Experimental Particle and Astroparticle Physics 2021/2022

Klein-Gordon Equation

1. Consider the Klein-Gordon (KG) equation

$$\left(\Box + \frac{m_0^2c^2}{\hbar^2}\right)\psi = \left(\frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2c^2}{\hbar^2}\right)\psi = 0$$

   a) show that $\psi=\exp(-i/\hbar\, p_\mu x^\mu)$ is a possible solution of the KG equation.
   b) could the static-field solution $\psi=[\exp(-\alpha\, r)]/r$ be a possible solution? (assume α is some sort of constant)
   c) What could be the physical interpretations of theses solutions, if they are possible?

2. Obtain expressions for the probability density and the current, for the KG equation. Explain the significance of the result.

3. Show the KG equation admits positive and negative energy solutions.
4. Is the KG equation Lorentz invariant?
5. Show the probability density derived from the KG equation, may not be positive definite.
6. Consider the non-relativistic limit of the KG equation solution, where the total energy (E) of the particle is similar to its rest mass (m_0c^2). By using the ansatz

$$\psi(\boldsymbol{r}, t) = \varphi(\boldsymbol{r}, t)\exp\left(-\frac{\mathrm{i}}{\hbar}m_0c^2t\right)$$

   where the time dependent solution (ψ) is split into a term (φ) that doesn't depend on the mass, plus a mass term, show the KG is consistent with a free Schrodinger equation for spinless particles.

7. In 1934 Pauli and Weisskopf revised the KG equation and inserted the charge e into the four-current j^μ allowing interpreting it as the charged-current density of the electron. Later Stuckelberg (in 1941) and Feynman (in 1948) (the Feynman-Stuckelberg interpretation) expressed the idea that both energy solutions are associated to particles and anti-particles (with opposite charge of particles). Show, using the free wave ansatz,

$$\psi = A\exp\left(-\frac{\mathrm{i}}{\hbar}p_\mu x^\mu\right) = A\exp\left[\frac{\mathrm{i}}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{x} - Et)\right]$$

   how you can construct solutions of the KG equation for positive and negative charges, associated to positive and negative charge densities.

8. Show that if a complex field (with a real and imaginary component)

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[\varphi_1(x) + \mathrm{i}\varphi_2(x)\right]$$

   (where φ_1 and φ_2 are real) obeys the KG equation, then each φ_1 and φ_2 also obey a KG equation, and reversely if the KG equation mass is the same $m=m_1=m_2$, for all fields.