Generalizing Weinberg’s compositeness relations

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Based on:

Yan Li, FKG, Jin-Yi Pang, Jia-Jun Wu, *Generalization of Weinberg's Compositeness Relations*, arXiv:2110.02766

Mini-workshop on Effective Range for X(3872), 12 Nov. 2021

Weinberg's compositeness relations

- Compositeness for S-wave shallow bound state as derived in Weinberg's paper, X_W, expressed in terms of scattering length and effective range

$$a = -\frac{2X_W}{1+X_W}R + O(m_\pi^{-1})$$
$$r = -\frac{1-X_W}{X_W}R + O(m_\pi^{-1})$$

$$R \equiv \frac{1}{\sqrt{2\mu|E_B|}}$$

Binding energy

- Effective coupling: $$g^2 = \frac{8\pi^2}{\mu^2 R}X_W$$

- Applied to the deuteron case

($E_B = -2.22$ MeV, $R = 4.31$ fm, $a = -5.42$ fm, $r = 1.77$ fm), $X_W = 1.68 > 1$

- Assumptions used in the derivations
  - ☐ Neglecting the non-pole term from the Low equation
  - ☐ Approximating the form factor by a constant

$$T_{p,k} = V_{p,k} + \frac{g(p)\, g^*(k)}{h_k - E_B} + \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{T_{p,q}T^*_{k,q}}{h_k + i\varepsilon - h_q} \qquad \text{w/ } h_k \equiv k^2/(2\mu)$$

Question: for ERE up to $\mathcal{O}(p^2)$, is a constant $g(p)$ a consistent approximation?
Inconsistency already pointed out in I. Matuschek et al., EPJA57, 101 (2021); see Christoph's talk

Generalization

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- The constant form factor assumption can be replaced by a more general separable ansatz

$$T_{p,k} = t_k \, g(p) \, g^*(k)$$

Twice-subtracted dispersion relation ⇒

$$t^{-1}(W) = (W - E_B) + (W - E_B)^2 \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|g(q)|^2}{(h_q - E_B)^2 (h_q - W)}$$

Then, we get

$$t(W) = \frac{1}{1 - F(W)} \frac{1}{W - E_B}, \quad F(W) \equiv (W - E_B) \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|g(q)|^2}{(h_q - E_B)^2 (W - h_q)}$$

- Compositeness emerges

$$F(\infty) = \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|\langle q|\hat{V}|B\rangle|^2}{(h_q - E_B)^2} = \int_0^\infty \frac{q^2 dq}{(2\pi)^3} |\langle q|B\rangle|^2 = X$$

- Introducing

$$F_1(W) \equiv \frac{\ln[1 - F(W)]}{W - E_B}, \qquad \mathrm{Im}\, F_1(E + i\varepsilon) = -\frac{\delta_B(E)}{E - E_B} \theta(E)$$

here δ_B is the phase of the T-matrix with the nonpole term neglected (convention: $\delta_B(0) = 0$)

$$\delta_B(E = h_p) \equiv \arg T_{p,p} = -\arg(1 - F(E + i\varepsilon)) \qquad \delta_B \in [-\pi, 0]$$

⇑

$F(0) \leq 0$, $\mathrm{Im}\, F(E + i\varepsilon) \leq 0$ for $E \geq 0$

Generalization

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- From the dispersion relation for $F_1(W)$, we obtain a solution:

$$F(W) = 1 - \exp\left(\frac{W - E_B}{\pi}\int_0^\infty dE\, \frac{-\delta_B(E)}{(E-W)(E-E_B)}\right)$$

and an expression for the compositeness

$$X = 1 - \exp\left(\frac{1}{\pi}\int_0^\infty dE\, \frac{\delta_B(E)}{E-E_B}\right) \in [0,1]$$

- Using $\mathrm{Im}\, F\left(h_p + i\epsilon\right) = -\frac{\pi p \mu}{(2\pi)^3}\frac{|g(p)|^2}{h_p - E_B}$, we get

$$|g(p)|^2 = -\frac{(2\pi)^3}{\pi\, p\mu}(h_p - E_B)\,\sin\delta_B(E)\exp\left[\frac{h_p - E_B}{\pi}\,⨍_0^\infty dE\,\frac{-\delta_B(E)}{(E-h_p)(E-E_B)}\right]$$

- Consider ERE $p\cot\delta_B \approx -\frac{8\pi^2}{\mu}\mathrm{Re}T^{-1}(h_p) = \frac{1}{a} + \frac{r}{2}p^2 + \mathcal{O}(p^4)$, we finally get

$$g^2(p) = \frac{8\pi^2}{\mu^2 R}\times\begin{cases} X_W + \mathcal{O}(p^4) & \text{for } a\in[-R,0]\ \&\ r\le 0 \\ \frac{a^2}{R^2}\frac{1}{1+(a+R)^2p^2} + \mathcal{O}(p^4) & \text{for } a < -R\ \&\ r > 0 \end{cases}$$

constant

contains $\mathcal{O}(p^2)$ terms, thus not self-consistent if using a constant g^2 but still work up to $\mathcal{O}(p^2)$ in ERE. Weinberg's relations do not hold in this case

Compositeness

Y. Li, FKG, J.-Y. Pang, J.-J. Wu, arXiv:2110.02766

- Poles of the T-matrix with ERE up to $\mathcal{O}(p^2)$: $\frac{1}{a}+\frac{r}{2}p^2 - i\,p = \frac{r}{2}(p-p_+)(p-p_-)$

  $$p_- = \frac{i}{R}, \qquad p_+ = -\frac{i}{R+a}$$

  with $R = \frac{1}{\sqrt{2\mu|E_B|}}$; r is expressed as $r = \frac{2R}{a}(R+a)$

- For $a \in [-R, 0]$, then $r < 0$, one bound state and one virtual state pole

  $$g^2(p) = \frac{8\pi^2}{\mu^2 R} X_W + \mathcal{O}(p^4), \qquad X = X_W \simeq \sqrt{\frac{1}{1+2r/a}}$$

- For $a < -R$, then $r > 0$, two bound state poles (the remote one $\sim i/\beta$ is unphysical)

  $$g^2(p) = \frac{8\pi^2}{\mu^2 R}\frac{a^2}{R^2}\frac{1}{1+(a+R)^2 p^2} + \mathcal{O}(p^4), \qquad X \simeq 1 - e^{-\infty} = 1$$

For the deuteron, $R = 4.31$ fm, $a = -5.42$ fm, $a + R \sim \beta^{-1} \sim m_\pi^{-1}$

$$X = 1 - \exp\left(\frac{1}{\pi}\int_0^\infty dE \frac{\delta_B(E)}{E-E_B}\right) \in [0,1]$$

$$p\cot\delta_B = \frac{1}{a} + \frac{r}{2}p^2 \Rightarrow \delta_B(\infty) = 0 \text{ for } r<0, \text{ and } \delta_B(\infty) = -\pi \text{ for } r > 0$$

- For extension of the Weinberg's relations to virtual state and near-threshold resonance, see I. Matuschek et al., EPJA57, 101 (2021); Christoph's talk

Thank you for your attention
Stay safe