New exact solutions of relativistic, viscous hydrodynamics

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Outline

New, exact solutions of relativistic Navier-Stokes (NS) and Israel-Stewart (IS) theory

→ spherically symmetric Hubble-flow: great amount of freedom of dissipative coefficients

Asymptotically perfect fluid solutions

→ effects of dissipative coefficients in final state measurments?

Applications of the new, relativistic, dissipative solutions

→ indirect description of experimental data

→ producing new, non relativistic solutions (my next presentation)

I. New, relativistic, dissipative solutions

Relativistic hydrodynamics (Navier-Stokes)

Local conservation of the four momentum and the particle number:

$$\partial_\mu \left(n u^\mu\right) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

The energy-momentum tensor is:

$$T^{\mu\nu} = \left(\varepsilon + p\right) u^\mu u^\nu - p g^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} - \Delta^{\mu\nu} \Pi$$

The heat current (with the heat conductivity λ):

$$q^\mu = \lambda \left(g^{\mu\nu} - u^\mu u^\nu\right) \left(\partial_\nu T - T u^\rho \partial_\rho u_\nu\right)$$

ζ: bulk viscosity

η: shear viscosity

The following terms describes the viscous effects:

$$\pi^{\mu\nu} = \eta \left[\Delta^{\mu\rho} \partial_\rho u^\nu + \Delta^{\nu\rho} \partial_\rho u^\mu\right] - \frac{2}{d} \eta \Delta^{\mu\nu} \partial_\rho u^\rho$$

$$\Pi = -\zeta \partial_\rho u^\rho$$

Relativistic hydrodynamics (Israel-Stewart)

Local conservation of the four momentum and the particle number:

$$\partial_\mu (n u^\mu) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

The energy-momentum tensor is:

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - p g^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} - \Delta^{\mu\nu} \Pi$$

The heat current (with the heat conductivity λ):

$$q^\mu = \lambda (g^{\mu\nu} - u^\mu u^\nu)(\partial_\nu T - T u^\rho \partial_\rho u_\nu)$$

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$$\Pi = -\zeta \partial_\rho u^\rho - \tau_\Pi u_\rho \partial^\rho \Pi$$

Relativistic hydrodynamics (Israel-Stewart)

Local conservation of the four momentum and the particle number:

$$\partial_\mu \left(n u^\mu\right) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

The energy-momentum tensor is:

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu - p g^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} - \Delta^{\mu\nu} \Pi$$

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To close the equation system:

EoS: $\varepsilon = \kappa p = c_s^{-2} p$

In this work: *κ=const.*

ζ: bulk viscosity
η: shear viscosity

Hubble-type solutions: scale variable

Hubble-type velocity field: $u^\mu = \frac{x^\mu}{\tau} = \gamma\left(1, \frac{r_x}{t}, \frac{r_y}{t}, \frac{r_z}{t}\right)$

Scale equation: $u^\mu \partial_\mu s = 0$

Directional scale variables: $s_x = \frac{r_x}{t},\ s_y = \frac{r_y}{t},\ s_z = \frac{r_z}{t}$

Satisfy the scale equation separately: $u^\mu \partial_\mu s_i = \partial_\tau s_i = 0$

Hubble-type solutions: equations to solve

Navier-Stokes theory

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

Energy conservation $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right) \frac{d}{\tau} p = \frac{d^2}{\tau^2} \frac{\zeta}{\kappa}$

Euler-equation: $p\tau - \zeta d = \phi(\tau)$

Entropy equation: $\partial_\tau \sigma + \frac{d}{\tau} \sigma = \frac{d^2}{\tau^2} \frac{\zeta}{T} \geq 0$

Ansatz for bulk viscosity: $\zeta = \zeta_0 \frac{p}{p_0}$

M. Nagy, M. Csanád, Z. Jiang, T. Csörgő: arXiv:1909.02498

T. Csörgő, G. K.: arXiv:2003.08859

Israel-Stewart theory

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

Energy conservation: $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right) \frac{d}{\tau} p = -\frac{d}{\tau} \frac{\Pi}{\kappa}$

Bulk pressure: $\Pi = -\zeta \frac{d}{\tau} - \tau_\Pi \dot{\Pi}$

Euler-equation: $p + \Pi = \Psi(\tau)$

Entropy equation: $\partial_\tau \sigma + \frac{d}{\tau} \sigma = -\frac{d}{\tau} \frac{\Pi}{T} \geq 0$

Ansatz for bulk viscosity: $\zeta = \Pi \frac{\zeta_0}{\Pi_0}$

Hubble-type solutions: equations to solve

Navier-Stokes theory

$\tau_\Pi \rightarrow 0$, or Π is constant

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

Energy conservation $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right) \frac{d}{\tau} p = \frac{d^2}{\tau^2} \frac{\zeta}{\kappa}$

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Euler-equation: $p + \Pi = \Psi(\tau)$

Entropy equation: $\partial_\tau \sigma + \frac{d}{\tau} \sigma = -\frac{d}{\tau} \frac{\Pi}{T} \geq 0$

Ansatz for bulk viscosity: $\zeta = \Pi \frac{\zeta_0}{\Pi_0}$

Heat conduction and shear viscosity cancelled!

Hubble-type solutions: equations to solve

Navier-Stokes theory

τ_Π → 0, or Π is constant

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

Energy conservation $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right)\frac{d}{\tau} p = \frac{d^2}{\tau^2}\frac{\zeta}{\kappa}$

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Analytic solutions of NS equations, with κ=const

The solution of the pressure is: $p(\tau) = p_0 \left(\frac{p_A}{p_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa}\right)}, \quad \frac{p_A}{p_0} = f_{A,0} = \exp\left[\frac{d^2\zeta_0}{\kappa_0 p_0 \tau_0}\right]$

The temperature has a generalized form: $T = T_0 \left(\frac{T_A}{T_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$

T. Csörgő, G. K.: arXiv:2003.08859

Conserved charge, μ > 0

$$p = nT$$

$$\frac{T_A}{T_0} = \frac{p_A}{p_0} = f_{0,A} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0}\right)$$

$$n = n_0 \left(\frac{\tau_0}{\tau}\right)^d \mathcal{V}\left(s_x, s_y, s_z\right)$$

No conserved charge, μ = 0

$$p = \frac{T\sigma}{1+\kappa}$$

$$\frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0}\frac{1}{1+\kappa_0}\right)$$

$$\sigma = \sigma_0 \left(\frac{\sigma_A}{\sigma_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^d \mathcal{V}\left(s_x, s_y, s_z\right), \quad \frac{\sigma_A}{\sigma_0} = f_{0,A}^{\frac{\kappa_0}{1+\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{p_0 \tau_0}\frac{1}{1+\kappa_0}\right)$$

Analytic solutions of NS equations, with κ=const

The solution of the pressure is: $p(\tau) = p_0 \left(\frac{p_A}{p_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa}\right)}, \quad \frac{p_A}{p_0} = f_{A,0} = \exp\left[\frac{d^2\zeta_0}{\kappa_0 p_0 \tau_0}\right]$

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$$n = n_0 \left(\frac{\tau_0}{\tau}\right)^d \boxed{\mathcal{V}\left(s_x, s_y, s_z\right)}$$

No conserved charge, μ = 0

$$\boxed{\mathcal{T}\left(s_x, s_y, s_z\right) = \frac{1}{\mathcal{V}\left(s_x, s_y, s_z\right)}}$$

$$p = \frac{T\sigma}{1+\kappa}$$

$$\frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

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Temperature dependence of the kinematic bulk viscosity

Conserved charge, μ > 0

$$p = nT$$

$$\frac{\zeta}{n}(T) = \frac{\zeta}{p}T = \frac{\zeta_0}{p_0}T \longrightarrow \frac{\zeta}{n}(T) \propto T$$

No conserved charge, μ = 0

$$p = \frac{T\sigma}{1+\kappa}$$

$$\frac{\zeta}{\sigma}(T) = \frac{\zeta}{p}\frac{T}{1+\kappa} = \frac{\zeta_0}{p_0}\frac{T}{1+\kappa} \longrightarrow \frac{\zeta}{\sigma}(T) \propto T$$

The kinematic bulk viscosity depends linarly on the temperature

Analytic solutions of NS equations, with temperature dependent κ= κ(T)

T. Csörgő, G. K.: arXiv:2003.08859

Conserved charge, μ > 0

$$\partial_\tau n + \frac{d}{\tau} n = 0$$

$$\frac{1}{T}\left[\frac{d}{dT}(\kappa T)\right]\partial_\tau T + \frac{d}{\tau} = \frac{d^2}{\tau^2}\frac{\zeta}{Tn}$$

$\frac{\zeta}{n}(T) = \frac{\zeta_0}{p_0} T$ → still linear in *T*

No conserved charge, μ = 0

$$\partial_\tau \sigma + \frac{d}{\tau}\sigma = \frac{d^2}{\tau^2}\frac{\zeta}{T} + \sigma(1+\kappa)\left[\frac{d}{dT}\left(\frac{\kappa T}{1+\kappa}\right)\right]\partial_\tau T$$

$$\frac{1+\kappa}{T}\left[\frac{d}{dT}\left(\frac{\kappa T}{1+\kappa}\right)\right]\partial_\tau T + \frac{d}{\tau} = \frac{d^2}{\tau^2}\frac{\zeta}{T\sigma}$$

$\frac{\zeta}{\sigma}(T) = \frac{\zeta_0}{p_0}\frac{T}{1+\kappa(T)}$ → not linear in *T*

κ=constant case → the coefficient of $\partial_\tau T$ was constant → temperature equation can be solved

κ=κ(T) case → if the coefficient of $\partial_\tau T$ is constant → temperature equation can be solved

Constraint for κ(T) → ***temperature dependence of κ(T) can be obtained***

T. Csörgő, G. K.: arXiv:1610.02197

Analytic solutions of NS equations, with temperature dependent κ= κ(T)

T. Csörgő, G. K.: arXiv:2003.08859

Conserved charge, μ > 0

$$\partial_\tau n + \frac{d}{\tau} n = 0$$

$$\frac{1}{T}\left[\frac{d}{dT}(\kappa T)\right]\partial_\tau T + \frac{d}{\tau} = \frac{d^2}{\tau^2}\frac{\zeta}{Tn}$$

$$\frac{\zeta}{n}(T) = \frac{\zeta_0}{p_0} T$$ → still linear in *T*

No conserved charge, μ = 0

$$c_s^{-2} = (1+\kappa)\left[\frac{d}{dT}\left(\frac{\kappa T}{1+\kappa}\right)\right]$$

$$\partial_\tau \sigma + \frac{d}{\tau}\sigma = \frac{d^2}{\tau^2}\frac{\zeta}{T} + \sigma\,(1+\kappa)\left[\frac{d}{dT}\left(\frac{\kappa T}{1+\kappa}\right)\right]\partial_\tau T$$

$$\frac{1+\kappa}{T}\left[\frac{d}{dT}\left(\frac{\kappa T}{1+\kappa}\right)\right]\partial_\tau T + \frac{d}{\tau} = \frac{d^2}{\tau^2}\frac{\zeta}{T\sigma}$$

$$\frac{\zeta}{\sigma}(T) = \frac{\zeta_0}{p_0}\frac{T}{1+\kappa(T)}$$ → not linear in *T*

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Analytic solutions of NS equations, with temperature dependent κ= κ(T)

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$$c_s^{-2} = (1+\kappa)\left[\frac{d}{dT}\left(\frac{\kappa T}{1+\kappa}\right)\right]$$

Conserved charge, μ > 0

$$\partial_\tau n + \frac{d}{\tau} n = 0$$

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$$\partial_\tau \sigma + \frac{d}{\tau}\sigma = \frac{d^2}{\tau^2}\frac{\zeta}{T} + \frac{\sigma}{c_s^2(T)}\partial_\tau T$$

$$\frac{1}{Tc_s^2(T)}\partial_\tau T + \frac{d}{\tau} = \frac{d^2}{\tau^2}\frac{\zeta}{T\sigma}$$

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$$\partial_\tau n + \frac{d}{\tau} n = 0$$

$$\frac{1}{T}\left[\frac{d}{dT}(\kappa T)\right]\partial_\tau T + \frac{d}{\tau} = \frac{d^2}{\tau^2}\frac{\zeta}{Tn}$$

$$\frac{\zeta}{n}(T) = \frac{\zeta_0}{p_0}T$$ → still linear in *T*

$$\kappa_{HM}(T) = \frac{\kappa_c T_c - \kappa_f T_f}{T_c - T_f} - \frac{\kappa_c - \kappa_f}{T_c - T_f}\frac{T_c T_f}{T}$$

No conserved charge, μ = 0

$$\partial_\tau \sigma + \frac{d}{\tau}\sigma = \frac{d^2}{\tau^2}\frac{\zeta}{T} + \frac{\sigma}{c_s^2(T)}\partial_\tau T$$

$$\frac{1}{Tc_s^2(T)}\partial_\tau T + \frac{d}{\tau} = \frac{d^2}{\tau^2}\frac{\zeta}{T\sigma}$$

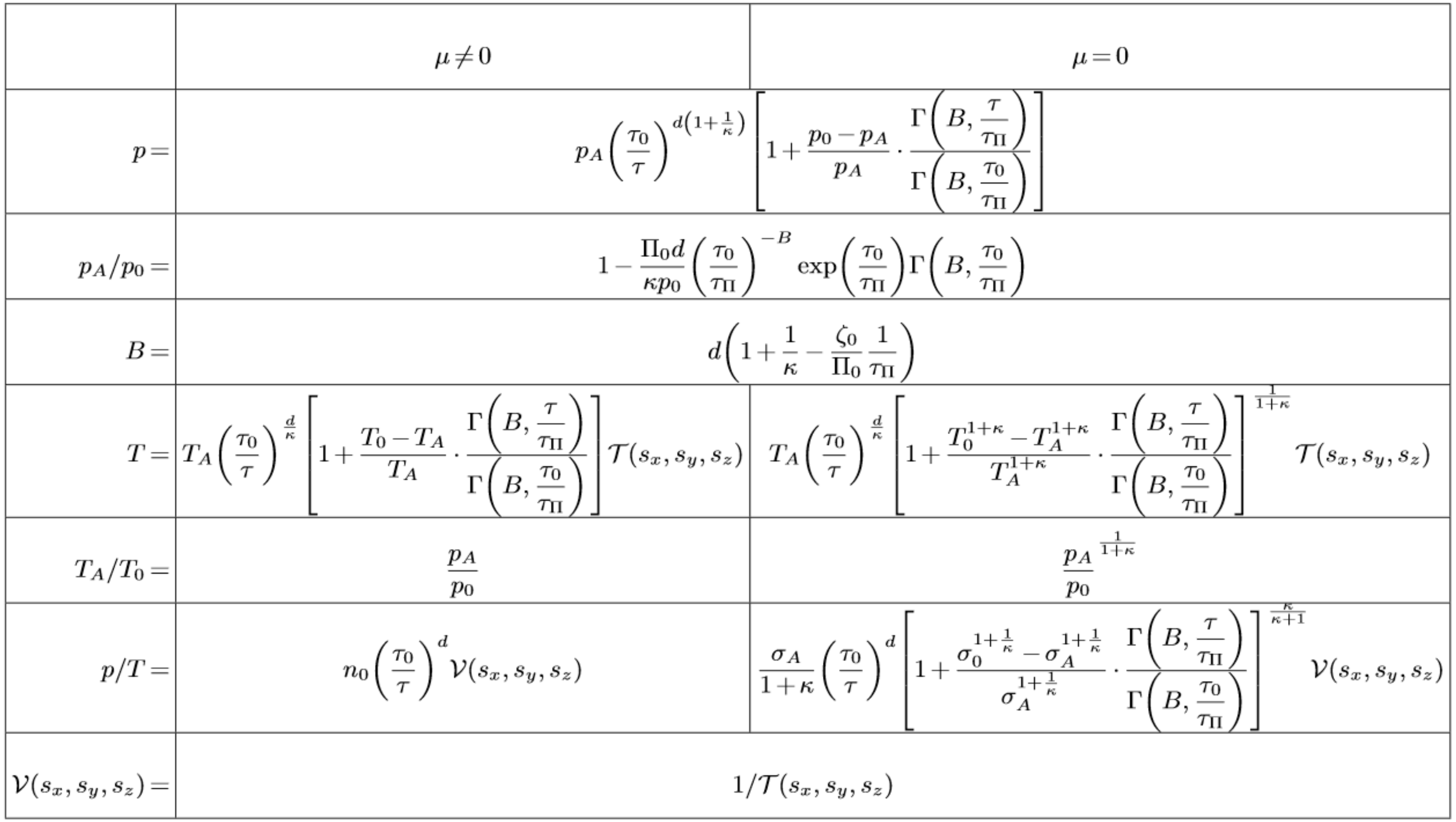
$$\frac{\zeta}{\sigma}(T) = \frac{\zeta_0}{p_0}\frac{T}{1+\kappa(T)}$$ → not linear in *T*

$$\kappa_{QM}(T) = \frac{\kappa_Q\left(\frac{T}{T_c}\right)^{1+\kappa_Q} + \frac{\kappa_c-\kappa_Q}{\kappa_c+1}}{\left(\frac{T}{T_c}\right)^{1+\kappa_Q} - \frac{\kappa_c-\kappa_Q}{\kappa_c+1}}$$

Constraint for κ(T) → ***temperature dependence of κ(T) can be obtained***

T. Csörgő, G. K.:
arXiv:1610.02197

Analytic solutions of IS equations



<table>
<tr><th></th><th>$\mu\neq 0$</th><th>$\mu=0$</th></tr>
<tr><td>$p=$</td><td colspan="2">$p_A\left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa}\right)}\left[1+\frac{p_0-p_A}{p_A}\cdot\frac{\Gamma\left(B,\frac{\tau}{\tau_\Pi}\right)}{\Gamma\left(B,\frac{\tau_0}{\tau_\Pi}\right)}\right]$</td></tr>
<tr><td>$p_A/p_0=$</td><td colspan="2">$1-\frac{\Pi_0 d}{\kappa p_0}\left(\frac{\tau_0}{\tau_\Pi}\right)^{-B}\exp\left(\frac{\tau_0}{\tau_\Pi}\right)\Gamma\left(B,\frac{\tau_0}{\tau_\Pi}\right)$</td></tr>
<tr><td>$B=$</td><td colspan="2">$d\left(1+\frac{1}{\kappa}-\frac{\zeta_0}{\Pi_0}\frac{1}{\tau_\Pi}\right)$</td></tr>
<tr><td>$T=$</td><td>$T_A\left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa}}\left[1+\frac{T_0-T_A}{T_A}\cdot\frac{\Gamma\left(B,\frac{\tau}{\tau_\Pi}\right)}{\Gamma\left(B,\frac{\tau_0}{\tau_\Pi}\right)}\right]\mathcal{T}(s_x,s_y,s_z)$</td><td>$T_A\left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa}}\left[1+\frac{T_0^{1+\kappa}-T_A^{1+\kappa}}{T_A^{1+\kappa}}\cdot\frac{\Gamma\left(B,\frac{\tau}{\tau_\Pi}\right)}{\Gamma\left(B,\frac{\tau_0}{\tau_\Pi}\right)}\right]^{\frac{1}{1+\kappa}}\mathcal{T}(s_x,s_y,s_z)$</td></tr>
<tr><td>$T_A/T_0=$</td><td>$\frac{p_A}{p_0}$</td><td>$\frac{p_A}{p_0}^{\frac{1}{1+\kappa}}$</td></tr>
<tr><td>$p/T=$</td><td>$n_0\left(\frac{\tau_0}{\tau}\right)^d\mathcal{V}(s_x,s_y,s_z)$</td><td>$\frac{\sigma_A}{1+\kappa}\left(\frac{\tau_0}{\tau}\right)^d\left[1+\frac{\sigma_0^{1+\frac{1}{\kappa}}-\sigma_A^{1+\frac{1}{\kappa}}}{\sigma_A^{1+\frac{1}{\kappa}}}\cdot\frac{\Gamma\left(B,\frac{\tau}{\tau_\Pi}\right)}{\Gamma\left(B,\frac{\tau_0}{\tau_\Pi}\right)}\right]^{\frac{\kappa}{\kappa+1}}\mathcal{V}(s_x,s_y,s_z)$</td></tr>
<tr><td>$\mathcal{V}(s_x,s_y,s_z)=$</td><td colspan="2">$1/\mathcal{T}(s_x,s_y,s_z)$</td></tr>
</table>

The κ=κ(T) case can be derived along the same logic that is used in the NS case!

M. Nagy, M. Csanád,
Z. Jiang, T. Csörgő:
arXiv:1909.02498

T. Csörgő, G. K.:
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Analytic solutions of IS equations, with temperature dependent κ= κ(T)

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Conserved charge, μ > 0

$$\partial_\tau n + \frac{d}{\tau} n = 0$$

$$\frac{1}{T}\left[\frac{d}{dT}(\kappa T)\right]\partial_\tau T + \frac{d}{\tau} = \frac{d\cancel{^2}}{\tau\cancel{^2}}\frac{\cancel{\zeta}}{Tn}\cdot\Pi$$

$$\frac{\zeta}{n}(T) = \frac{\zeta_0}{p_0}T \longrightarrow$$ still linear in *T*

$$\kappa_{HM}(T) = \frac{\kappa_c T_c - \kappa_f T_f}{T_c - T_f} - \frac{\kappa_c - \kappa_f}{T_c - T_f}\frac{T_c T_f}{T}$$

No conserved charge, μ = 0

$$\partial_\tau \sigma + \frac{d}{\tau}\sigma = \frac{d\cancel{^2}}{\tau\cancel{^2}}\frac{\cancel{\zeta}\cdot\Pi}{T} + \frac{\sigma}{c_s^2(T)}\partial_\tau T$$

$$\frac{1}{Tc_s^2(T)}\partial_\tau T + \frac{d}{\tau} = \frac{d\cancel{^2}}{\tau\cancel{^2}}\frac{\cancel{\zeta}}{T\sigma}\cdot\Pi$$

$$\frac{\zeta}{\sigma}(T) = \frac{\zeta_0}{p_0}\frac{T}{1+\kappa(T)} \longrightarrow$$ not linear in *T*

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Constraint for κ(T) → ***temperature dependence of κ(T) can be obtained***

T. Csörgő, G. K.: arXiv:1610.02197

II. Asymptotically perfect fluid solutions

Asymptotically perfect fluid solutions

In the $\tau \gg \tau_0$ limit, both the NS and IS cases lead to the same asymptotic perfect fluid temperature profile and pressure:

$$T \sim T_A \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right) \qquad p \sim p_A \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa_0}\right)}$$

If μ=0 the entropy density asymptotically equals to a perfect fluid form (and if μ≠0 the particle density is unchanged):

$$\sigma \sim \sigma_A \left(\frac{\tau_0}{\tau}\right)^{d} \mathcal{V}\left(s_x, s_y, s_z\right) \qquad \frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

The bulk viscosity is absorbed to the asymptotic normalization constants!

The effect of bulk viscosity is scaled out!

T. Csörgő, L. P. Csernai, Y. Hama, T. Kodama:
arXiv:nucl-th/0306004

Asymptotically perfect fluid solutions

In the τ >> τ_0 limit, both the NS and IS cases lead to the same asymptotic perfect fluid temperature profile and pressure:

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If μ=0 the entropy density asymptotically equals to a perfect fluid form (and if μ≠0 the particle density is unchanged):

$$\sigma \sim \boxed{\sigma_A}\left(\frac{\tau_0}{\tau}\right)^{d} \mathcal{V}\left(s_x, s_y, s_z\right) \qquad \boxed{\frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0}\frac{1}{1+\kappa_0}\right)}$$

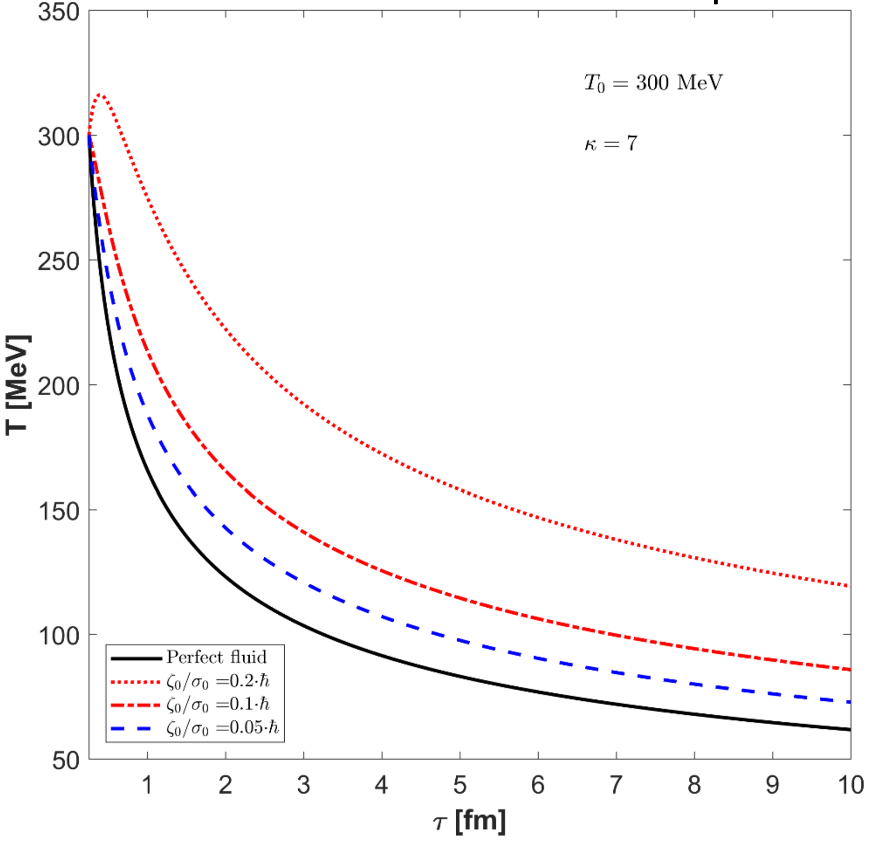
The bulk viscosity is absorbed to the **asymptotic normalization constants**!

The effect of bulk viscosity is scaled out!

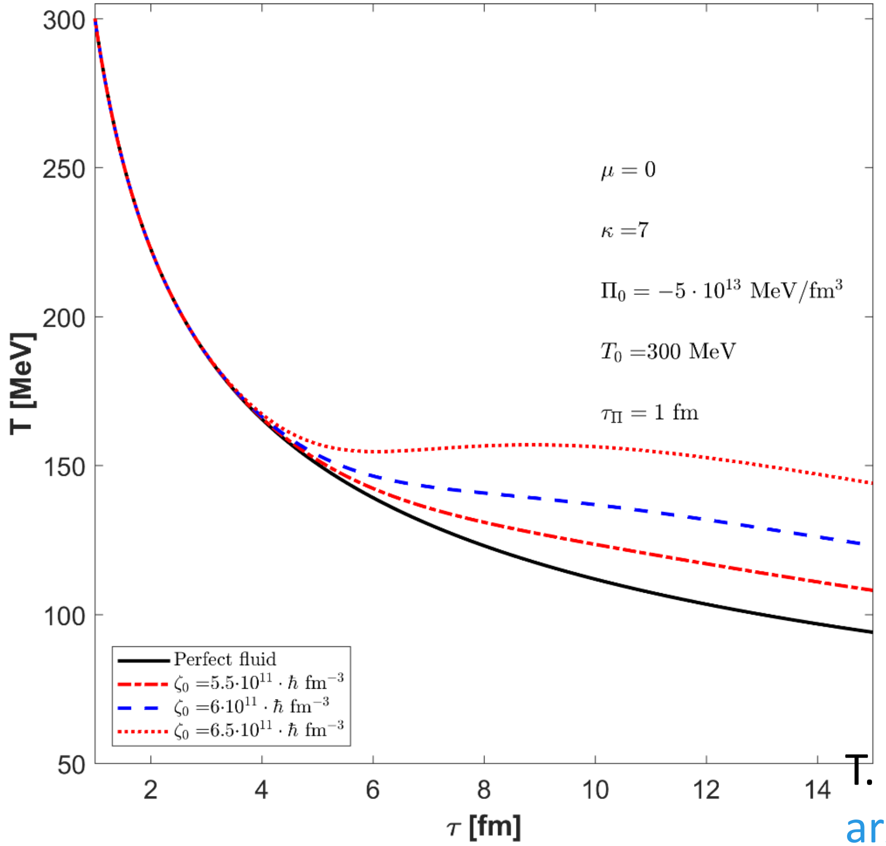
T. Csörgő, L. P. Csernai, Y. Hama, T. Kodama:
arXiv:nucl-th/0306004

Evolution of the temperature: same initial conditions

Solutions of Navier-Stokes eqs.



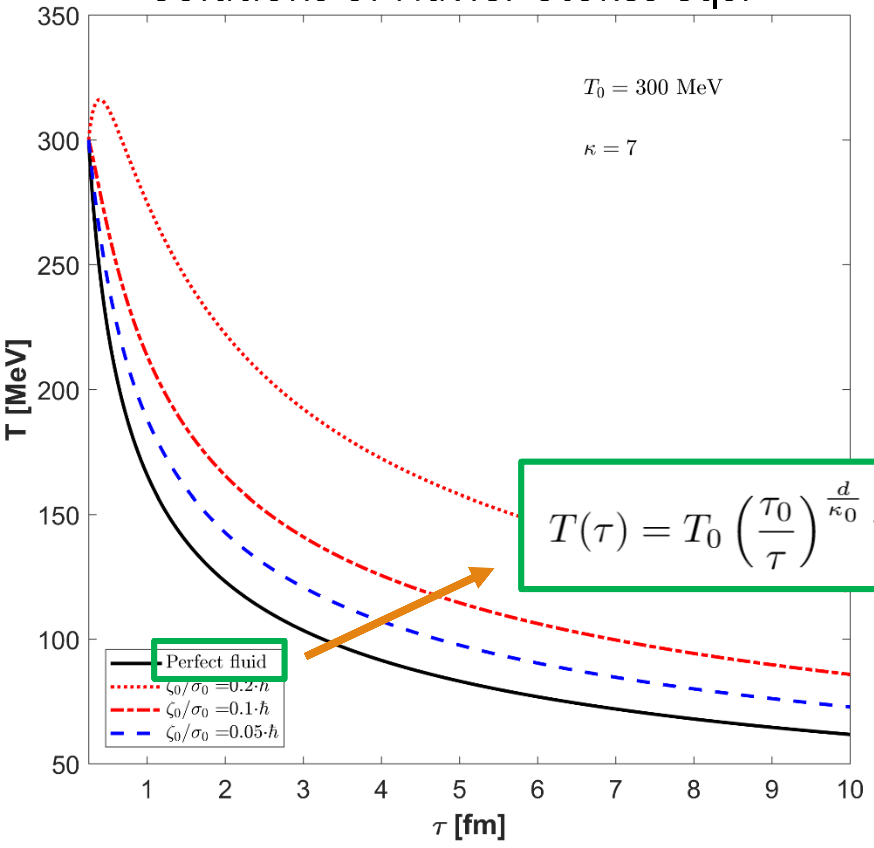
Solutions of Isreal-Stewart eqs.



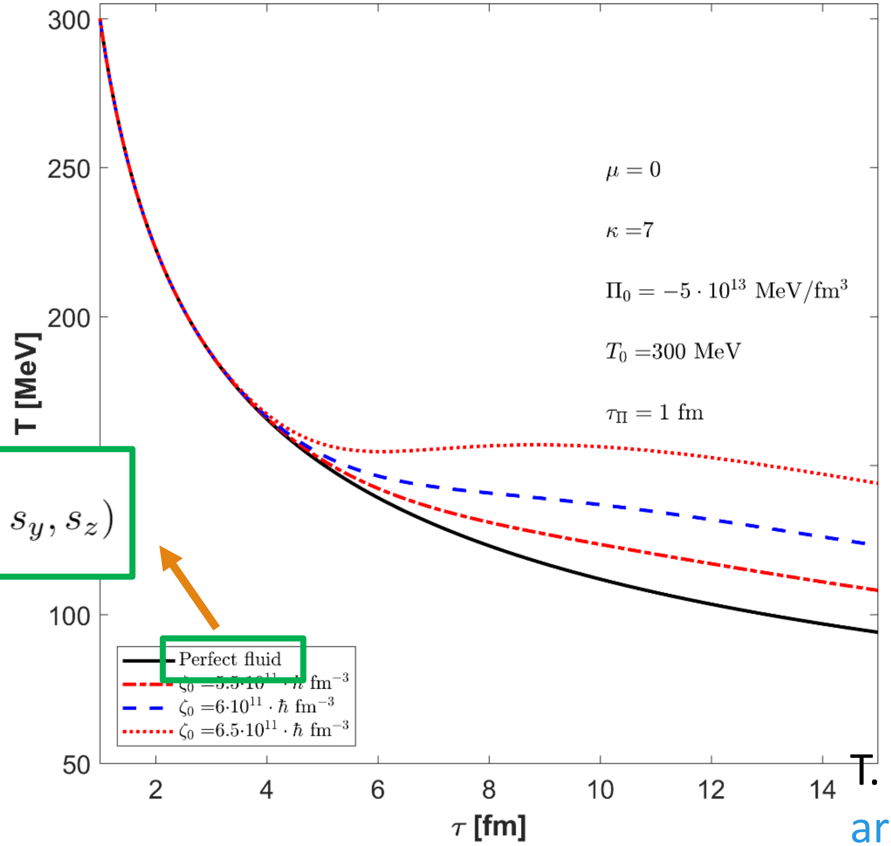
T. Csörgő, G. K.: arXiv:2003.08859

Evolution of the temperature: same initial conditions

Solutions of Navier-Stokes eqs.



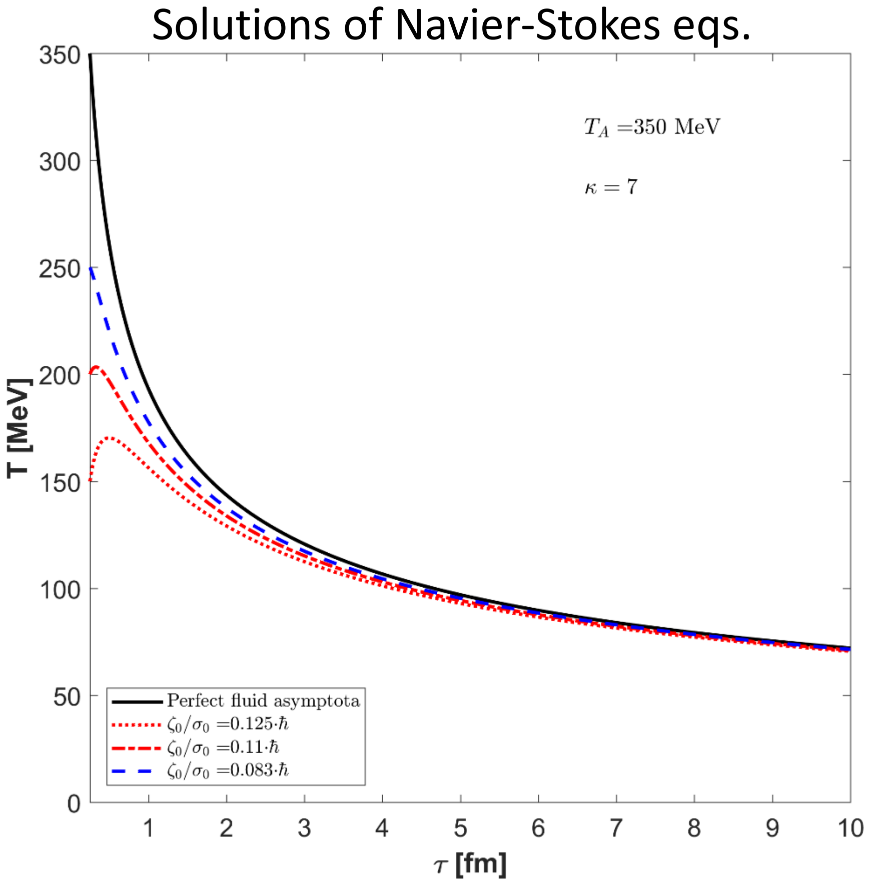
Solutions of Isreal-Stewart eqs.

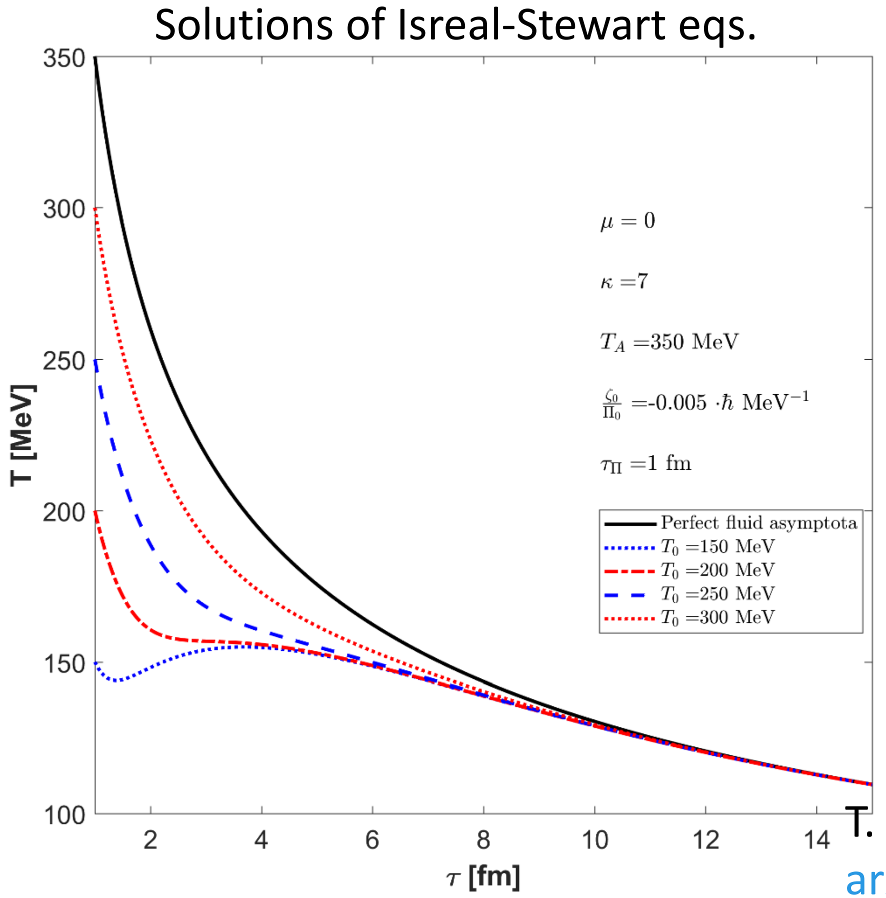


$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$$

T. Csörgő, G. K.: arXiv:2003.08859

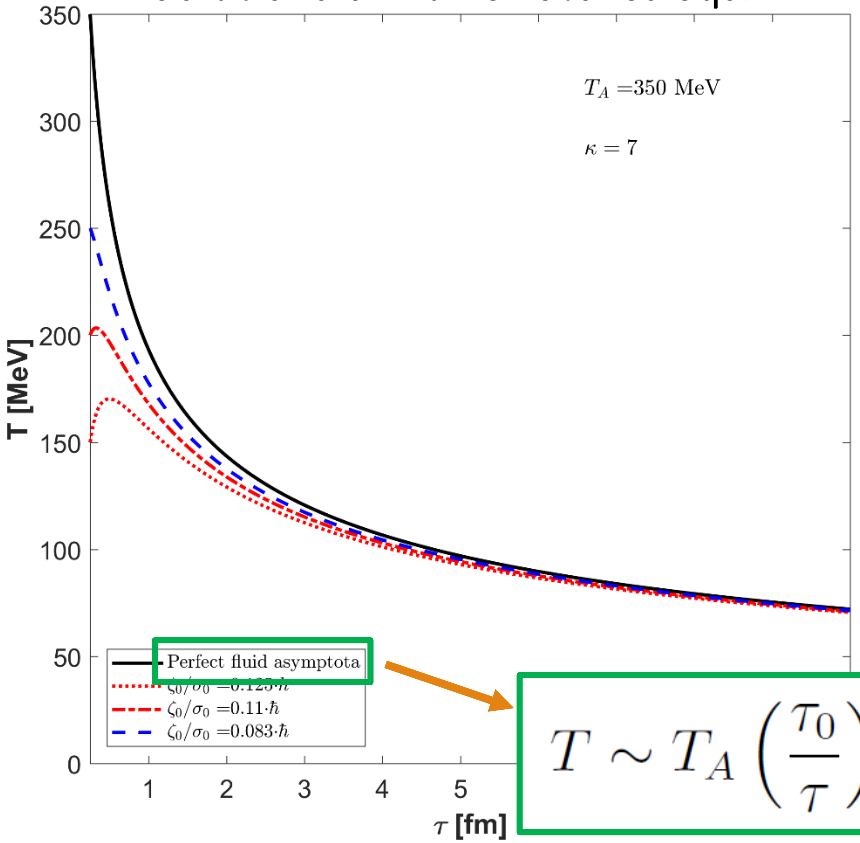
Evolution of the temperature: same attractor

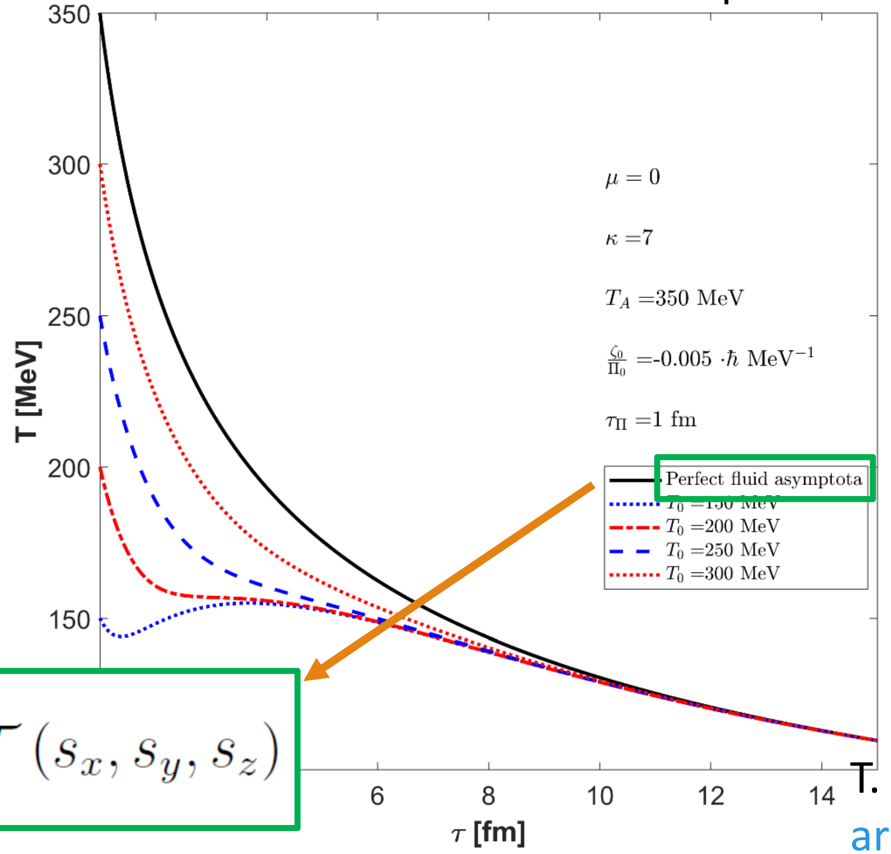

Solutions of Navier-Stokes eqs.


Solutions of Isreal-Stewart eqs.

T. Csörgő, G. K.: arXiv:2003.08859

Evolution of the temperature: same attractor

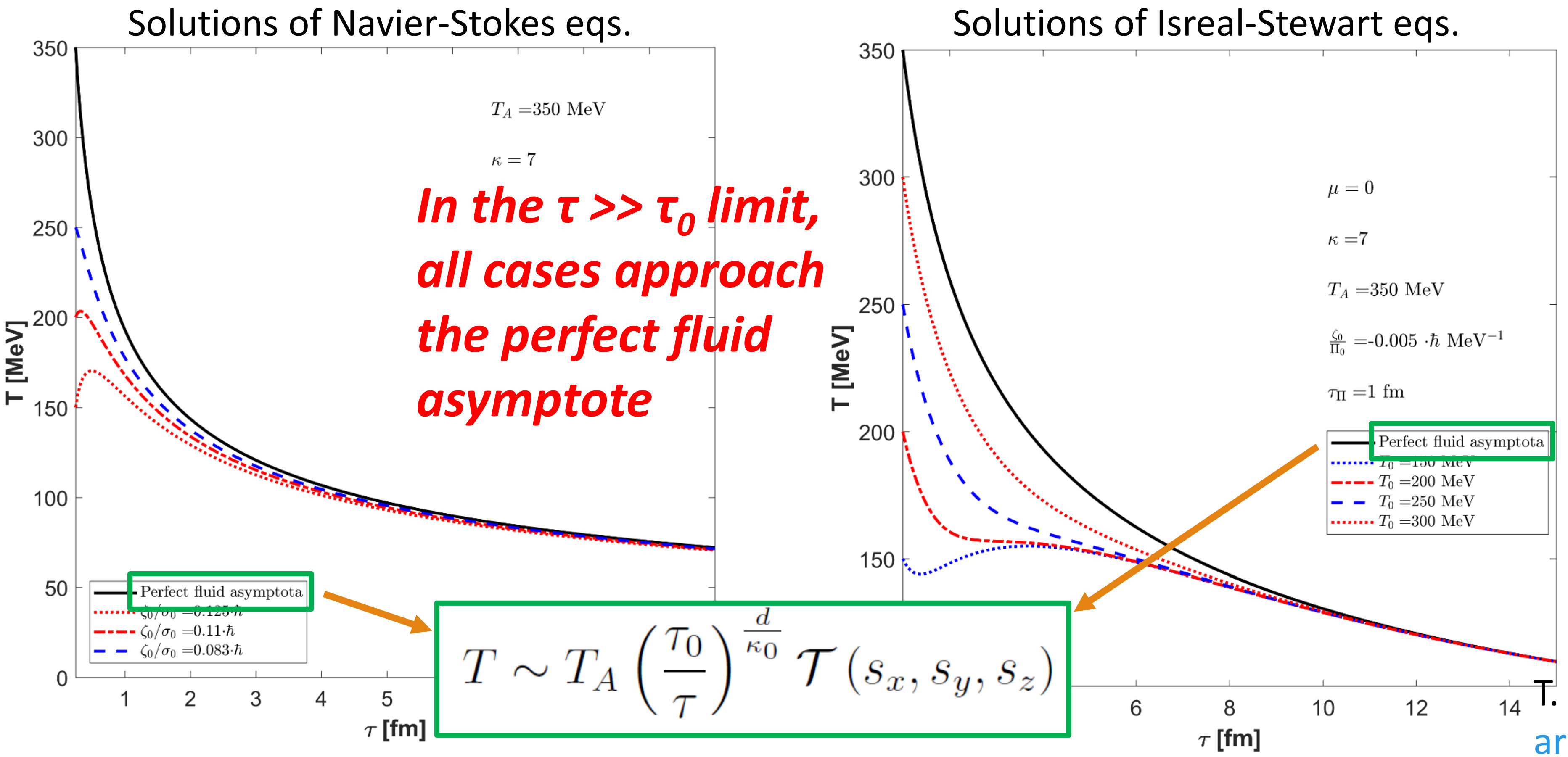

Solutions of Navier-Stokes eqs.


Solutions of Isreal-Stewart eqs.

$$T \sim T_A \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$$

T. Csörgő, G. K.: arXiv:2003.08859

Evolution of the temperature: same attractor



Solutions of Navier-Stokes eqs.

Solutions of Isreal-Stewart eqs.

In the τ >> τ_0 limit, all cases approach the perfect fluid asymptote

$$T \sim T_A \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa 0}} \mathcal{T}\left(s_x, s_y, s_z\right)$$

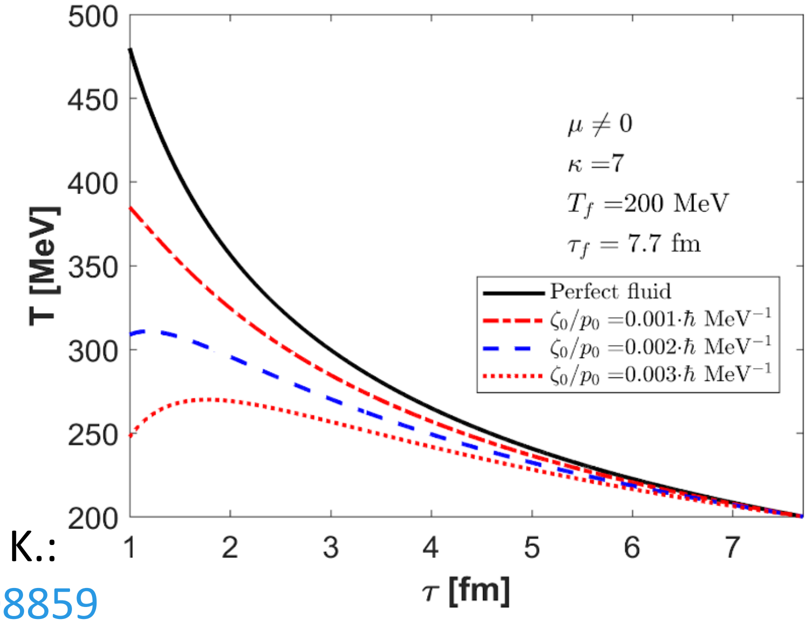
T. Csörgő, G. K.: arXiv:2003.08859

III. Applications

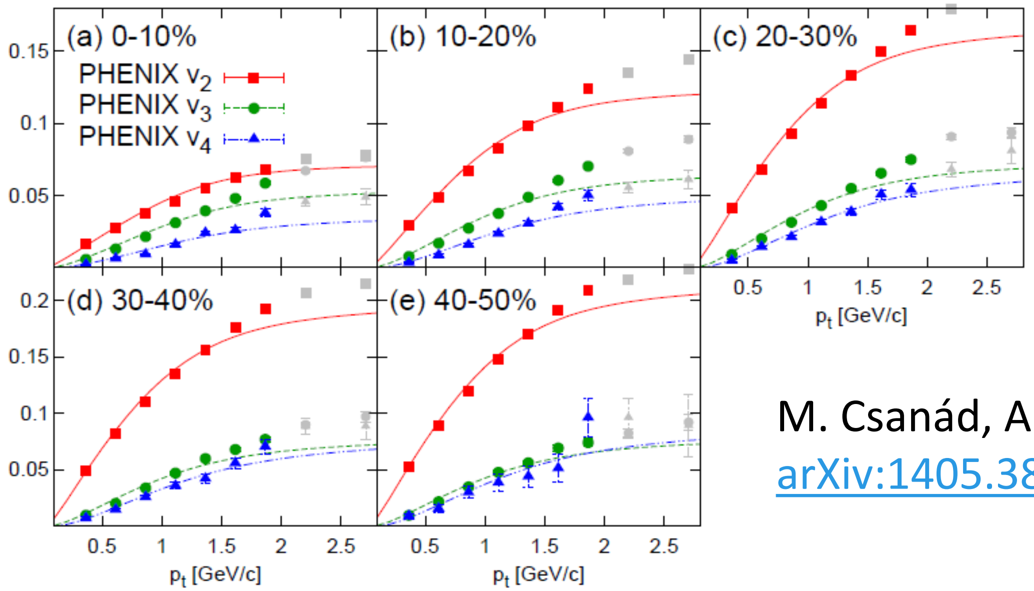
1st application of the solutions of NS eqs.

In arXiv:1405.3877: v_2, v_3 and v_4 were reproduced for $s_{NN}^{1/2}$ = 200 GeV Au+Au collisions with τ_f=7.7 fm/c and T_f=200 MeV final state parameters

We co-varied the initial conditions so that exactly the same freeze-out parameters are obtained



T. Csörgő, G. K.: arXiv:2003.08859



M. Csanád, A. Szabó: arXiv:1405.3877

Further applications of the solutions of NS eqs.

Producing new, dissipative solutions of non relativistic hydro

- See more details in my next presentation

Cosmological approach?

- Initial viscosity effects can be important in the early stage evolution of the Universe

Summary

New, analytic, exact solutions of relativistic Navier-Stokes and Israel-Stewart equations with spherically symmetric Hubble-flow

The effect of shear viscosity cancel because of the velocity field

The solutions are causal and asymptotically perfect (the effect of bulk viscosity cancels for late times), both for a finite and vanishing μ

These exact solutions tend to the Csörgő-Csernai-Hama-Kodama perfect fluid solution

Cannot decide from final state measurements that the medium evolved as a perfect fluid with higher initial temperature (T_A) or as a viscous fluid with lower initial temperature (T_0)

We were able to reproduce the experimental data in $s_{NN}^{1/2}$ = 200 GeV Au+Au collisions on v_2, v_3 and v_4

Non relativistic limit: new solutions of non relativistic Navier-Stokes theory

Thank you for your attention!