The Weinberg Compositeness Criterion

Definition, Extension and Application

Christoph Hanhart

Forschungszentrum Jülich

Based on

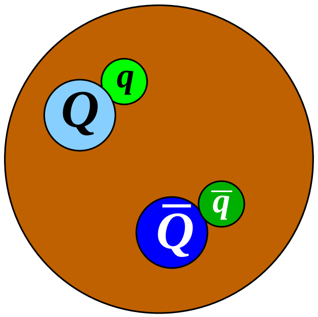
F.-K. Guo et al. "Hadronic molecules," Rev. Mod. Phys. 90(2018)015004

I. Matuschek et al., "On the nature of near-threshold bound and virtual states," Eur. Phys. J. A 57 (2021) no.3, 101

V. Baru et al., "Effective range expansion for narrow near-threshold resonances," [arXiv:2110.07484 [hep-ph]].

Overview

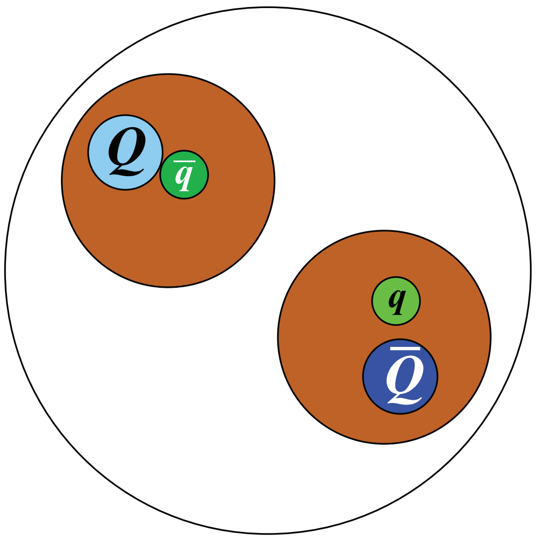
We want to disentangle e.g.



Tetraquarks

$\rightarrow$ Compact object formed from diquarks, (Qq) and $(\bar{Q}\bar{q})$

from



Hadronic-Molecules

$\rightarrow$ Extended object made of $(\bar{Q}q)$ and $(Q\bar{q})$

Bohr radius = $1/\gamma = 1/\sqrt{2\mu E_b}$
$\gg 1$ fm $\gtrsim$ confinement radius

for near threshold states

Tool: The Weinberg compositeness criterion

Definition using non–rel. QM

Landau (1960), Weinberg (1963), Baru et al. (2004)

Expand in terms of non–interacting quark and meson states

$$|\Psi\rangle = \begin{pmatrix} \lambda|\psi_0\rangle \\ \chi(\mathbf{p})|h_1h_2\rangle \end{pmatrix},$$

here $|\psi_0\rangle$ = elementary state and $|h_1h_2\rangle$ = two–hadron cont., then

$\lambda^2 = |\langle\psi_0|\Psi\rangle|^2 =$ probability to find bare state in physical state

$\rightarrow$ λ^2 is the quantity of interest!

The Schrödinger equation reads

$$\hat{\mathcal{H}}|\Psi\rangle = E|\Psi\rangle, \;\; \hat{\mathcal{H}} = \begin{pmatrix} \hat{H}_c & \hat{V} \\ \hat{V} & \hat{H}^0_{hh} \end{pmatrix} \longrightarrow \;\; \chi(p) = \lambda \frac{f(p^2)}{E - p^2/(2\mu)}$$

introducing the transition form factor $\langle\psi_0|\hat{V}|hh\rangle = f(p^2)$,

Note: $\hat{H}^0_{hh}$ contains only meson kinetic terms!

Effective Coupling

Therefore

$$|\Psi\rangle = \lambda \begin{pmatrix} |\psi_0\rangle \\ -\frac{f(p^2)}{E_B+p^2/(2\mu)}|h_1h_2\rangle \end{pmatrix} ,$$

For the normalization of the physical state we get

$$1 = \langle\Psi|\Psi\rangle = \lambda^2 \left(1 + \int \frac{d^3p}{(2\pi)^3} \frac{f^2(p^2)}{(E_B + p^2/(2\mu))^2}\right)$$

using

$$\int \frac{f^2(p^2)d^3p}{(p^2/(2\mu) + E_B)^2} = \frac{4\pi^2\mu^2 f(0)^2}{\sqrt{2\mu E_B}} + \mathcal{O}\left(\frac{\sqrt{E_B\mu}}{\beta}\right)$$

for s–waves; $1/\beta$= range of forces; $\mu f(0)^2/(2\pi) = g^2$; $\gamma = \sqrt{2\mu E_B}$

$$1 = \lambda^2 \left(1 + \frac{\mu g^2}{\gamma} + \mathcal{O}\left(\frac{\gamma}{\beta}\right)\right) \implies \boxed{g^2 = \frac{\gamma}{\mu}\left(\frac{1-\lambda^2}{\lambda^2}\right)}$$

Thus...

using for residue $g^2_{\text{eff(NR)}} = (2\pi/\mu)\lambda^2 g^2$ or for rel. norm.

$$\frac{g^2_{\text{eff(NR)}}}{4\pi} = (1-\lambda^2)\frac{\gamma}{2\mu^2} \leq \frac{\gamma}{2\mu^2}$$

$(1-\lambda^2)$ = Quantifies molecular component in physical state

The structure information is hidden in the

effective coupling, extracted from experiment,

independent of the phenomenology

used to introduce the pole(s)

Connecting to effective range expansion

The scattering amplitude is in terms of the previous parameters

$$T_{\rm NR}(E) = \frac{2\pi}{\mu}\frac{g^2}{E + E_B + g^2(ik+\gamma)}$$

where $k^2 = 2\mu E$ & $g^2 = \infty$ for molecule / $g^2 = 0$ for compact state

The effective range expansion reads:

$$T_{\rm NR}(E) = -\frac{2\pi}{\mu}\frac{1}{1/a + (r/2)k^2 - ik}$$

and we get from matching coefficients

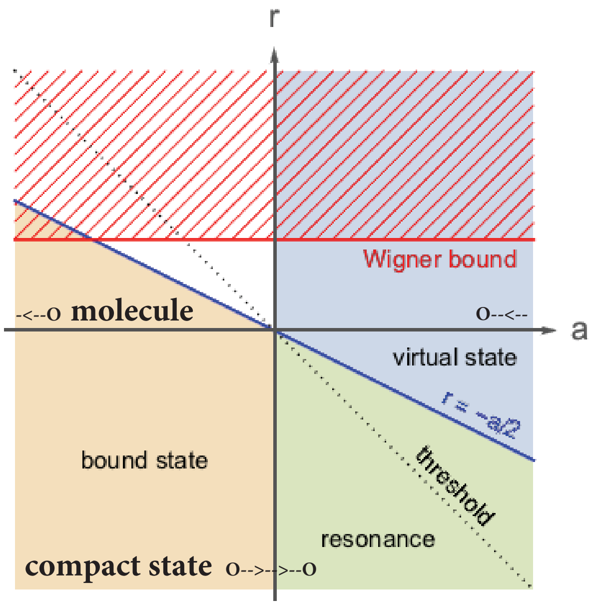
$$\frac{1}{a} = -\frac{E_B}{g^2} + \gamma \qquad \Longrightarrow a = -2\frac{1-\lambda^2}{2-\lambda^2}\left(\frac{1}{\gamma}\right) + \mathcal{O}\left(\frac{1}{\beta}\right)$$

$$r = -\frac{1}{g^2\mu} \qquad \Longrightarrow r = -\frac{\lambda^2}{1-\lambda^2}\left(\frac{1}{\gamma}\right) + \mathcal{O}\left(\frac{1}{\beta}\right)$$

Weinbergs analysis and a generalisation

I. Matuschek et al., EPJA57(2021)3

Assume attractive interaction (bound state $a<0$, all others $a>0$)



Weinberg (for bound states):

Molecules:

$|a| \gg |r|$ and $|r| \simeq$ range

Compact states:

$|a| \ll |r|$ and $r<0$ with $|r| \gg$ range

What happens when a changes sign? (r fixed)

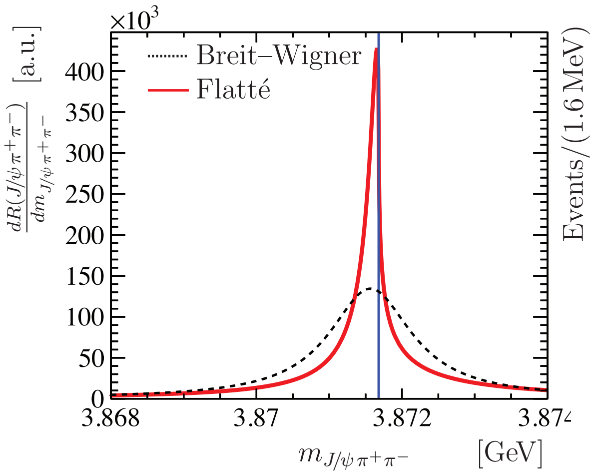
Molecule: turns into a virtual state (and eventually a resonance)

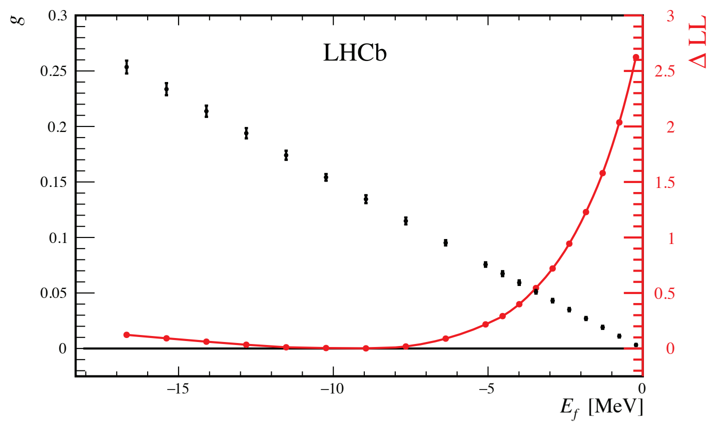
Compact state: turns into a resonance directly

Subsummed in compositness: $\bar{X} = 1/\sqrt{1+|2r/a|}$

other approaches: Sekihara, Hyodo, Oset, Oller, Nieves, Jido ...
mostly relying on on-shell factorisation of the potential; little about virtual states

$\chi_{c1}(3872)$ also known as $X(3872)$





LHCb, PRD102(2020)092005

C.H. at al., PRD76(2007)034007

Data analysed employing for the rate

$$\frac{\Gamma_\rho(E)}{\left|E - E_f + \frac{i}{2}\left[g^2\left(\sqrt{2\mu_1 E} + \sqrt{2\mu_2(E-\delta)}\right) + \Gamma_\rho(E) + \Gamma_\omega(E) + \Gamma_0\right]\right|^2}$$

with E_f fixed to -7.18 MeV: $g^2 = 0.108 \pm 0.003$ such that

$$-r = 2/(\mu_1 g^2) + \sqrt{\mu_2/(2\mu_1^2\delta)} \simeq (3.8 + 1.4) \text{ fm} \gg 1/M_\pi$$

Does this mean $\chi_{c1}(3872)$ is a compact state? A. Esposito et al.,2108.11413

Effective range and Weinberg criterion

The second term in

$$-r = 2/(\mu_1 g^2) + \sqrt{\mu_2/(2\mu_1^2\delta)} \simeq (3.8 + 1.4)\ \text{fm} \gg 1/M_\pi$$

comes from the expansion

$$ik_2 = \sqrt{2\mu_2(\delta - k_1^2/(2\mu_1))} = \sqrt{2\mu_2\delta} - \frac{1}{2}\sqrt{\frac{\mu_2}{2\mu_1^2\delta}}\, k_1^2 + \mathcal{O}\left(\left(\frac{k_1^2}{\mu_1\delta}\right)^2\right)$$

which "measures" the contribution from the charged channel and does not have a proper isospin limit ($\delta \to 0$)

$\rightarrow$ we thus see that this contribution is sizable

$\rightarrow$ to understand $\chi_{c1}(3872)$ as isoscalar state: take isospin limit

Thus the quantity relevant for the Weinberg analysis is then

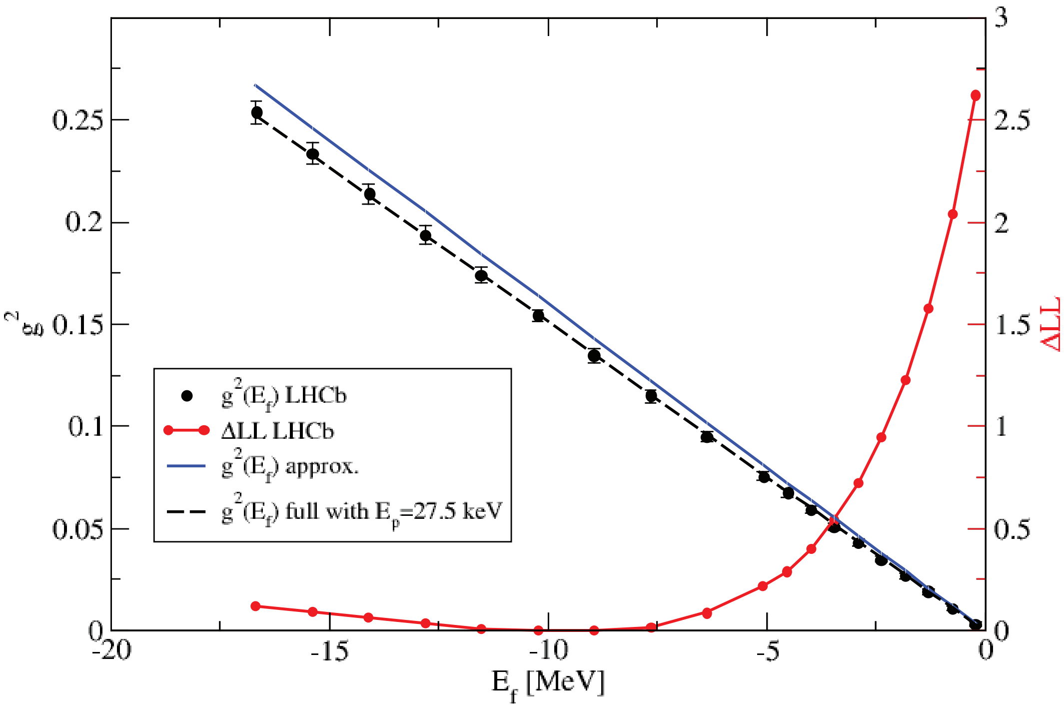
$$-r_{\text{eff.}} = 1/(\mu_1 g^2) \simeq 1.9\ \text{ fm } \approx 1/M_\pi$$

Regarding the correlations

$$\frac{\Gamma_\rho(E)}{\left|E - E_f + \frac{i}{2}\left[g^2(\sqrt{2\mu_1 E} + \sqrt{2\mu_2(E-\delta)}) + \Gamma_\rho(E) + \Gamma_\omega(E) + \Gamma_0\right]\right|^2}$$

Parts in red define E_p, the real part of pole location:

$$E_p = E_f + \frac{g^2}{2}\left(\sqrt{2\mu_1|E_p|} + \sqrt{2\mu_2(\delta + |E_p|)}\right) \implies g^2(E_f, E_p)$$



Since $E_p \ll \delta$ one may approximate correlation parameter free

$$g^2(E_f, 0) = -\sqrt{\frac{2}{\mu_2 \delta}} E_f$$

To remove correlation:

Express E_f by E_p

Consequence:

The formula that should be used in the analysis:

$$\frac{\Gamma_\rho(E)}{\left|E - E_p + \frac{i}{2}\left[g^2\left(\sqrt{2\mu_1 E} - i\gamma_1 + \sqrt{2\mu_2(E-\delta)} - i\gamma_2\right) + \Gamma_{\rm inel.}(E)\right]\right|^2}$$

where $\gamma_1 = \sqrt{2\mu_1|E_p|}$ and $\gamma_2 = \sqrt{2\mu_2(\delta + |E_p|)}$

The LHCb data only provides lower bound for g

If one allows for $\Delta LL = 1$, one finds $g^2 > 0.1$ and accordingly

$$-r_{\rm eff.} < 2\ \text{fm} \quad \text{and} \quad \bar{X} = \frac{1}{\sqrt{1 + 2|r_{\rm eff.}/\Re(a)|}} > 0.94\ ,$$

fully consistent with a molecular interpretation

Similar numbers emerge for the T_{cc} state ...

How to get positive r?

→ The formulas were derived neglecting finite range corrections

→ The Wigner bound (causality!) requires $r < R \sim 1/\beta$

E.P. Wigner, Phys. Rev. 98, 145 (1955)

$\Longrightarrow$ Zero range interactions call for neg. effective ranges

The longest range interaction is the one π exchange, however in the charm system $\pi D\bar{D}$ can go on-shell

$\Longrightarrow$ no fixed sign of potential

Three-body calculation reveals (for T_{cc}): $r_{\rm OPE} = +0.4$ fm

M. L. Du et al., [arXiv:2110.13765 [hep-ph]].

Conclusion

At present the data on $\chi_{c1}(3872)$ aka $X(3872)$ are consistent with a molecular interpretation, but so far a sizeable compact component cannot be excluded.

We need

- $\rightarrow$ Reanalysis of LHCb data with correlations removed
- $\rightarrow$ Combined analysis of inelastic and elastic channels
- $\rightarrow$ Direct measurement of line shape (PANDA?)
- $\rightarrow$ Information on (iso)spin partner states

... thank you very much for your attention