Bhabha scattering
– Status of Theory Predictions –

Tord Riemann

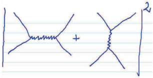
Based on work done in collaboration with S. Actis, J. Gluza, M. Worek et al.

DESY, Zeuthen, Germany

Research Workshop of the Israel Science Foundation
High precision measurements of luminosity at future linear colliders and polarization of lepton beams
3 - 5 October 2010, School of Physics and Astronomy, Tel Aviv University, Tel Aviv, Israel

In 2010, we experience the 75^{th} anniversary of Bhabha scattering

http://en.wikipedia.org/wiki/Homi_J._Bhabha
http://de.wikipedia.org/wiki/Homi_Jehangir_Bhabha



H. Bhabha
"The Scattering of Positrons by Electrons with Exchange on Dirac's Theory of the Positron"
[1] Proc. Roy. Soc. A154 (1936) 195

Homi J. Bhabha (1909 – 1966) was an Indian nuclear physicist who played a major role in the development of the Indian atomic energy program and is considered to be the father of India's nuclear program.
Bhabha was born into a prominent family, through which he was related to . . . Dorab Tata.
Early education at Bombay schools and at the Royal Institute of Science,
He attended Caius College of Cambridge University to pursue studies in mechanical engineering.
Studies under Paul Dirac to complete the Mathematics Tripos.
Doctorate in theoretical physics under R. H. Fowler.
Groundbreaking research on absorption of cosmic rays and electron shower production, . . . series of papers
1935 – Bhabha published [1] – the first calculation of the the cross section of electron-positron scattering.
1945 – He established the Tata Institute of Fundamental Research in Bombay . . .
1948 – . . . and the Atomic Energy Commission of India
He died when Air India Flight 101 crashed near Mont Blanc in January 24, 1966.
Many possible theories have been advanced for the aircrash, including a conspiracy theory in which CIA is involved in order to paralyze Indian nuclear weapon programme.

Bhabha's article

Here is the first page of [1]
H. Bhabha
The Scattering of Positrons by Electrons with Exchange on Dirac's Theory of the Positron
Proc. Roy. Soc., A154, p. 195 (1936)

Scattering of Positrons by Electrons 195

molecules have orientations similar to the dibenzyl orientation, and the other two can approximately be derived from them by a rotation of 180° about the a axis, and a translation of $\frac{1}{2}c$. The resulting structure explains the pseudo-orthorhombic properties, the approximate halvings, and the principal X-ray intensities. It is contrary to a structure previously deduced from magnetic measurements by Krishnan, Guha, and Banerjee, who predicted a twisted and distorted molecule; but it is shown that the new structure is equally capable of explaining the magnetic data. Detailed measurements have not yet been made on tolane and azobenzene, but the preliminary data are sufficient to show that they are both closely similar to the stilbene structure.

The Scattering of Positrons by Electrons with Exchange on Dirac's Theory of the Positron

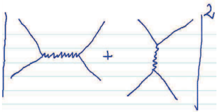
By H. J. Bhabha, Ph.D., Gonville and Caius College

(*Communicated by R. H. Fowler, F.R.S.—Received October* 20, 1935)

It has been shown by Mott† that exchange effects play a considerable part in the collision and consequent scattering of one electron by another. Mott's original calculation was non-relativistic, and there the exchange effect vanishes when the two electrons have their spins pointing in opposite directions. Møller‡ later developed relativistically invariant expressions for the collision of two charged particles with spin, and it may be seen directly from Møller's general formula for the collision cross-section that, in the collision of two identical particles, the effect of exchange does not in general vanish even when the two colliding particles initially have their spins pointing in opposite directions. It tends however to zero in this case as the relative velocity of the particles becomes small compared to c, the velocity of light, in agreement with the calculation of Mott.

The effect of exchange in the general relativistic case will still be con-

Bhabha's formula I



- $|\mathcal{M}_s + \mathcal{M}_t|^2$
- **simple process, $m_e = 0$**
- **strong forward peak $\sim 1/t^2$**
- **But: μ-pair etc. production advantageous ?**

$$\frac{d\sigma_0}{2\pi d\vartheta} = \frac{\alpha^2}{s}\left(\frac{s}{t} + 1 + \frac{t}{s}\right)^2$$

Beam energy E, scattering angle ϑ, and s, t:

$$\begin{aligned} s &= 4E^2 \\ t &= \frac{s}{2}\,(1 - \cos\vartheta) \end{aligned}$$

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where we have inserted in expressions like (11) the values of $E_1'^*$, $\mathbf{p}_1'^*$, etc., in terms of E^*, p^*, $\mathbf{p}'^*$. The spurs in (14) are easily evaluated if we remember that the spurs of all the Dirac matrices and their products are zero, excepting that of the unit matrix.

We get finally for the differential effective cross-section dQ^* for the scattering of the electron through an angle between θ^* and $\theta^* + d\theta^*$ in the system L^* the expression

$$\begin{aligned} dQ^* = \frac{\pi}{8}\frac{e^4}{m^2c^4\gamma^{*2}}\Big[& \frac{1}{(\gamma^{*2}-1)^2\sin^4\frac{1}{2}\theta^*}\{1 + 4(\gamma^{*2}-1)\cos^2\tfrac{1}{2}\theta^* \\ & \qquad + 2(\gamma^{*2}-1)^2(1+\cos^4\tfrac{1}{2}\theta^*)\} \\ & + \frac{1}{\gamma^{*4}}\{3 + 4(\gamma^{*2}-1) + (\gamma^{*2}-1)^2(1+\cos^2\theta^*)\} \\ & - \frac{1}{\gamma^{*2}(\gamma^{*2}-1)\sin^2\frac{1}{2}\theta^*}\{3 + 4(\gamma^{*2}-1)(1+\cos\theta^*) \\ & \qquad + (\gamma^{*2}-1)^2(1+\cos\theta^*)^2\}\Big].\sin\theta^*\,d\theta^*. \quad (15) \end{aligned}$$

This is just dQ. We may, if we choose, express it in terms of θ and γ by using the relations (1) and (2). This would only lead to very complicated expressions, and it is more convenient to leave it in its present form. dQ is the differential effective cross-section for the scattering of the electron through an angle between θ and $\theta + d\theta$ in the system in which the positron is initially at rest. But (15) is clearly quite symmetrical between the positron and electron, so that dQ also gives the effective cross-section for the scattering of the positron through an angle between θ and $\theta + d\theta$ in the system in which the electron is initially at rest. We shall henceforth use L to denote any system in which either the electron or the positron is initially at rest.

For many purposes it is more convenient to express the scattering in terms of the number of particles initially at rest which after the collision receive a certain fraction ε of the kinetic energy of the colliding particle. Let E'_R denote in the system L the energy after the collision of the particle which was initially at rest. (It may be either an electron or a positron.) Then E'_R is connected with θ by the usual relativistic formula†

Outline

Earlier presentations at FCAL meetings:

[2] 2005 Tel Aviv *Forward Bhabha Scattering – Theoretical Problems*
[3] 2007 Zeuthen *Two-loop Heavy Fermion Corrections to Bhabha Scattering*
[4] 2009 Zeuthen *NNLO contributions to Bhabha scattering*

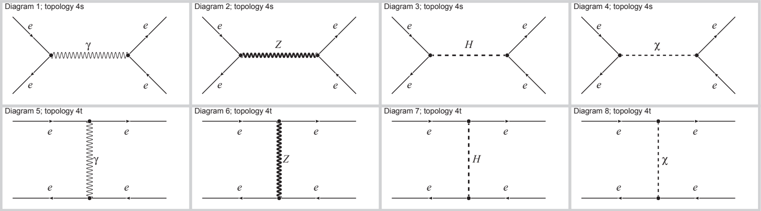
Mini-review [5] 2003 Mini-review Jadach, *Theoretical error of luminosity cross section at LEP*

Recent work:

[6, 7] 2006/08 Actis,Czakon,Gluza,Riemann
Virtual Hadronic and Heavy-Fermion $\mathcal{O}(alpha^2)$ Corrections to Bhabha Scattering
[8] 2009 Working Group Report . . . on Rad.Corrs. and MC Generators for Low Energies
Quest for precision in hadronic cross sections at low energy: Monte Carlo tools vs. experimental data.

Electroweak Born diagrams

The Born diagrams



$s = M_Z^2, 500, 3000 \ \ \mathrm{GeV}^2$ $\qquad \vartheta_{\min} = 26, 39, 44 \ \ \mathrm{mrad}$

$t_{\min} = \frac{s}{2}(1 - \cos\vartheta_{\min}) = 1.5, 0.2, 1.5 \quad \mathrm{GeV}^2$

$\frac{m_e^2}{t} \leq 2 \times 10^{-6}$ safely neglect m_e

$\frac{m_\mu^2}{t} \leq 6 \times 10^{-2}$ care about heavier masses

Higgs diagrams: play no role [even at meson factories]

Z diagrams: contribute at GigaZ and ILC, see later.

Electroweak Born cross-sections

The Bhabha Born cross-section:

$$\frac{d\sigma_{ew}}{d\Omega} = \frac{\alpha^2}{4s}\left(T_s + T_{st} + T_t\right),$$

with Z axial coupling $a = 1$ and vector coupling $v = 1 - 4s_w^2$,

$$\begin{aligned}
T_s &= (1+\cos^2\theta)\left[1 + 2\mathrm{Re}\chi(s)\left(v^2\right) + |\chi(s)|^2\left(1+v^2\right)^2\right] + 2\cos\theta\left[2\mathrm{Re}\chi(s) + |\chi(s)|^2\left(4v^2\right)\right],\\
T_{st} &= -2\frac{(1+\cos\theta)^2}{(1-\cos\theta)}\left\{1 + [\chi(t) + \mathrm{Re}\chi(s)]\left(1+v^2\right) + \chi(t)\mathrm{Re}\chi(s)\left[(1+v^2)^2 + 4v^2\right]\right\},\\
T_t &= 2\,\frac{(1+\cos\theta)^2}{(1-\cos\theta)^2}\left\{1 + 2\chi(t)\left(1+v^2\right) + \chi(t)^2\left[(1+v^2)^2 + 4v^2\right]\right\} + \frac{8}{(1-\cos\theta)^2}\left[1 - \chi(t)\left(1-v^2\right)\right]^2.
\end{aligned}$$

We choose the following conventions:

$$\begin{aligned}
\chi(s) &= \frac{G_F}{\sqrt{2}}\,\frac{M_Z^2}{8\pi\alpha}\,\frac{s}{s - M_Z^2 + iM_Z\Gamma_Z},\\
\chi(t) &= \frac{G_F}{\sqrt{2}}\,\frac{M_Z^2}{8\pi\alpha}\,\frac{t}{t - M_Z^2}, \qquad t < 0.
\end{aligned}$$

Input quantities

Beware:
Among the quantities

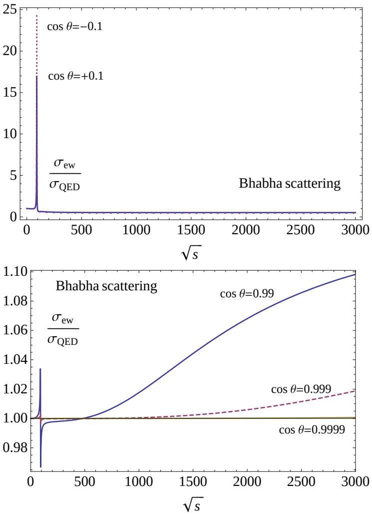
$$\alpha, \quad G_F, \quad s_w^2, \quad M_Z, \quad \Gamma_Z, \quad M_W$$

there are only three independent, and Γ_Z is predicted by the theory.

The phrasing *effective Born cross-section* means here that we use, in the Born estimates the following five input variables:

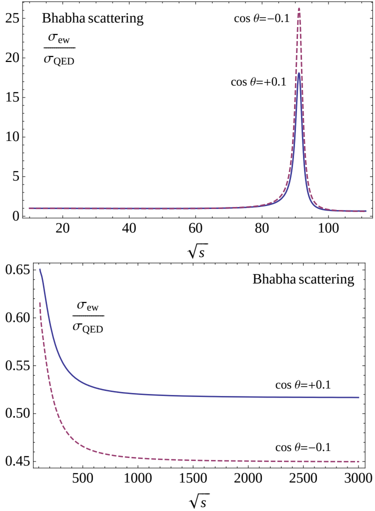
$$\begin{aligned}
\alpha &= 1/137.036, \\
s_w^2 &= 0.23, \\
M_Z &= 91.1876 \pm 0.0021 \,\mathrm{GeV}, \\
\Gamma_Z &= 2.4952 \pm 0.0023 \,\mathrm{GeV}, \\
G_F &= (1.16637 \pm 0.00001) \times 10^{-5} \,\mathrm{GeV}^{-2}.
\end{aligned}$$

Born_ew/Born_QED at large and small angles [7]



Ratio of electroweak to QED Bhabha scattering cross-sections at large angles (left) and small angles (right) as a function of $\sqrt{s}$.

Born_ew/Born_QED at different energies; large angles [7]



Ratio of electroweak to QED Bhabha scattering cross-section at large angles in the energy ranges of LEP1/GigaZ (up) and ILC (down).

Beyond Born: The role of NLO and NNLO contributions

For more details see e.g. [9] K. Mönig *Bhabha scattering at the ILC* talk at Mini-Workshop on Bhabha scattering, Univ. Karlsruhe, April 2005
http://www-zeuthen.desy.de/~moenig//bhabha_ilc.pdf

- ILC: $e^+e^- \to W^+W^-, f\bar{f}$ with $O(10^6)$ events $\to 10^{-3}$
- ILC: $e^+e^- \to e^+e^-$, a probe for New Physics with $O(10^5)$ events/year $\to 10^{-3}$
- GigaZ: $e^+e^- \to Z \to$ had, l^+l^-, the latter with $O(10^8)$ events $\to 10^{-4}$

Conclude: will need:

$$\Delta\mathcal{L}/\mathcal{L} \approx 2 \times 10^{-4}$$

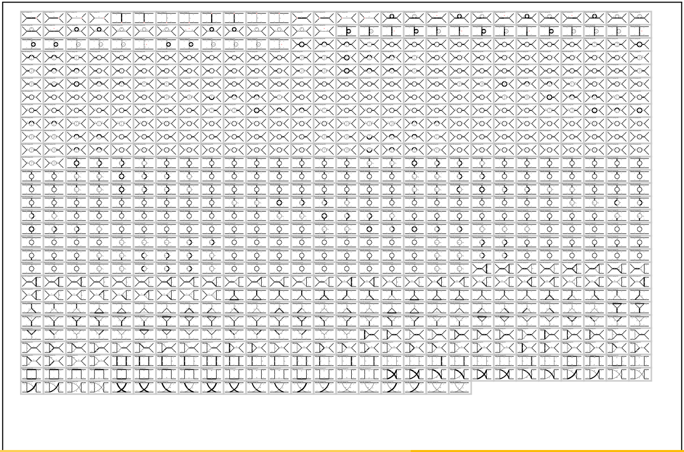
Luminosity L – from very forward Bhabha scattering
Need a theoretical cross-section prediction
for small angles: with up to 5 significant digits
for large angles: with up to 4 significant digits
Compare the perturbative orders:

$$\begin{aligned} (\alpha/\pi) &= 2 \times 10^{-3} && \text{NLO – 1-loop + real} \\ (\alpha/\pi)^2 &= 0.6 \times 10^{-5} && \text{NNLO – 2-loop + real} \end{aligned}$$

Additional factors of the type $\log\left(\frac{s}{m_e}\right)^n, \quad n = 1, 2\ [, 3]$ at two loops

804 electroweak 1-loop Bhabha diagrams

The calculation of electroweak 1-loop corrections plus real QED bremsstrahlung is fully controlled and automatized.
QED bremsstrahlung → MC programs: there are other experts, not me.



1991 – can do better now, but yet standard in production

Table 2:
The differential Bhabha cross section in nbarn as function of the scattering angle and the cms-energy.
$M_Z = 91.16$ *GeV*, $m_t = 150$ *GeV*, $M_H = 100$ *GeV*.
Upper rows: DZ, lower rows: H.
δ_m: *largest relative deviation in per mille.*

$\sqrt{s}$ (GeV) / θ	60	89	91.16	93	200
15°	129.6 129.6	65.11 65.11	57.93 57.93	49.00 49.00	11.82 11.82
45°	1.451 1.451	1.376 1.377	1.755 1.756	.4833 .4837	11.67 11.68
60°	.4303 .4305	.6124 .6129	1.125 1.126	.2697 .2699	.03075 .03077
75°	.1717 .1718	.3627 .3630	.8718 .8720	.2232 .2233	.01072 .01072
90°	.08873 .08876	.2768 .2769	.7790 .7787	.2088 .2087	.004862 .004855
105°	.05917 .05918	.2690 .2690	.8082 .8074	.2157 .2157	.002858 .002853
120°	.04906 .04906	.3053 .3051	.9323 .9309	.2429 .2426	.002077 .002074
135°	.04671 .04672	.3626 .3624	1.111 1.109	.2838 .2833	.001743 .001742
165°	.04839 .04839	.4638 .4635	1.425 1.422	.3590 .3584	.001539 .001540
δ_m	0.6	0.8	1.8	2.0	1.7

Bhabha scattering

Bardin,Hollik,T.R., Z.PhysikC49(1991)485 The 1991

result is yet the state of the art in e.g. the programs ZFITTER and BHWIDE.
ZFITTER news page at DESY Zeuthen:
`http://www-zeuthen.desy.de/theory/research/zfitter/`

A dedicated comparison: altalc versus Feynarts

The 1991 result is yet the state of the art in e.g. the programs ZFITTER and BHWIDE.
Now, such calculations of O(1000) diagrams are better than to 10 digits.

Results: Numerical comparison in all $f\bar{f}$

Bhabha $e^- e^+ \to e^- e^+ \, (\gamma)$ **at LC:** $\sqrt{s} = 500 \text{ GeV}, E_{\max}(\gamma_{\text{soft}}) = \frac{\sqrt{s}}{10}$

$\cos\theta$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{Born}}$ (pb)	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\mathcal{O}(\alpha^3)=\text{Born+QED+weak+soft}}$	Group
-0.9999	0.21482 70434 05632 5	0.14889 12125 78083 7	aITALC
-0.9999	0.21482 70434 05632 6	0.14889 12189 28404 0	*FeynArts*
-0.9	0.21699 88288 10920 5	0.19344 50785 26863 6	aITALC
-0.9	0.21699 88288 10920 0	0.19344 50785 26862 2	*FeynArts*
-0.9	0.21699 88288 41513 1	0.19344 50785 62637 9	$m_e = 0$
$+0.0$	0.59814 23072 50330 3	0.54667 71794 69423 1	aITALC
$+0.0$	0.59814 23072 50329 4	0.54667 71794 69421 8	*FeynArts*
$+0.0$	0.59814 23072 88584 4	0.54667 71794 99961 4	$m_e = 0$
$+0.9$	$0.18916\,03223\,32270\,6 \cdot 10^3$	$0.17292\,83490\,66507\,2 \cdot 10^3$	aITALC
$+0.9$	$0.18916\,03223\,32270\,6 \cdot 10^3$	$0.17292\,83490\,66508\,0 \cdot 10^3$	*FeynArts*
$+0.9$	$0.18916\,03223\,31848\,5 \cdot 10^3$	$0.17292\,83490\,61347\,4 \cdot 10^3$	$m_e = 0$
$+0.9999$	$0.20842\,90676\,46142\,9 \cdot 10^9$	$0.19140\,17861\,11341\,6 \cdot 10^9$	aITALC
$+0.9999$	$0.20842\,90676\,46436\,4 \cdot 10^9$	$0.19140\,17861\,11979\,0 \cdot 10^9$	*FeynArts*

Great independent agreement up to 14 digits! : limit in double precision

Previous agreement with *FeynArts*: 11 digits hep-ph/0307132**, SANC: 10 digits** hep-ph/0207156

Thanks to T. Hahn, numbers supplied with *FeynArts* + *FormCalc* + *LoopTools*

Bhabha scattering at $\sqrt{s} = 500$ GeV – EWSM versus QED

[10] Fleischer, Lorca, Riemann, LCWS Paris 2004, Automatized calculation of 2-fermion production with DIANA and aITALC

$e^+e^- \to e^+e^- \quad \sqrt{s} = 500$ GeV			
$\cos\theta$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{Born}}$ /pb	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{B+1-loop}}$ /pb	Model
−.9000	0.216 998	0.144 359	EWSM
−.9000	0.523 873	0.387 798	QED
−.5000	0.261 360	0.181 086	EWSM
−.5000	0.611 600	0.471 451	QED
0.0000	0.598 142	0.431 573	EWSM
0.0000	$0.117\,253 \cdot 10^1$	0.916 946	QED
0.5000	$0.421\,272 \cdot 10^1$	$0.320\,045 \cdot 10^1$	EWSM
0.5000	$0.550\,440 \cdot 10^1$	$0.435\,535 \cdot 10^1$	QED
0.9000	$0.189\,160 \cdot 10^3$	$0.150\,885 \cdot 10^3$	EWSM
0.9000	$0.189\,118 \cdot 10^3$	$0.152\,861 \cdot 10^3$	QED
0.9900	$0.206\,555 \cdot 10^5$	$0.170\,576 \cdot 10^5$	EWSM
0.9900	$0.206\,381 \cdot 10^5$	$0.170\,818 \cdot 10^5$	QED
0.9990	$0.208\,236 \cdot 10^7$	$0.176\,139 \cdot 10^7$	EWSM
0.9990	$0.208\,242 \cdot 10^7$	$0.176\,190 \cdot 10^7$	QED
0.9999	$0.208\,429 \cdot 10^9$	$0.180\,172 \cdot 10^9$	EWSM
0.9999	$0.208\,430 \cdot 10^9$	$0.180\,178 \cdot 10^9$	QED

Running of α_{em} has been switched off here

Bhabha scattering at $\sqrt{s} = M_Z$ GeV

[11] Fleischer Gluza Lorca Riemann EPJC 2006, First order radiative corrections to Bhabha scattering in d dimensions

$\cos\theta$	Born EWSM	$\mathcal{O}(\alpha)$ EWSM
-0.9	$0.12201 \cdot 10^4$	$0.11767 \cdot 10^4$
-0.7	$0.10099 \cdot 10^4$	$0.95012 \cdot 10^3$
-0.5	$0.85685 \cdot 10^3$	$0.79246 \cdot 10^3$
0	$0.73164 \cdot 10^3$	$0.64561 \cdot 10^3$
$+0.5$	$0.10701 \cdot 10^4$	$0.91360 \cdot 10^3$
$+0.7$	$0.16162 \cdot 10^4$	$0.13917 \cdot 10^4$
$+0.9$	$0.70112 \cdot 10^4$	$0.63472 \cdot 10^4$
$+0.99$	$0.62198 \cdot 10^6$	$0.57186 \cdot 10^6$
$+0.999$	$0.62612 \cdot 10^8$	$0.57540 \cdot 10^8$
$+0.9999$	$0.62666 \cdot 10^{10}$	$0.57822 \cdot 10^{10}$

Differential cross-sections in pbarn for Bhabha scattering at $\sqrt{s} = m_Z$. Born contribution and the $\mathcal{O}(\alpha)$ correction are shown; the maximum soft-photon energy is $\sqrt{s}/10$.

Bhabha scattering at $\sqrt{s} = 500$ GeV

[11] Fleischer Gluza Lorca Riemann EPJC 2006, First order radiative corrections to Bhabha scattering in d dimensions

rad	$\cos\theta$	Born EWSM	$\mathcal{O}(\alpha)$ EWSM		$\mathcal{O}(\alpha)$ QED $N_f = 9$	
2.691	-0.9	$2.16999 \cdot 10^{-1}$	$1.93445 \cdot 10^{-1}$	-10.85%	$4.69800 \cdot 10^{-1}$	116.50
2.346	-0.7	$2.30098 \cdot 10^{-1}$	$2.08843 \cdot 10^{-1}$	-9.24%	$5.03879 \cdot 10^{-1}$	118.98
2.094	-0.5	$2.61360 \cdot 10^{-1}$	$2.38707 \cdot 10^{-1}$	-8.67%	$5.66238 \cdot 10^{-1}$	116.65
1.571	0	$5.98142 \cdot 10^{-1}$	$5.46677 \cdot 10^{-1}$	-8.60%	$1.09322 \cdot 10^{0}$	82.77
1.047	$+0.5$	$4.21273 \cdot 10^{0}$	$3.81301 \cdot 10^{0}$	-9.49%	$5.13530 \cdot 10^{0}$	21.90
0.795	$+0.7$	$1.58240 \cdot 10^{1}$	$1.43357 \cdot 10^{1}$	-9.41%	$1.64548 \cdot 10^{1}$	3.99
0.451	$+0.9$	$1.89160 \cdot 10^{2}$	$1.72928 \cdot 10^{2}$	-8.58%	$1.76464 \cdot 10^{2}$	-6.71
0.142	$+0.99$	$2.06556 \cdot 10^{4}$	$1.90607 \cdot 10^{4}$	-7.72%	$1.91774 \cdot 10^{4}$	-7.16
0.045	$+0.999$	$2.08236 \cdot 10^{6}$	$1.91624 \cdot 10^{6}$	-7.98%	$1.92546 \cdot 10^{6}$	-7.53
0.014	$+0.9999$	$2.08429 \cdot 10^{8}$	$1.91402 \cdot 10^{8}$	-8.17%	$1.92270 \cdot 10^{8}$	-7.75

Differential cross-sections in pbarn for Bhabha scattering at $\sqrt{s} = 500$. Born contribution, the $\mathcal{O}(\alpha)$ correction, and also a QED prediction are shown; the maximum soft-photon energy is $\sqrt{s}/10$.

Bhabha scattering at $\sqrt{s} = 500$ GeV

[12] Gluza Lorca Riemann ACAT 2003, KEK Tsukuba, Automated use of DIANA for two-fermion production at colliders

Table: Cross-sections for $e^- e^+ \to e^- e^+ \; (\gamma)$ at $\sqrt{s} = 500 \text{ GeV}$ with a photon energy cut $E^{\max}_{\gamma_{\text{soft}}} = \sqrt{s}/10$.

$\cos\theta$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\mathcal{O}(\alpha^3)}$ (pb)	Group
-0.9	0.19344 50785 26863 6	aïTALC
-0.9	0.19344 50785 26862 2	FA/FC/LT
-0.9	0.19344 50785 62637 9	$m_e = 0$
0.0	0.54667 71794 69423 1	aïTALC
0.0	0.54667 71794 69421 8	FA/FC/LT
0.0	0.54667 71794 99961 4	$m_e = 0$
0.9	$0.17292\,83490\,66507\,2 \cdot 10^3$	aïTALC
0.9	$0.17292\,83490\,66508\,0 \cdot 10^3$	FA/FC/LT
0.9	$0.17292\,83490\,61347\,4 \cdot 10^3$	$m_e = 0$

Overview

The evaluation of the NNLO corrections is nearly finished.
The present MC-programs contain “the most important of the NNLO corrections” already – whatever this means.
What it means is not in all cases evident.
Why?

(I) The leading NNLLO [next-to-next-to–leading-Log] corrections are proportional to

$$\frac{\alpha^2}{\pi^2}\log^2(t/m_e^2), \quad \frac{\alpha^2}{\pi^2}\log(t/m_e^2)$$

But we seem to need also the constant terms [in m_e^2/s], without lop-factors

$$\frac{\alpha^2}{\pi^2}\log^2(t/m_e^2), \quad \frac{\alpha^2}{\pi^2}$$

(II) There are types of diagrams not considered so far.
It may come out that the so far uncovered terms stay small and negligible also with the new accuracy demands.
But maybe – not

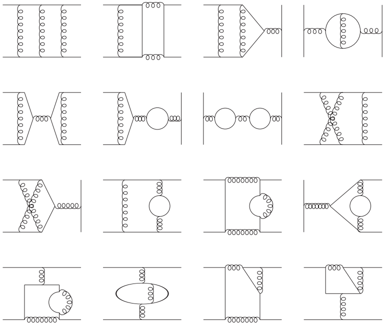
New theoretical techniques are needed to finalize the theoretical predictions.

Adapted from Z.Bern, LoopFest 2002

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.



The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Have in mind:

- pure photonic virtual corrections
- $m_e = 0$ assumed here (not seen)
- $N_f = 1$: only electrons no hadrons, μ, τ, *quarks*

Add the real corrections

- real 2-photon emission $\to$ JG
- electron pair emission $\to$ JG

Recent progress for the $N_f = 1$ massive predictions

Massive next-to-next-to-leading–Logarithmic virtual corrections

[13] Glover, Tausk, van der Bij, 2001
Second order contributions to elastic large-angle Bhabha scattering

In massless QED: Constant term $\left(m_e^2\right)^0$ without Logs

[14] Bern, Dixon, Ghinculov, 2002
Two-loop correction to Bhabha scattering

In massive QED: Constant term $\left(m_e^2\right)^0$ without Logs

[15] Penin, 2005
Two-loop corrections to Bhabha scattering

Massive electron loop insertion at NNLO: $N_f = 1$ corr^s

[16] Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij, 2004 *Two-loop $N_f = 1$ QED Bhabha scattering differential cross section*
also, but a bit later: [17] Czakon, Gluza, Riemann (2004)

Real electron pair emissions

[18] Burgers, 1985
and present studies by M. Worek, J. Gluza, TR et al. 2010

Backup-slide for virtual 2-loops with $m_e \neq 0$ I

Status 2005

Know the constant term ($m_e = 0$)
from 2-loop Bhabha scattering

A. Penin, Two-Loop Corrections to Bhabha Scattering, hep-ph/0501120 v.3, $\rightarrow$ PRL
Transform the massless 2-loop results of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by $D = 4 - 2\epsilon$ into the on-mass-shell renormalization with $m_e \rightarrow 0$ and IR regulation by $\lambda = m_\gamma \neq 0$
Use IR-properties of amplitudes (see Penin):

- [A] Exponentiation of the IR logarithms (Sudakov 1956,...)
- [B] Factorization of the collinear logarithms into expernal legs (Frenkel, Taylor 1976)
- [C] Non-renormalization of the IR exponents (YFS 1961, ...)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani et al. 2005, Penin)

If all this is correct, the constant term in m_e is known for the MCs (but the radiative one-loops with 5-point functions).

The real electron pair emission diagrams

Not all of them contain true electron pairs

At high energies, the true pairs develop large logarithms and should dominate

Without MC program hopeless

As mentioned, see talk of J. Gluza:

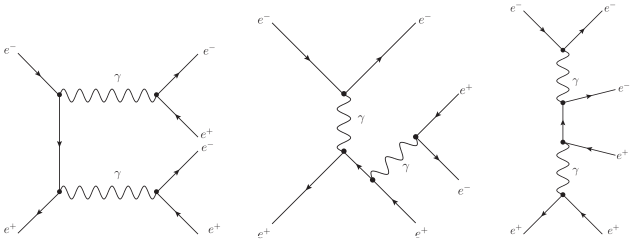


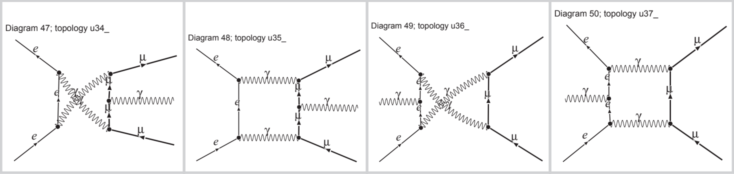
Figure: Samples of the 36 diagrams contributing to $e^+e^- \to e^+e^-(e^+e^-)$.

Not done for NNLO $N_f = 1$: loop-by-loop contributions from real photon emission

The one-loop diagrams for real photon emission:

$$e^+ + e^- \rightarrow e^+ + e^- + \gamma \qquad \text{at QED 1-loop}$$

In preparation: Gluza, Riemann, Yundin and Czyz et al.
Show here the μ-pair sub-sample of diagrams:



First results:
[19] Actis et al., (2008)
NLO QED Corrections to Hard-Bremsstrahlung Emission in Bhabha Scattering
[20] Kajda, Sabonis, Yundin (2008)
QED Pentagon Contributions to $e^+e^- \rightarrow \mu^+\mu^-\gamma$

Note: Need 5-point tensor reduction [21] Fleischer, Riemann + C++ by Yundin

What remains: the virtual and real $N_f = 2$ corrections I

Show only the non-factorizing, truly complicated diagrams and the real corrections

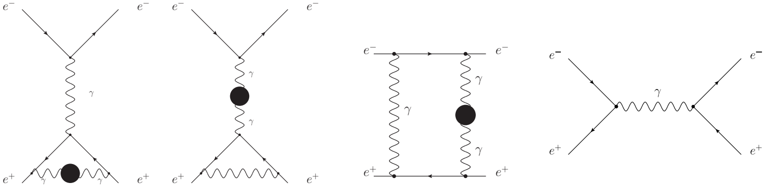


Figure: (a)–(c) are sample two-loop diagrams

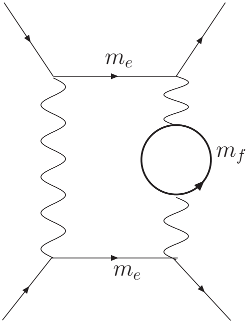
Backup: virtual $N_f = 2$ corrections I

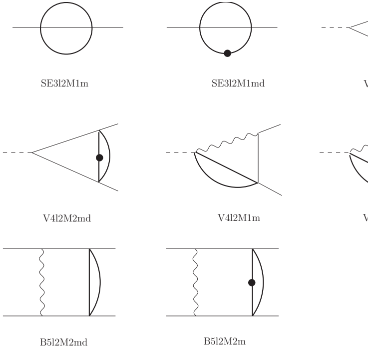
. **The $n_f = 2$ contributions have been determined in 2007**

- Self-energies are not a two-masses-problem
- 2-vertices are known (for $m_e^2 = m_f^2$ and $m_e^2 << m_f^2$): G. Burgers PLB 164 (1885), Kniehl, Krawczyk, Kühn, Stuart PLB 209 (1988)
- What is really new: the 2-boxes with two different fermions involved

Box-master integrals: Actis, Czakon, Gluza, TR (ACGR), [17] PRD 71 (2005)

Backup: virtual $N_f = 2$ corrections II





Backup: virtual $N_f = 2$ corrections III

- $m_e^2 << m_f^2 << s, t$: Becher,Melnikov JHEP 6 (2007) and [22] ACGR NPB 786 (2007)
- $m_e^2 << m_f^2, s, t$: ACGR 0710.5111 $->$ [23] APP B38 (2007)
  and Bonciani,Ferroglia,Penin 0710.4775 (2007)
- $m_e^2 << m_{hadrons}^2, s, t$: ACGR 0711.3847 $->$ [6] PRL 100 (2008)
  and Kuehn et al. 0807.1284 (2008)

Backup: virtual $N_f = 2$ corrections I

How to evaluate the $N_f = 2$ diagrams?

We did it in 2 ways

- Decompose the 2-loop integrals to master integrals, solve them.
  Here: In the limit $m_e^2 << m_f^2 << s, t, u$
  This was done in hep-ph/07042400v2 $\longrightarrow$ ACGR, [22] NPB 786 (2007)
- Alternatively, rewrite the 2-loop integrals as dispersion integrals.
  Decompose the loop integrals afterwards into master integrals
  The master integrals are simpler, of one-loop type, but the numerical dispersion integration remains then.

Advantages of the dispersion integrals:

- get easily the range $m_e^2 << m_f^2, s, t, u$
- method applies also to hadronic insertions

Backup: virtual $N_f = 2$ corrections I

Dispersion Integrals

$$\frac{g_{\mu\nu}}{q^2 + i\,\delta} \rightarrow \frac{g_{\mu\alpha}}{q^2 + i\,\delta} \left(q^2\, g^{\alpha\beta} - q^\alpha\, q^\beta\right) \Pi_{\rm had}(q^2)\, \frac{g_{\beta\nu}}{q^2 + i\,\delta},$$

the once-subtracted dispersion integral

$$\Pi_{\rm had}(q^2) = -\frac{q^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dz}{z}\, \frac{\mathrm{Im}\,\Pi_{\rm had}(z)}{q^2 - z + i\,\delta}.$$

Finally, one relates $\mathrm{Im}\,\Pi_{\rm had}$ to the hadronic cross-section ratio $R_{\rm had}$,

$$\mathrm{Im}\Pi_{\rm had}(z) = -\frac{\alpha}{3} R_{\rm had}(z) = -\frac{\alpha}{3}\, \frac{\sigma_{e^+e^-\to\mathrm{hadrons}}(z)}{(4\pi\alpha^2)/(3z)},$$

For heavy fermion insertions, we have instead of $R_{\rm had}(z)$:

$$R_f(z) = Q_f^2 C_f (1 + 2m_f^2/z)\sqrt{1 - 4m_f^2/z},$$

Replacing the $\Pi_{\rm had}(q^2)$ in a vertex or in box diagram by the z-dispersion integral and exchanging the $\int d^4k$ with the $\int dz$ creates one-loop diagrams with a subsequent z-integration.

Backup: virtual $N_f = 2$ corrections I

The kernel functions for the dispersion integrals

$$\Delta\alpha(x) = \Delta\alpha^{(5)}_{\rm had}(x) + \Pi_e(x) + \sum_{f=\mu,\tau,t} \Pi_f(x)$$

$$\Delta\alpha^{(5)}_{\rm had}(x) = \frac{\alpha}{\pi}\frac{x}{3}\int_{4m_\pi^2}^{\infty} dz \frac{R^{(5)}_{\rm had}(z)}{z}\frac{1}{x-z+i\delta}$$

$$V_2(x) = V_{2e}(x) + V_{2\rm rest}(x)$$

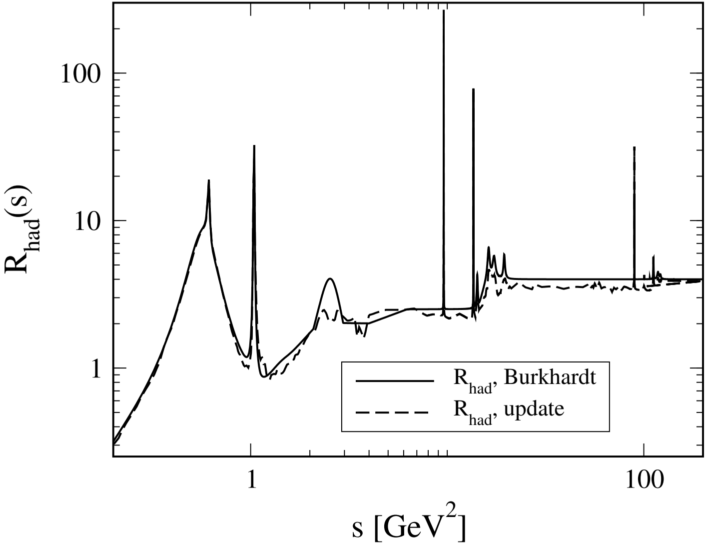
$$V_{2\rm rest}(x) = \int_{4M^2}^{\infty} dz \frac{R(z)}{z} K_V(x+i\delta; z)$$

$$K_V(x;z) = \frac{1}{3}\left\{-\frac{7}{8} - \frac{z}{2\,x} + \left(\frac{3}{4}+\frac{z}{2\,x}\right)\ln\left(-\frac{x}{z}\right) - \frac{1}{2}\left(1+\frac{z}{x}\right)^2\left[\zeta_2 - \mathrm{Li}_2\left(1+\frac{x}{z}\right)\right]\right.$$

$$B_i(x,y) = \int_{4M^2}^{\infty} dz \frac{R(z)}{z} K_{box,i}(x+i\delta, y+i\delta; z)$$

The $K_{box,i}(x,y;z)$ are determined as linear combinations of one-loop integrals with mass $z = M^2$.

Backup: virtual $N_f = 2$ corrections



A comparison of the parametrizations from [24] and [25].

The real μ pair corrections

Under study: Worek, Gluza, TR et al.

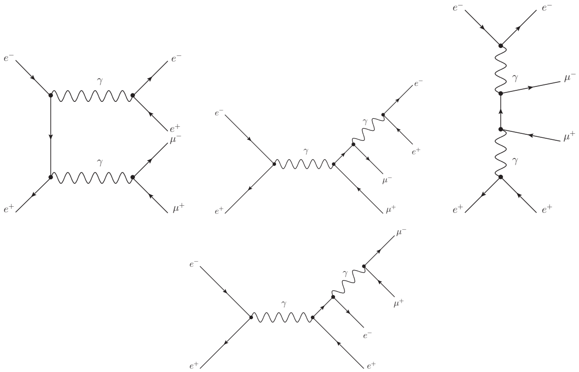
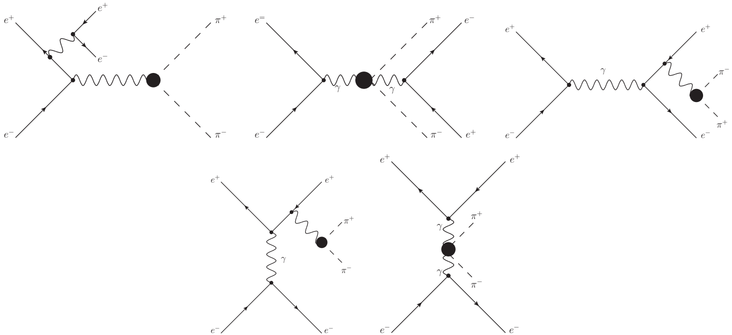


Figure: Samples of the 12 diagrams contributing to $e^+e^- \to e^+e^-\mu^+\mu^-$.

The real pion pair emission

Need also the complete real hadron emission at high energies
under study with Czyz, Gluza et al., pion emission only: [26]
See discussion in talk by J. Gluza



Sample diagrams with real pion pair emission, taken from [26].

Summary

We have plenty of time, but we have to use it dedicatedly . . .

- Thank you for spelling out the need of NNLO results !
- Much non-trivial progress reached in last 10 years
- Did not yet find the way into the MC-programs
- A bit new stuff is to do yet
- Understanding details and combining them will take another effort

"You know, we're just not reaching that guy."

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To be added

- Formulae for figures
- 2-loop ew: Sudakov remarks for ILC, due to small M_W, M_Z
- 2-loop ew: form factors for GigaZ, due to large m_t, M_H
- 2-loop ew: complete needed? For large and/or small angles at which energy?
- comment on status for meson factories
- real pairs plus irreducible vertex: some Logs compensate, and if the Logs are huge, this becomes improtant/interesting; it plays absolutely no role for meson factories; often the peripheral pairs are not discussed at all.