Precision Higgs Phenomenology at N3LO and Beyond

Gherardo Vita

Higgs 2022
Pisa, 10 Nov 2022

Based on:

“Collinear expansion for color singlet cross sections”
M.Ebert, B.Mistlberger, GV
[2006.03055]

“TMD PDFs at N3LO”
M.Ebert, B.Mistlberger, GV
[2006.05329]

“N-jettiness beam functions at N3LO”
M.Ebert, B.Mistlberger, GV
[2006.03056]

“Soft Integrals and Soft Anomalous Dimensions at N3LO and Beyond”
C.Duhr, B.Mistlberger, GV
[2205.04493]

“The Four-Loop Rapidity Anomalous Dimension and Event Shapes to Fourth Logarithmic Order”
C.Duhr, B.Mistlberger, GV
[2205.02242]
Testing the Higgs at Colliders

- Differential Higgs measurements at the moment are limited by statistics...

...but situation will improve dramatically with HL-LHC

See also talks by Caterina Vernieri and Alessandro Tarabini
Improving Theoretical Predictions

Precision Higgs Phenomenology requires higher order theory predictions

\[ \sigma_{pp\to X} \sim \int f_a(x_1) f_b(x_2) \otimes \hat{\sigma}_{ab\to X} \]

\[ \hat{\sigma}_{ab\to X} = \sigma_0 \text{ LO} + \alpha_s \sigma_1 \text{ NLO} + \alpha_s^2 \sigma_2 \text{ NNLO} + \alpha_s^3 \sigma_3 \text{ N^3LO} \]

Projected theory error reduction by a factor of 2 in HL-LHC analysis will require tremendous effort from the theory community...

Theory errors are projected to be a major limiting factor for Higgs precision program

See also overview talks by G. Zanderighi and R. Harlander...
Improving Theoretical Predictions

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Note: N3LO corrections are sizable not only for Higgs! They are necessary ingredients for the precision program at LHC and future colliders.

**“The Path Forward to N3LO”**
Snowmass Whitepaper
[Caola, Chen, Duhr, Liu, Mistlberger, Petriello, GV, Weinzierl]

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n3loxs [Baglio, Duhr, Mistlberger, Szafron ‘22]
Predictions for Differential Cross Sections

\[ \sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2 \]

- Cross sections are obtained via **phase space integrals** over **amplitudes** (squared) convoluted with Parton Distribution Functions (PDFs)

- Bottlenecks are present for each ingredient. In particular:
  - Efficiently calculate and evaluate multiloop scattering amplitudes
  - Handling of kinematics limits and phase space singularities
  - Extracting N3LO PDFs

- Note: this is just for the hard scattering process, many details also needed for full fledged MC prediction (parton shower, hadronization, ...)


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Predictions for Differential Cross Sections: IR singularities

\[ \sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2 \]

- Cross sections require integration over phase space
- Complexity of infrared singularities grows with loop order
- Extremely challenging to systematize their treatment order by order
- Cancellation of infrared singularities at a given order necessary for fixed order prediction
- Kinematic regions close to these singular boundaries give physical effects not captured by standard perturbation theory
- Several observables necessity of all order treatment of soft and collinear QCD radiation (resummation), (eg. Higgs \( p_T \) at \( p_T < 30 \text{ GeV} \))
Differential Cross Sections using Slicing

\[ \sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2 \]

- One way to deal with IR singularities in cross sections calculations are EFT subtractions
  - \( q_T \) Subtraction: [Catani, Grazzini '07]
  - N-Jettiness: [Boughezal, Focke, Liu, Petriello '15]
  - [Gaunt, Stahlhofen, Tackmann, Walsh '15]

\[ \sigma(\mathbf{X}) = \int_0^{q_T \text{cut}} dq_T \frac{d\sigma^{\text{sing}}(\mathbf{X})}{dq_T} + \int_{q_T \text{cut}} dq_T \frac{d\sigma(\mathbf{X})}{dq_T} + \Delta\sigma(\mathbf{X}, q_T \text{cut}) \]

- IR singularities controlled via factorization theorem in singular region

\[ \frac{d\sigma}{dQ^2 dY d^2 q_T} = \sigma_0 \sum_{a,b} H_{ab}(Q^2, \mu) \int \frac{d^2 \vec{b}_T}{(2\pi)^2} e^{i \vec{q}_T \cdot \vec{b}_T} \tilde{B}_a \left( x_1^B, b_T, \mu, \frac{b_T \omega_a}{\nu} \right) \tilde{B}_b \left( x_2^B, b_T, \mu, b_T \omega_b \nu \right) \]

- Above the cut region from lower order subtraction for H+1j

- Numerically challenging (tradeoff between stability of H+1j and size of residuals)

- Main challenge to extend slicing to N3LO for ggH was determination of N3LO gluon \( q_T \)

**Beam Functions** (aka gluon TMDPDF) [Ebert, Mistlberger, GV '20]
Beam Functions calculation at N3LO

- We calculated the beam function matching kernels at N3LO in SCET ~ closely related to fully differential partonic cross sections for ggH and Drell-Yan in collinear limit
  - Feynman diagrams leading to \( \sim 1 \text{ million} \) scalar integrals in \( d \) dimensions (dimensional regularization)
  - Collinear Expansion at the cross section level
    - “Collinear expansion for color singlet cross sections” [Ebert, Mistlberger, GV ‘20]
  - Reduction to basis of Master Integrals via Integration By Parts (IBPs) identities
  - 490 new Master Integrals to calculate

- Derived system of Differential Equations for the Master Integrals
- System has 2 non trivial scales with algebraic dependence on the variables (that is beyond what is solvable with algorithmic tools available in the literature)
- Algebraic sectors: constructed dlog integrand basis via calculation of leading singularities of candidate integrals on maximal cut surface
- Boundary values of differential eqn. from soft integrals and constraints on singular behavior

- One (complicated) calculation to obtain all color channels for:
  - Quark \( \tau \) beam functions (N-Jettiness)
  - Gluon \( \tau \) beam functions (N-Jettiness)
  - Quark TMDPDF \( (q_T) \)
  - Unpolarized Gluon TMDPDF \( (q_T) \)
Renormalization and Cross checks

**Gluon TMD Beam Function at N3LO**
(necessary for Higgs in gluon fusion at N3LO)

- Bare result contains: logs of transverse momentum, Harmonic Polylogs up to weight 5 in the collinear splitting variable, poles in dim-reg, rapidity divergences

- Renormalization steps:
  - Coupling renormalization
  - Zero-bin subtraction via Soft Function
  - Effective Field Theory (SCET II) renormalization (with rapidity divergences)
  - N3LO DGLAP counterterms subtractions

- All IR divergences cancelled
- Logarithmically enhanced terms match RGE prediction
- Eikonal limit matches soft approximation

[Billis, Ebert, Michel, Tackmann '19]

[ Luo, Yang, Zhu, Zhu]
Slicing at N3LO

- $q_T$ beam functions at N3LO were last missing ingredient for:
  - $q_T$ subtraction for differential and fiducial Drell-Yan and Higgs production at N3LO
  - $q_T$ resummation at N3LL

- Many new exciting phenomenological results at N3LO employing them!

And many more:

- [Ju, Schönherr '21]
- [Camarda, Cieri, Ferrera '21]
- [Re, Rottoli, Torrielli '21]
- [Chen, Gehrmann, Glover, Huss, Monni, Re, Rottoli, Torrielli '22]
- [Neumann, Campbell '22]
Resummation for Higgs $p_T$ distribution

\[
\frac{d\sigma^{\text{res}}}{dQdYdq_T} \sim \tilde{\sigma}_0 \int_0^\infty d(b_T Q)^2 J_0(b_T Q) H_{gg}(Q, \mu_H) B_g \left( x_1, b_T, \mu_B, \frac{Q e^{-y_b T}}{v_n} \right) B_g \left( b_T, \mu_B, Q e^Y b_T v_n \right) \\
\times \exp \left[ \int_{\mu_H}^\mu \frac{d\mu'}{\mu'} \gamma_H^g(Q, \mu') + 2 \int_{\mu, I}^\mu \frac{d\mu'}{\mu'} \widetilde{\gamma}_B^g(\mu', Q/\mu) \right] \\
\text{RG evolution of Hard function} \quad \mu\text{-RG evolution of Beam functions} \quad \text{Rapidity RG evolution}
\]

- Key ingredient for the resummation of large logarithms in transverse momentum distributions (including the Higgs $p_T$) is the \textbf{rapidity anomalous dimension}, AKA Collins Soper Kernel
  - NNLO: known for a long time. [Davies, Webber, Stirling ’85]
  - N3LO: determined in 2016 [de Florian, Grazzini ’00]
  - N4LO: C.Duhr, B.Mistlberger, GV [2205.02242]
    (see also [Moult, Zhu, Zhu ’22])
Rapidity Anomalous Dimension to Four Loops

- Brute force N4LO calculation for this quantity way beyond current fixed order technology (would require N4LO differential distribution in soft/collinear limit)
  
  \[ \gamma_{r,4} = C_A^2 C_R \left( -\frac{21164}{9} \zeta_2^2 - \frac{26104}{9} \zeta_2 \zeta_3 + \frac{4228}{3} \zeta_4 \right) \frac{C_A}{3} \left( \frac{2752}{3} \zeta_2 \zeta_5 + \frac{1201744}{27} \zeta_6 + \frac{11071}{3} \zeta_7 - \frac{28290079}{2187} \frac{b_4^4}{12} \right) \]

- Used EFT consistency rules and conformal identities to relate N4LO an. dim. to N3LO TMD Soft Function at higher order in dimensional regularization
  
  \[ C_A C_R \left( \frac{224}{9} \zeta_3^2 \zeta_2 + \frac{6752}{9} \zeta_2^3 - \frac{22256}{9} \zeta_2 \zeta_3 - \frac{160}{9} \zeta_4 + \frac{1472}{9} \right) + C_R \left( -\frac{898033}{2916} + \frac{160}{9} \zeta_3 \zeta_2 - \frac{160}{9} \zeta_4 + \frac{10432}{9} \right) \]

- Obtained results at N4LO for quark and gluon anomalous dimension.
  
  - applicable both to Higgs transverse momentum distribution in ggH and bbH as well as for Drell-Yan (di-lepton production, Z/W)

- Full N4LO logarithmic structure at small \( p_T \) determined up to PDF evolution

\[ \alpha_s(m_Z) = 0.118 \]

\[ b_T \text{[GeV}^{-1}] \]

\[ \gamma_{r,4}^f = C_A^2 C_R \left( -\frac{21164}{9} \zeta_2^2 + \frac{26104}{9} \zeta_2 \zeta_3 + \frac{4228}{3} \zeta_4 \right) \frac{C_A}{3} \left( \frac{2752}{3} \zeta_2 \zeta_5 + \frac{1201744}{27} \zeta_6 + \frac{11071}{3} \zeta_7 - \frac{28290079}{2187} \frac{b_4^4}{12} \right) \]

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➢ Introduced motivations and techniques for theoretical predictions at N3LO

➢ Discussed the calculation of gluon Beam Function at N3LO via collinear expansion of cross sections

\[
\lambda^2 - 4\epsilon \quad \text{and} \quad -\lambda^2 + O(\lambda^3)
\]

➢ Presented the computation of the Rapidity Anomalous Dimension at N4LO, necessary for the resummation of the Higgs transverse momentum distributions beyond N3LO

Quark Rapidity Anomalous Dimension

- NNLL
- N^3LL
- N^4LL
➢ Introduced motivations and techniques for theoretical predictions at N3LO

➢ Discussed the calculation of gluon Beam Function at N3LO via collinear expansion of cross sections

\[ \lambda^2 - 4e \left[ \begin{array}{c} \lambda_1 \\ 0 \\ \lambda_2 \\ 0 \\ \lambda_3 \\ 0 \end{array} \right] - \lambda^2 + O(\lambda^3) \]

➢ Presented the computation of the Rapidity Anomalous Dimension at N4LO, necessary for the resummation of the Higgs transverse momentum distributions beyond N3LO

15 Thank you!
Backup
Collinear expansion of cross sections: Applications

Approximation of differential distributions (e.g. Higgs rapidity at LHC, DY, ...)

“Collinear expansion for color singlet cross sections”
Ebert, Mistlberger, GV

Fixed order QCD beyond leading power (Data for subleading power RGEs, improvement for slicing methods, ...)

Collinear expansion of cross sections

Anomalous dimensions (Splitting functions, collinear, rapidity)

Universal objects of QCD IR at high perturbative order

Differential counterterms for local subtractions

Observable dependent initial state radiation dynamics (TMD PDFs, Beam Functions, double differential Beam Func, ...)

Observable dependent final state radiation dynamics (TMD Fragmentation Functions, EEC, thrust jet functions, ...)

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The calculation of the Rapidity anomalous dimension to 4 loop by brute force would require the calculation of differential object (e.g. TMD soft function) to 4 loop.

This is beyond the current technology for fixed order calculations.

Anomalous dimensions known at 4 loops

- **Hard/Collinear** Anomalous Dimension to 4 loops [von Manteuffel, Panzer, Schabinger - 2002.04617]

\[
\mu^2 \frac{d}{d\mu^2} H_{ij}^B(\mu^2) = \gamma_H^r(\alpha_S(\mu^2), \mu^2) H_{ij}^B(\mu^2),
\]

\[
\gamma_H^r(\alpha_S(\mu^2), \mu^2) = \Gamma_{\text{cusp}}^r(\alpha_S(\mu)) \ln \frac{Q^2}{\mu^2} + \frac{1}{2} \gamma_H(\alpha_S(\mu^2))
\]

- **Virtual** Anomalous Dimension to 4 loops [Das, Moch, Vogt - 1912.12920]

\[
\mu^2 \frac{d}{d\mu^2} f_i^{th}(z, \mu^2) = \gamma_f^r(z, \alpha_S(\mu^2)) \otimes z f_i^{th}(z, \mu^2),
\]

\[
\gamma_f^r(z, \alpha_S(\mu^2)) = \Gamma_{\text{cusp}}^r(\alpha_S(\mu^2)) \left[ \frac{1}{1-z} \right] + \frac{1}{2} \gamma_f^r(\alpha_S(\mu^2)) \delta(1-z)
\]
Rapidity Anomalous Dimension to Four Loops

- We also know there is a Rapidity/Threshold Correspondence for conformal theories, which holds at the critical dimension of QCD [Vladimirov - 1610.05791]

\[ \gamma^i_r[\alpha_s, \epsilon^*] + \gamma^i_{th}[\alpha_s, \epsilon^*] = 0 \]

\[ \beta[\alpha_s, \epsilon] = -2\alpha_s \left[ \epsilon + \frac{\alpha_s}{4\pi} \beta_0 + \left( \frac{\alpha_s}{4\pi} \right)^2 \beta_1 + \ldots \right] \]

\[ \epsilon^* = - \left[ \left( \frac{\alpha_s}{4\pi} \right) \beta_0 + \left( \frac{\alpha_s}{4\pi} \right)^2 \beta_1 + \ldots \right] \]

Critical dimension of QCD

- Threshold anomalous dimension is part of RGE of soft function

\[ \mu \frac{d}{d\mu} \ln S_i(\vec{b}_T, \mu, \nu) = 4\Gamma^i_{cusp}[\alpha_s(\mu)] \ln \mu/\nu + \gamma^i_{th}[\alpha_s] \]

\[ \nu \frac{d}{d\nu} \ln S_i(\vec{b}_T, \mu, \nu) = -4 \int_{b_0/b_T}^{\mu} \frac{d\mu'}{\mu'} \Gamma^i_{cusp}[\alpha_s(\mu')] + \gamma^i_{th}[\alpha_s] \]

- Via SCET I consistency relations, relate it to Virtual and Collinear an.dim.

\[ \gamma^r_{thr.} (\alpha_s(\mu^2)) = -2\gamma^r_f (\alpha_s(\mu^2)) - \gamma^r_H (\alpha_s(\mu^2)) \]
Difference between threshold and Rapidity anomalous dimension comes from higher orders in dimensional regularization evaluated at critical point!

\[ \epsilon^* = - \left[ \left( \frac{\alpha_s}{4\pi} \right) \beta_0 + \left( \frac{\alpha_s}{4\pi} \right)^2 \beta_1 + \ldots \right] \]

To obtain these terms it is necessary to calculate the Soft Function at N3LO to higher orders in dimensional regularization.

We obtained this in

“Soft Integrals and Soft Anomalous Dimensions at N3LO and Beyond”
C.Duhr, B.Mistlberger, GV [2205.04493]

Key point: Use method of differential equations and fix boundaries by relations between differential and inclusive threshold integrals.
Improving Theoretical Predictions

Precision Higgs Phenomenology requires higher order theory predictions

\[ \sigma_{pp \rightarrow X} \sim \int f_a(x_1) f_b(x_2) \otimes \hat{\sigma}_{ab \rightarrow X} \]

\[ \hat{\sigma}_{ab \rightarrow X} = \sigma_0 + \alpha_s \sigma_1 + \alpha_s^2 \sigma_2 + \alpha_s^3 \sigma_3 + \ldots \]

Important! Convergence turns out to be slower than naive estimate

\[ \Rightarrow \text{N3LO gives few percent (not per-mille) shift} \]

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n3loxs
[Baglio, Duhr, Mistlberger, Szafron '22]
Additional ingredients for percent level LHC pheno

\[
\sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2 + \mathcal{O}(\Lambda^2/Q^2)
\]

1. **Accessibility and User Friendliness:** Creating frameworks that make N³LO (and NNLO) predictions easily accessible for comparison to experimental data.

2. **Corrections beyond QCD:** EWK and masses.

3. **Factorisation Violation at N³LO:** tops, PDFs.

4. **Parton Showers:** Consistent combination of parton showers with fixed order perturbative computations at N³LO.

5. **Resummation:** Complementing N³LO computations and resummation techniques for infrared sensitive observables.

6. **Uncertainties:** Deriving / defining reliable uncertainty estimates for theoretical computations at the percent level.

7. **Beyond Leading Power Factorisation:** Exploring the limitations of leading power perturbative descriptions of hadron collision cross sections.
How to go forward: PDFs

\[ \sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2 \]

- Currently only NNLO PDFs available
- For **N3LO PDFs:**
  - Evolution of PDFs at N3LO: 4-loop splitting functions
    - First results: [Moch,Ruijl,Ueda,Vermaseren,Vogt] [MSHT20aN3LO PDF]
  - N3LO predictions for Global Dataset required.
  - Numerical capabilities to perform PDF fits.
  - EWK corrections / resummation / etc. in PDFs.

Note: Parton (gluon) Luminosity already improving significantly thanks to LHC data

Proton structure at the precision frontier

N3LO TMD Beam Function Results

Quark $q_T$ beam function (necessary for Drell-Yan) and Gluon $q_T$ beam function (necessary for Higgs in gluon fusion) at N3LO

$$\tilde{I}_{ij}^{\text{TMD}}(z, b_T, \mu, \mu^2)\bigg|_{\text{N3LO}} = \tilde{I}_{ij}^{\text{TMD}} (3,0) (z) + \sum_{n=1}^{6} \tilde{I}_{ij}^{\text{TMD}} (3,n) (z) \left[ \ln \left( \frac{b_T \mu}{b_0} \right) \right]^n$$

- Matching kernels have only HPLs
  - We provide expansions for kernels up to 50 orders for fast numerical evaluation
- N3LO corrections have non trivial $z$ dependence, $\sim 0.5$-2% deviation from NNLO

- First calculation of gluon beam functions at N3LO
- Found small error in one color structure for all quark-quark channels in quark TMD in the literature  
  [Luo, Yang, Zhu, Zhu 1912.05778]