + \left(-\frac{1}{2}\int_{S^3/\Gamma} \breve{t} \star \breve{t}\right) \int_{\mathcal{M}^8} \breve{\gamma}_4 \breve{B}_2^2 \,.$$

We can identify the term in brackets (times $\breve{B}_2^2$), with the fractional instanton number n_{inst}. In particular $r_{\mathfrak{g}}/2$ is given by the classical level $-\frac{1}{2}$ spin-Chern-Simons invariant of S^3/Γ evaluated on a flat connection:

$$\frac{r_{\mathfrak{g}}}{2} = -\frac{1}{2}\int_{S^3/\Gamma} \breve{t} \star \breve{t} \,.$$

This geometrizes field theory results in [Witten '00], [Córdova, Freed, Lam, Seiberg '19], so it allows us to compute anomalies in the space of coupling constants for non-Lagrangian theories.

2-groups

So far I have discussed “ordinary” p-form symmetries. But by now we know that the general story is significantly more interesting, with generalisations of this structure in multiple directions.

2-groups

So far I have discussed “ordinary” p-form symmetries. But by now we know that the general story is significantly more interesting, with generalisations of this structure in multiple directions.

For instance, we can have 2-group symmetries. [Kapustin, Thorngren '13], [Sharpe '15], [Tachikawa '17], [Córdova, Dumitrescu, Intriligator '18], [Benini, Córdova, Hsin '18], [Córdova, Dumitrescu, Intriligator '20], [. . .]

2-groups

So far I have discussed "ordinary" p-form symmetries. But by now we know that the general story is significantly more interesting, with generalisations of this structure in multiple directions.

For instance, we can have 2-group symmetries. [Kapustin, Thorngren '13], [Sharpe '15], [Tachikawa '17], [Córdova, Dumitrescu, Intriligator '18], [Benini, Córdova, Hsin '18], [Córdova, Dumitrescu, Intriligator '20], [...]

These were interpreted geometrically in [Del Zotto, IGE, Schäfer-Nameki '22], [Cvetič, Heckman, Hübner, Torres '22], they follow from the non-triviality of certain Mayer-Vietoris exact sequence for the base of the cone. (But a SymmTFT description is lacking.)

Non-invertibles

During the last couple of years a number of $d > 3$ field theory examples have been found that have non-invertible symmetries:

$$\mathcal{N}(M_2) \times \mathcal{N}(M_2) \propto (1 + T(M_2)) \times (\text{condensations})$$

[Gaiotto, Johnson-Freyd '19], [Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela '21], [Kaidi, Ohmori, Zheng '21], [Choi, Córdova, Hsin, Lam, Shao '21], [Koide, Nagoya, Yamaguchi '21], [Roumpedakis, Seifnashri, Shao '22], [Bhardwaj, Bottini, Schäfer-Nameki, Tiwari '22], [Arias-Tamargo, Rodriguez-Gomez '22], [Choi, Córdova, Hsin, Lam, Shao '22], [Kaidi, Zafrir, Zheng '22], [Choi, Lam, Shao '22], [Córdova, Ohmori '22], [Bashmakov, Del Zotto, Hasan '22], [Aguilera Damia, Argurio, García-Valdecasas '22]

Non-invertibles

During the last couple of years a number of $d > 3$ field theory examples have been found that have non-invertible symmetries:

$$\mathcal{N}(M_2) \times \mathcal{N}(M_2) \propto (1 + T(M_2)) \times (\text{condensations})$$

[Gaiotto, Johnson-Freyd '19], [Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela '21], [Kaidi, Ohmori, Zheng '21], [Choi, Córdova, Hsin, Lam, Shao '21], [Koide, Nagoya, Yamaguchi '21], [Roumpedakis, Seifnashri, Shao '22], [Bhardwaj, Bottini, Schäfer-Nameki, Tiwari '22], [Arias-Tamargo, Rodriguez-Gomez '22], [Choi, Córdova, Hsin, Lam, Shao '22], [Kaidi, Zafrir, Zheng '22], [Choi, Lam, Shao '22], [Córdova, Ohmori '22], [Bashmakov, Del Zotto, Hasan '22], [Aguilera Damia, Argurio, García-Valdecasas '22]

In upcoming work with B. Heidenreich and S. Schäfer-Nameki we'll explain how this structure appears in string theory.

Conclusions

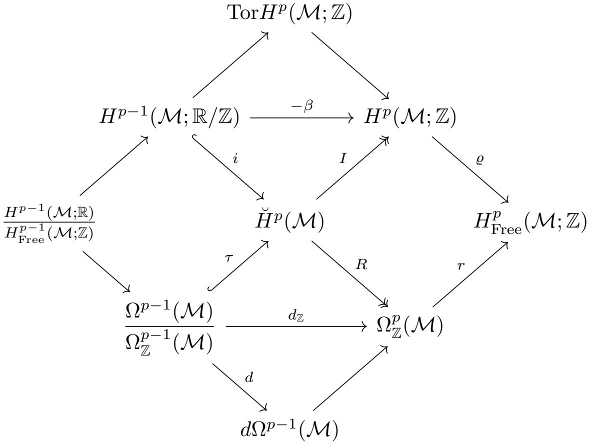
For geometrically engineered theories there is a close connection between the symmetries of a theory and the geometry. But crucially, the symmetries are often much easier to extract from the geometry than many other properties of the theory. This is particularly so for non-Lagrangian cases.

I have focused on the developments I understand best. There is a lot of recent beautiful literature developing complementary approaches, for example in the context of anomaly inflow. See for instance [Bah, Bonetti, Minasian '20].

We don't quite have a full systematic dictionary yet, but the general picture is gradually becoming clear.

Differential cohomology

The degree d differential cohomology group $\breve{H}^d(\mathcal{M})$ fits into:



and enjoys a product:

$$\breve{H}^p(\mathcal{M}) \star \breve{H}^q(\mathcal{M}) \to \breve{H}^{p+q}(\mathcal{M})\,.$$

Chern-Simons terms

The differential cohomology formulation of the M-theory Chern-Simons term "$C_3 \wedge G_4 \wedge G_4$" is

$$S_{\mathsf{CS}} = -\frac{1}{6} 2\pi i \int_{\mathcal{M}^{11}} \breve{G}_4 \star \breve{G}_4 \star \breve{G}_4 \,.$$

Chern-Simons terms

The differential cohomology formulation of the M-theory Chern-Simons term "$C_3 \wedge G_4 \wedge G_4$" is

$$S_{\mathsf{CS}} = -\frac{1}{6} 2\pi i \int_{\mathcal{M}^{11}} \breve{G}_4 \star \breve{G}_4 \star \breve{G}_4 \,.$$

In differential cohomology, for $\breve{x} \in \breve{H}^{d+1}(\mathcal{M}^d)$ we have

$$\int_{\mathcal{M}^d} \breve{x} \qquad \in \breve{H}^1(\mathrm{pt}) = \mathbb{R}/\mathbb{Z} \,.$$

Chern-Simons terms

The differential cohomology formulation of the M-theory Chern-Simons term "$C_3 \wedge G_4 \wedge G_4$" is

$$S_{\mathsf{CS}} = -\frac{1}{6} 2\pi i \int_{\mathcal{M}^{11}} \breve{G}_4 \star \breve{G}_4 \star \breve{G}_4 \,.$$

In differential cohomology, for $\breve{x} \in \breve{H}^{d+1}(\mathcal{M}^d)$ we have

$$\int_{\mathcal{M}^d} \breve{x} \qquad \in \breve{H}^1(\mathrm{pt}) = \mathbb{R}/\mathbb{Z} \,.$$

Note: The integral above is not well defined by itself because of the factor of $\frac{1}{6}$, but it is well known that the whole M-theory action is. [Witten '96] This subtlety plays an important role in our discussion (one needs to consider the full M-theory action to obtain the right field theory answer), but I'll not discuss it in detail.

The differential KK reduction

On $\mathcal{M}^8 \times S^3/\Gamma$ we can expand

$$\breve{G}_4 = \breve{\gamma}_4 \star \breve{1} + \breve{B}_2 \star \breve{t}_2 + \dots$$

with $t_2 \in H^2(S^3/\Gamma) = \Gamma^{\mathsf{ab}}$ and $\breve{t}_2$ a flat representative of t_2.

The differential KK reduction

On $\mathcal{M}^8 \times S^3/\Gamma$ we can expand

$$\breve{G}_4 = \breve{\gamma}_4 \star \breve{1} + \breve{B}_2 \star \breve{t}_2 + \ldots$$

with $t_2 \in H^2(S^3/\Gamma) = \Gamma^{\mathsf{ab}}$ and $\breve{t}_2$ a flat representative of t_2.

Then $-\frac{1}{6}\int \breve{G}_4^3$ contains a term

$$S_{\mathsf{symm}} = \ldots + \left(-\frac{1}{2}\int_{S^3/\Gamma} \breve{t}_2 \star \breve{t}_2\right) \int_{\mathcal{M}^8} \breve{\gamma}_4 \breve{B}_2^2 \,.$$

SymmTFTs in 5d

(2112.02092, with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki)

As another example, for 5d SCFTs obtained from M-theory on $X^6 = \mathcal{C}_{\mathbb{R}}(L^5)$ the resulting symmetry theory is:

$$S_{\mathsf{Sym}} = \int_{\mathcal{W}_6} \Big(K_{ij} B_2^{(i)} \cup \delta C_3^{(j)} + \Omega_{ijk} B_2^{(i)} \cup B_2^{(j)} \cup B_2^{(k)} + \Upsilon_{ij\alpha} B_2^{(i)} \cup B_2^{(j)} \cup F_2^{(\alpha)} \Big)$$

where the K, Ω, Υ coefficients are classical spin-Chern-Simons invariants on the L^5.

SymmTFTs in 5d

(2112.02092, with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki)

As another example, for 5d SCFTs obtained from M-theory on $X^6 = \mathcal{C}_{\mathbb{R}}(L^5)$ the resulting symmetry theory is:

$$S_{\mathsf{Sym}} = \int_{\mathcal{W}_6} \Big(K_{ij} B_2^{(i)} \cup \delta C_3^{(j)} + \Omega_{ijk} B_2^{(i)} \cup B_2^{(j)} \cup B_2^{(k)} + \Upsilon_{ij\alpha} B_2^{(i)} \cup B_2^{(j)} \cup F_2^{(\alpha)} \Big)$$

where the K, Ω, Υ coefficients are classical spin-Chern-Simons invariants on the L^5. We can compute these geometrically using differential cohomology (see also [Cvetič, Dierigl, Lin, Zhang '21]), and in cases where there is a geometric interpretation we can compare against field theory predictions.

SymmTFTs in 5d

(2112.02092, with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki)

As another example, for 5d SCFTs obtained from M-theory on $X^6 = \mathcal{C}_{\mathbb{R}}(L^5)$ the resulting symmetry theory is:

$$S_{\mathsf{Sym}} = \int_{\mathcal{W}_6} \Big(K_{ij} B_2^{(i)} \cup \delta C_3^{(j)} + \Omega_{ijk} B_2^{(i)} \cup B_2^{(j)} \cup B_2^{(k)} + \Upsilon_{ij\alpha} B_2^{(i)} \cup B_2^{(j)} \cup F_2^{(\alpha)} \Big)$$

where the K, Ω, Υ coefficients are classical spin-Chern-Simons invariants on the L^5. We can compute these geometrically using differential cohomology (see also [Cvetič, Dierigl, Lin, Zhang '21]), and in cases where there is a geometric interpretation we can compare against field theory predictions. For instance, for $SU(p)_q$ we get

$$K_{11} = \gcd(p,q) \ ; \ \Omega_{111} = \frac{q\, p\, (p-1)\, (p-2)}{6 \gcd(p,q)^3} \ ; \ \Upsilon_{111} = \frac{p\, (p-1)}{2 \gcd(p,q)^2}$$

in agreement with [Gukov, Pei, Hsin '20].