New exact solutions of non relativistic, viscous hydrodynamics

TAMÁS CSÖRGŐ, GÁBOR KASZA

7TH DAY OF FEMTOSCOPY, GYÖNGYÖS

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Further applications of the solutions of relativistic NS eqs.

Producing new, viscous solutions of non relativistic hydro → ***it's only an academic result***

1st step: Non relativistic limit of the relativistic solution → ***Spherically symmetric solution of non relativistic, viscous hydro***

2nd step: Spheroidal and ellipsoidal generalization

3rd step: Add rotation to the velocity field → $v = v_{Hubble} + v_{rot}$

Result: ***Ellipsoidally symmetric, rotating, viscous fireball solution of non relativistic hydro***

Why is non relativistic hydro important?

- The same effects can be understood in a much simpler formalism
- ***The basic equations of non relativistic, viscous hydro are fully clarified***

T. Csörgő, G. K.:
manuscript in preparation

Non relativistic, viscous hydrodynamics

Local conservation of the particle number, energy and momentum:

$$\partial_t n + \nabla(n\vec{v}) = 0$$

$$\partial_t \varepsilon + \nabla(\varepsilon\vec{v}) + p\nabla\vec{v} = \zeta(\nabla\vec{v})^2 + 2\eta\left[\mathrm{Tr}\left(D^2\right) - \frac{1}{3}(\nabla\vec{v})^2\right]$$

$$(\varepsilon + p)(\partial_t + \vec{v}\nabla)\vec{v} + \nabla p = \zeta\nabla(\nabla\vec{v}) + \eta\left[\Delta\vec{v} + \frac{1}{3}\nabla(\nabla\vec{v})\right]$$

Balance equation of entropy:

$$\partial_t \sigma + \nabla(\sigma\vec{v}) = \frac{\zeta}{T}(\nabla\vec{v})^2 + \frac{2\eta}{T}\left[\mathrm{Tr}\left(D^2\right) - \frac{1}{3}(\nabla\vec{v})^2\right] \geq 0$$

where: $D_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i}\right)$

To close the equation system:

EoS: $\varepsilon = \kappa p$

In this work: *κ=const.*

ζ: bulk viscosity

η: shear viscosity

Spherically symmetric, dissipative fireball solution

Velocity field
and self similarity:

$$\vec{v} = \frac{\dot{R}}{R}(r_x, r_y, r_z) \quad \rightarrow \quad (\partial_t + \vec{v}\nabla)s = 0, \qquad s = \frac{r^2}{R^2}$$

Particle density,
temperature and
ideal gas approach:

$$n(\vec{r},t) = n_0 \left(\frac{R_0}{R}\right)^d \mathcal{V}(s)$$

$$T(\vec{r},t) = T_0 f_T(t) \mathcal{T}(s)$$

$$p(\vec{r},t) = p_0 f_T(t) \left(\frac{R_0}{R}\right)^d \mathcal{V}(s)\mathcal{T}(s)$$

$$\mathcal{V}(s) = \frac{1}{\mathcal{T}(s)} \exp\left(-\frac{C_E}{2}\int_0^s \frac{du}{\mathcal{T}(u)}\right)$$

Energy and momentum
conservation:

$$\kappa \partial_t \ln(f_T) + d\frac{\dot{R}}{R} = \frac{\zeta d^2}{p}\left(\frac{\dot{R}}{R}\right)^2 \qquad R\ddot{R} = C_E \frac{T_0}{m} f_T(t)$$

Two possible solution of the energy conservation:

1. with homogeneous pressure: $v(s)\tau(s)=1$, $C_E=0$, $\zeta=\zeta(p)$
2. with inhomogeneous pressure: $\zeta \sim p$

Spherically symmetric, dissipative fireball solution
- with homogeneous pressure -

If the pressure is homogeneous, then $v(s)\tau(s)=1$, $C_E=0$ so the Euler equation and ζ are:

$$\ddot{R}=0 \longrightarrow \dot{R}=\text{const.} \longrightarrow R=\dot{R}t+R_0\sim\dot{R}t$$

$$\zeta\equiv\zeta(p(t))$$

With that, the energy conservation becomes:

$$\kappa\partial_t \ln(f_T)+\frac{d}{t}=\frac{\zeta(p(t))}{p(t)}\frac{d^2}{t^2}$$

If the bulk viscosity is linear in pressure: $\zeta(p(t))=\zeta_0\frac{p(t)}{p_0}$

$$p(t)=p_0\left(\frac{t_0}{t}\right)^{d\left(1+\frac{1}{\kappa}\right)}\exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\left[1-\frac{t_0}{t}\right]\right)$$

$$T(t,s)=T_0\left(\frac{t_0}{t}\right)^{\frac{d}{\kappa}}\exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\left[1-\frac{t_0}{t}\right]\right)\mathcal{T}(s)$$

$\tau_0 \ll \tau$ ⟶

Late time approximation:

perfect fluid asymptote

$$T(t)\sim T_A\left(\frac{t_0}{t}\right)^{\frac{d}{\kappa}}\mathcal{T}(s)$$

$$p(t)\sim p_A\left(\frac{t_0}{t}\right)^{d\left(1+\frac{1}{\kappa}\right)}$$

$$T_A=T_0\exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\right)$$

$$p_A=p_0\exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\right)$$

Spherically symmetric, dissipative fireball solution
- with inhomogeneous pressure -

If the pressure is inhomogeneous, then ζ has to be linear in pressure: $\zeta(t,s) = \zeta_0 \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t) \left(\frac{R_0}{R} \right)^{\frac{d}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{g_T}}{g_T} = \underbrace{\frac{\zeta_0 d^2}{\kappa p_0} \left(\frac{\dot{R}}{R} \right)^2}_{\text{effect of bulk viscosity}}$$

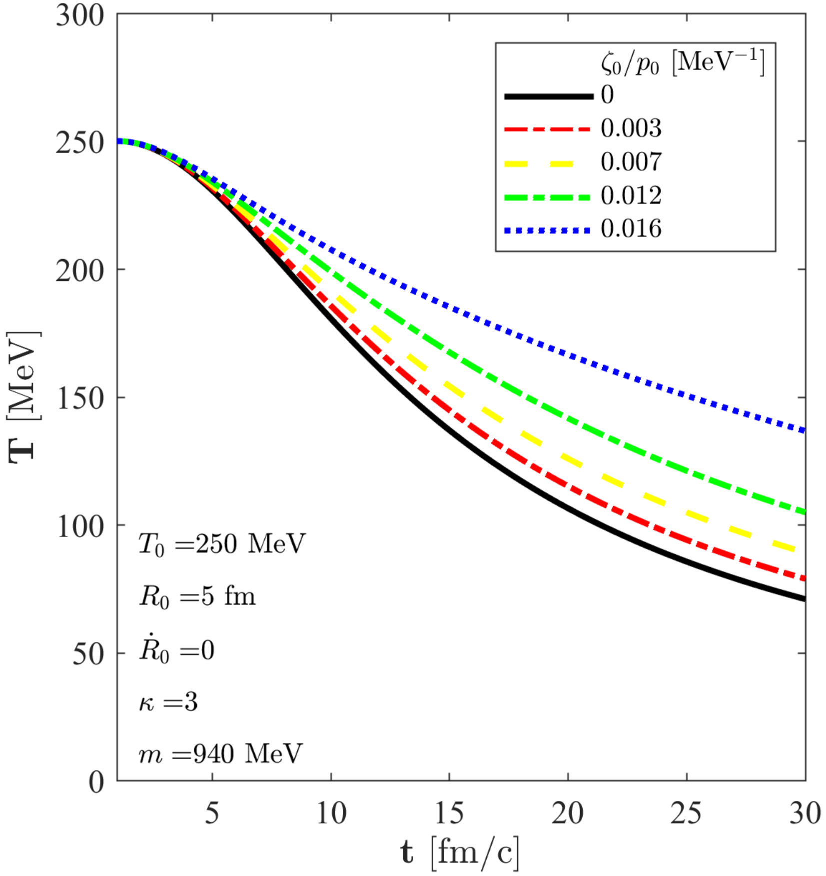
The Euler equation is:

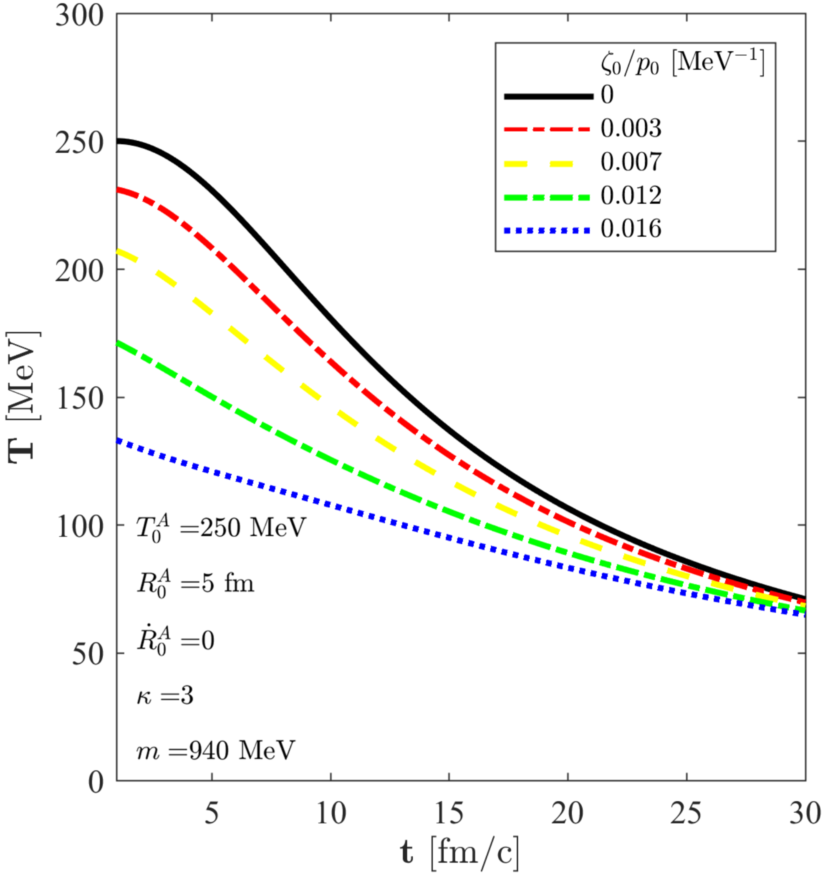
$$R\ddot{R} = C_E \frac{T_0}{m} \left(\frac{R_0}{R} \right)^{\frac{d}{\kappa}} \underbrace{g_T(t)}_{\text{effect of bulk viscosity}}$$

This set of differential equations is solved numerically (next slides)

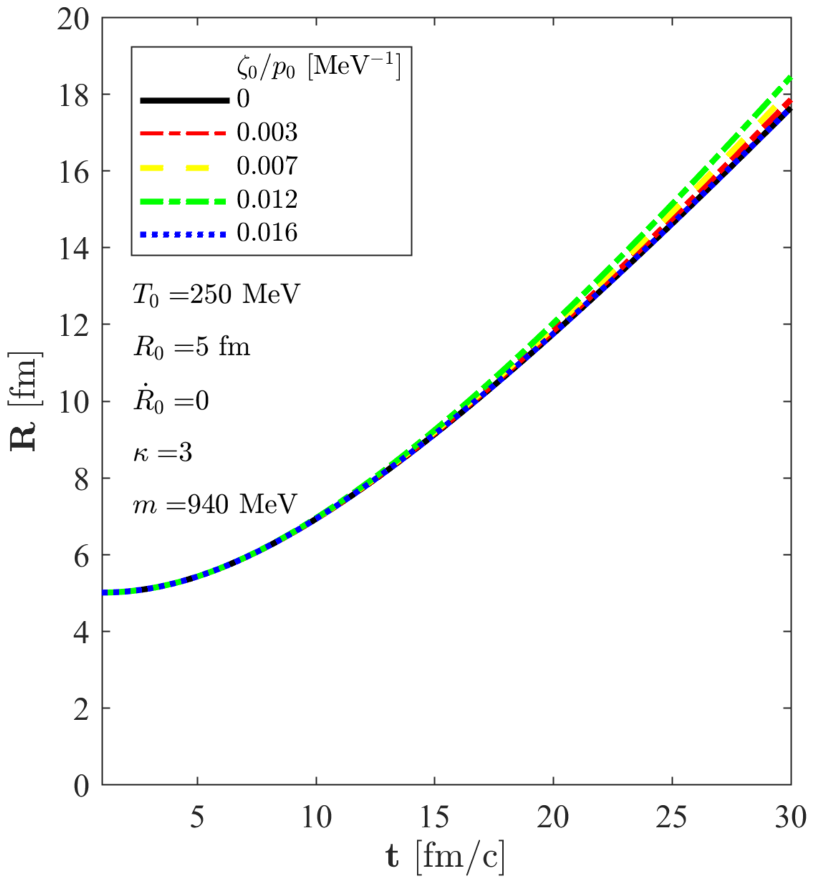
The asymptotic attractor is a perfect fluid solution again!

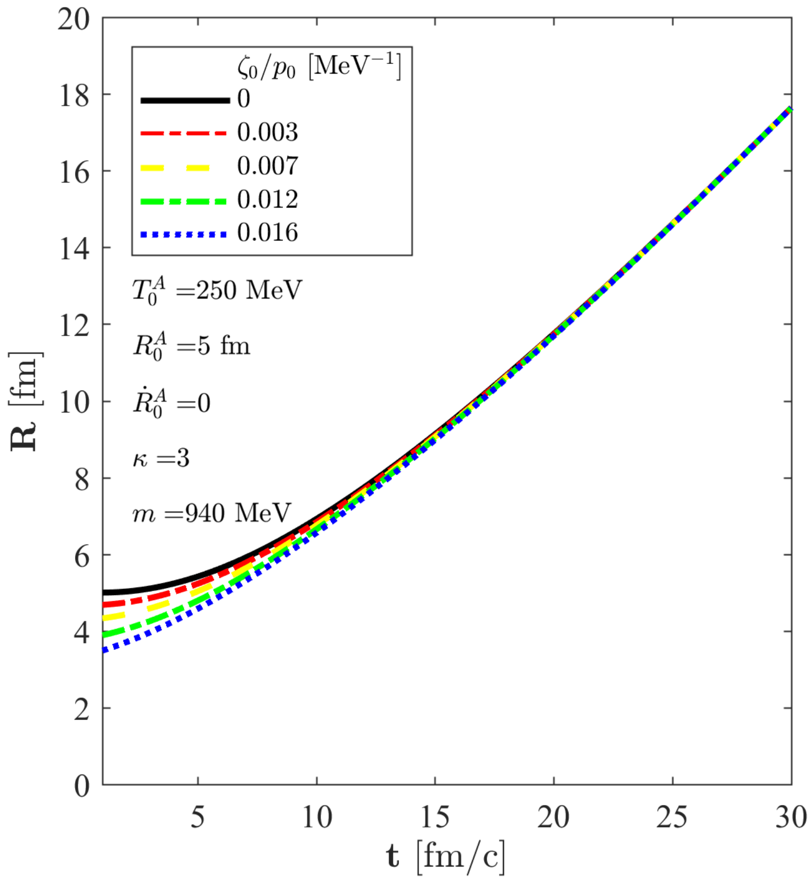
Spherically symmetric, dissipative fireball solution
- with inhomogeneous pressure -



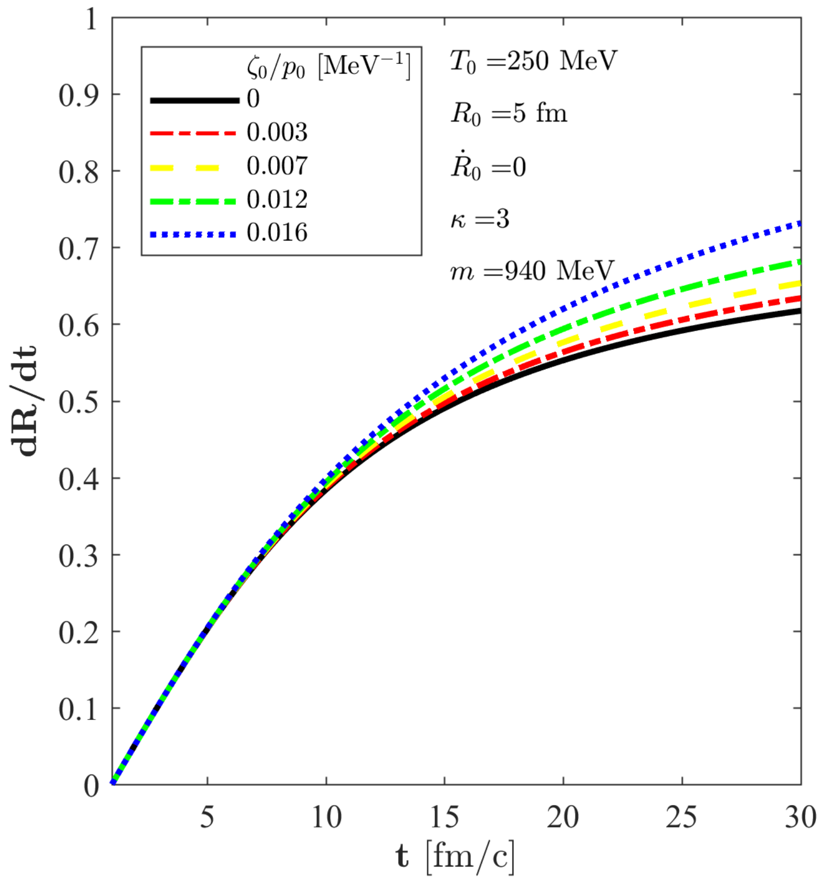


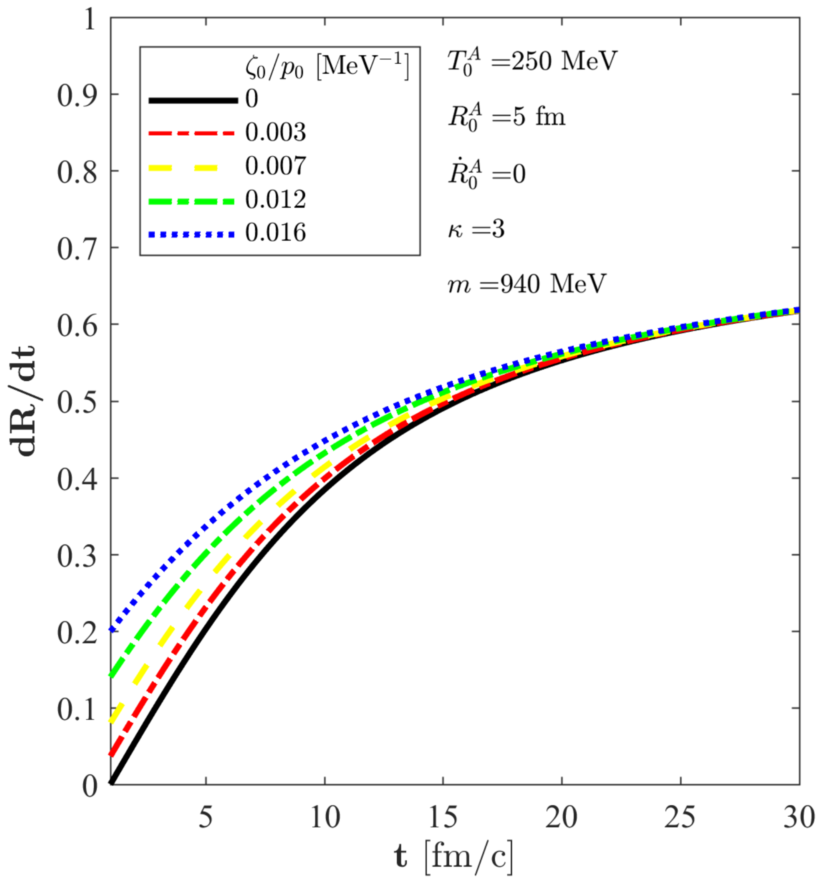
Spherically symmetric, dissipative fireball solution - with inhomogeneous pressure -





Spherically symmetric, dissipative fireball solution - with inhomogeneous pressure -





Generalization

The temperature and particle density can be given in a „symmetry independent" form:

$$n(\vec{r},t) = n_0 f_T^{1/\kappa}(t)\mathcal{V}(s)$$
$$T(\vec{r},t) = T_0 f_T(t)\mathcal{T}(s)$$
$$p(\vec{r},t) = p_0 f_T^{1+\frac{1}{\kappa}}(t)\mathcal{V}(s)\mathcal{T}(s)$$

The $f_T(t)$ function carries the symmetry and includes the dissipative effects

We assumed that it is the product of a perfect fluid term and a dissipative correction

Spheroidally symmetric, rotating, dissipative fireball solution

Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$

Hubble flow: $v_H(\vec{r}, t) = \left(\frac{\dot{R}}{R} r_x, \frac{\dot{Y}}{Y} r_y, \frac{\dot{R}}{R} r_z \right)$

Rotational term: $v_{rot}(\vec{r}, t) = \omega(r_z, 0, -r_x)$

The scales of the fireball: $R(t)$ and $Y(t)$

Angular velocity: $\dot{\vartheta} = \omega(t)$

M. I. Nagy, T. Csörgő: arXiv:1309.4390
M. I. Nagy, T. Csörgő: arXiv:1606.09160
T. Csörgő, M. I. Nagy, I. F. Barna: arXiv:1511.02593

Ellipsoidally symmetric, rotating, dissipative fireball solution

Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$

Hubble flow:
$$v_H(\vec{r},t) = \begin{pmatrix} \left(\frac{\dot{X}}{X}\cos^2\vartheta + \frac{\dot{Z}}{Z}\sin^2\vartheta\right) r_x \\ \frac{\dot{Y}}{Y} r_y \\ \left(\frac{\dot{X}}{X}\sin^2\vartheta + \frac{\dot{Z}}{Z}\cos^2\vartheta\right) r_z \end{pmatrix} + \left(\frac{\dot{Z}}{Z} - \frac{\dot{X}}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix} r_z \\ 0 \\ r_x \end{pmatrix}$$

Rotational term:
$$v_{rot}(\vec{r},t) = \dot{\vartheta}\begin{pmatrix} r_z \\ 0 \\ -r_x \end{pmatrix} + \dot{\vartheta}\begin{pmatrix} \left(\frac{X}{Z}\cos^2\vartheta + \frac{Z}{X}\sin^2\vartheta\right) r_z \\ 0 \\ -\left(\frac{X}{Z}\sin^2\vartheta + \frac{Z}{X}\cos^2\vartheta\right) r_x \end{pmatrix} + \dot{\vartheta}\left(\frac{X}{Z} - \frac{Z}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix} r_x \\ 0 \\ -r_z \end{pmatrix}$$

The scales of the fireball: $X(t)$, $Y(t)$ and $Z(t)$

Average transverse scale: $R = \frac{X+Z}{2}$

Angular velocity:
$$\dot{\vartheta} = \frac{\omega(t)}{2} = \frac{\omega_0}{2}\frac{R_0^2}{R(t)^2}$$

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Spherically symmetric, dissipative fireball solution
- with inhomogeneous pressure -

If the pressure is inhomogeneous, then ζ has to be linear in pressure: $\zeta(t,s) = \zeta_0 \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t) \left(\frac{R_0}{R} \right)^{\frac{d}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{g_T}}{g_T} = \underbrace{\frac{\zeta_0 d^2}{\kappa p_0} \left(\frac{\dot{R}}{R} \right)^2}_{\text{effect of bulk viscosity}}$$

The Euler equation is:

$$R\ddot{R} = C_E \frac{T_0}{m} \left(\frac{R_0}{R} \right)^{\frac{d}{\kappa}} \underbrace{g_T(t)}_{\text{effect of bulk viscosity}}$$

Spheroidally symmetric, dissipative fireball solution
- with inhomogeneous pressure -

If the pressure is inhomogeneous, then ζ and η has to be linear in pressure: $\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t)\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{g}_T}{g_T} = \underbrace{\frac{\zeta_0}{\kappa p_0}\left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y}\right)^2}_{\text{effect of bulk viscosity}} + \underbrace{\frac{2\eta_0}{\kappa p_0}\left[\frac{2\dot{R}^2}{R^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{3}\left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y}\right)^2\right]}_{\text{effect of shear viscosity}}$$

The Euler equation is:

$$R\ddot{R} = Y\ddot{Y} = C_E \frac{T_0}{m}\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa}} \underbrace{g_T(t)}_{\text{effect of bulk and shear viscosity}}$$

Spheroidally symmetric, dissipative fireball solution
- with inhomogeneous pressure and rotation -

If the pressure is inhomogeneous, then ζ and η has to be linear in pressure: $\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t) \left(\frac{R_0^2 Y_0}{R^2 Y} \right)^{\frac{1}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{g}_T}{g_T} = \underbrace{\frac{\zeta_0}{\kappa p_0} \left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} \right)^2}_{\text{effect of bulk viscosity}} + \underbrace{\frac{2\eta_0}{\kappa p_0} \left[\frac{2\dot{R}^2}{R^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{3} \left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} \right)^2 \right]}_{\text{effect of shear viscosity}}$$

The Euler equation is:

$$R\ddot{R} - \underbrace{R^2\omega^2}_{\text{effect of rotation}} = Y\ddot{Y} = C_E \frac{T_0}{m} \left(\frac{R_0^2 Y_0}{R^2 Y} \right)^{\frac{1}{\kappa}} \underbrace{g_T(t)}_{\text{effect of bulk and shear viscosity}}$$

Ellipsoidally symmetric, dissipative fireball solution
- with inhomogeneous pressure and rotation -

If the pressure is inhomogeneous, then ζ and η has to be linear in pressure: $\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t)\left(\frac{X_0 Y_0 Z_0}{XYZ}\right)^{\frac{1}{\kappa}}$$

With that, the energy conservation becomes:

effect of shear viscosity for ellipsoidally symmetric fireball

$$\frac{\dot{g}_T}{g_T} = \frac{\zeta_0}{\kappa p_0}\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}\right)^2 + \frac{2\eta_0}{\kappa p_0}\left[\frac{\dot{X}^2}{X^2}+\frac{\dot{Y}^2}{Y^2}+\frac{\dot{Z}^2}{Z^2}-\frac{1}{3}\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}\right)^2\right] + \frac{\eta_0 {\omega_0}^2}{4\kappa p_0}\frac{(X_0+Z_0)^4}{(X+Z)^4}\left(\frac{X}{Z}-\frac{Z}{X}\right)^2$$

effect of bulk viscosity

the only effect of shear viscosity for spheroidally symmetric fireball

effect of rotation, if $\eta \neq 0$

The Euler equation is (where $R=(X+Z)/2$):

$$X\left(\ddot{X}-R\omega^2\right) = Y\ddot{Y} = Z\left(\ddot{Z}-R\omega^2\right) = C_E\frac{T_0}{m}\left(\frac{X_0 Y_0 Z_0}{XYZ}\right)^{\frac{1}{\kappa}} g_T(t)$$

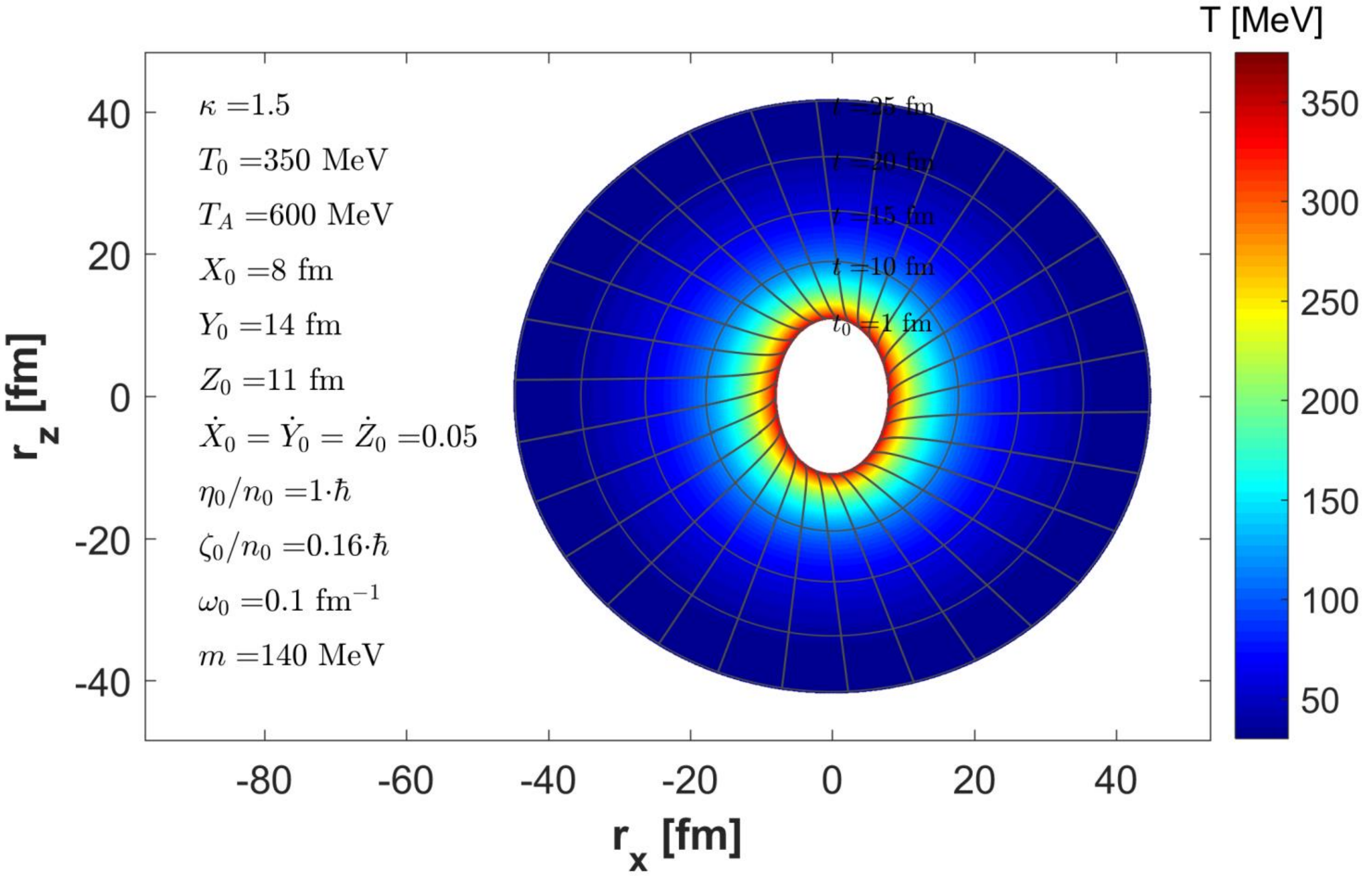
effect of rotation

effect of rotation

effect of rotation, bulk and shear viscosity

Trajectory and evolution

- Evolution of the edge of fireball
- From 1 to 25 fm
- Ellipsoidal symmetry
- Pion mass
- *η*, *ζ* and rotation are included
- Greater initial size



Trajectory and evolution

- Evolution of the edge of fireball
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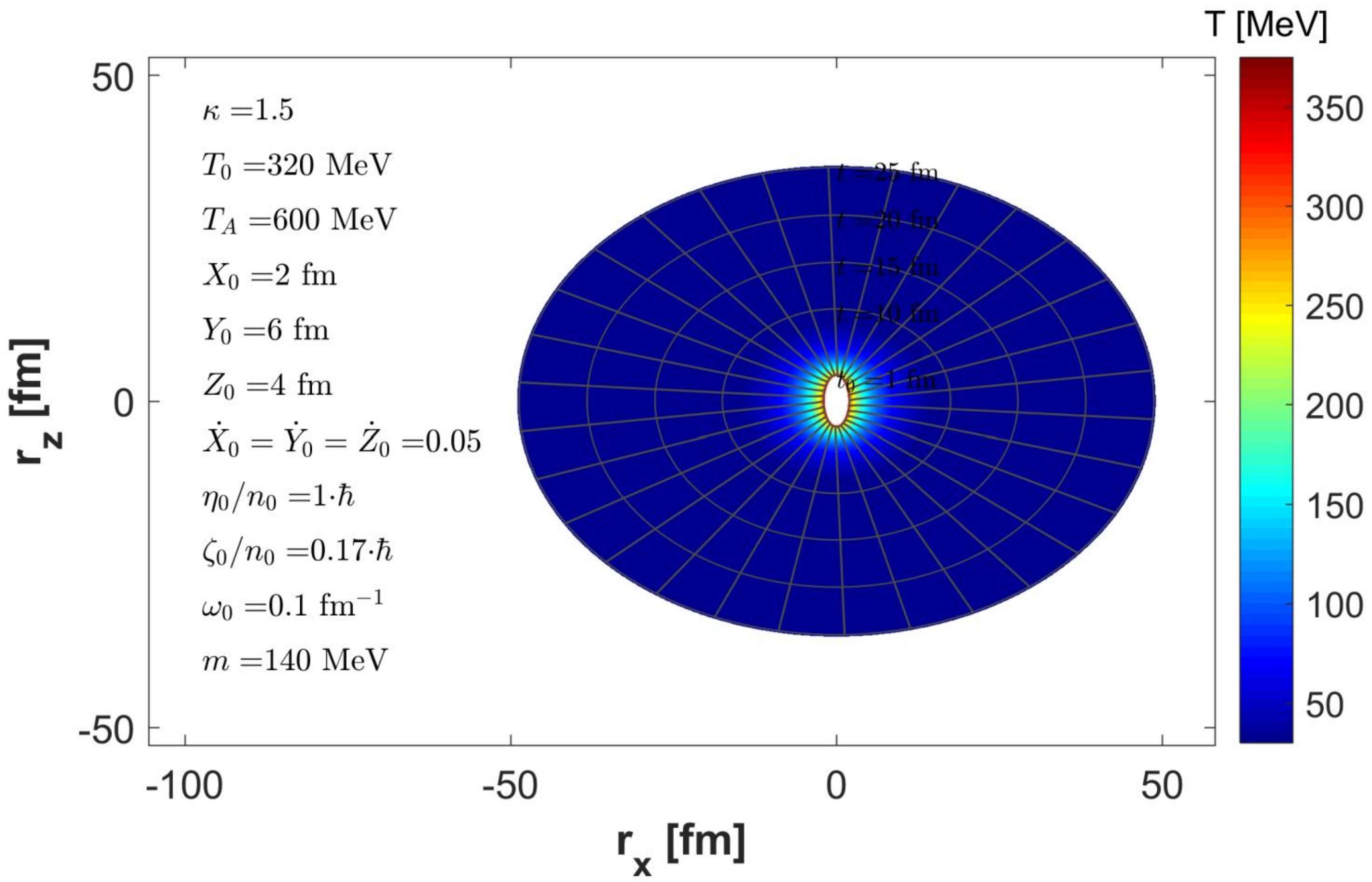
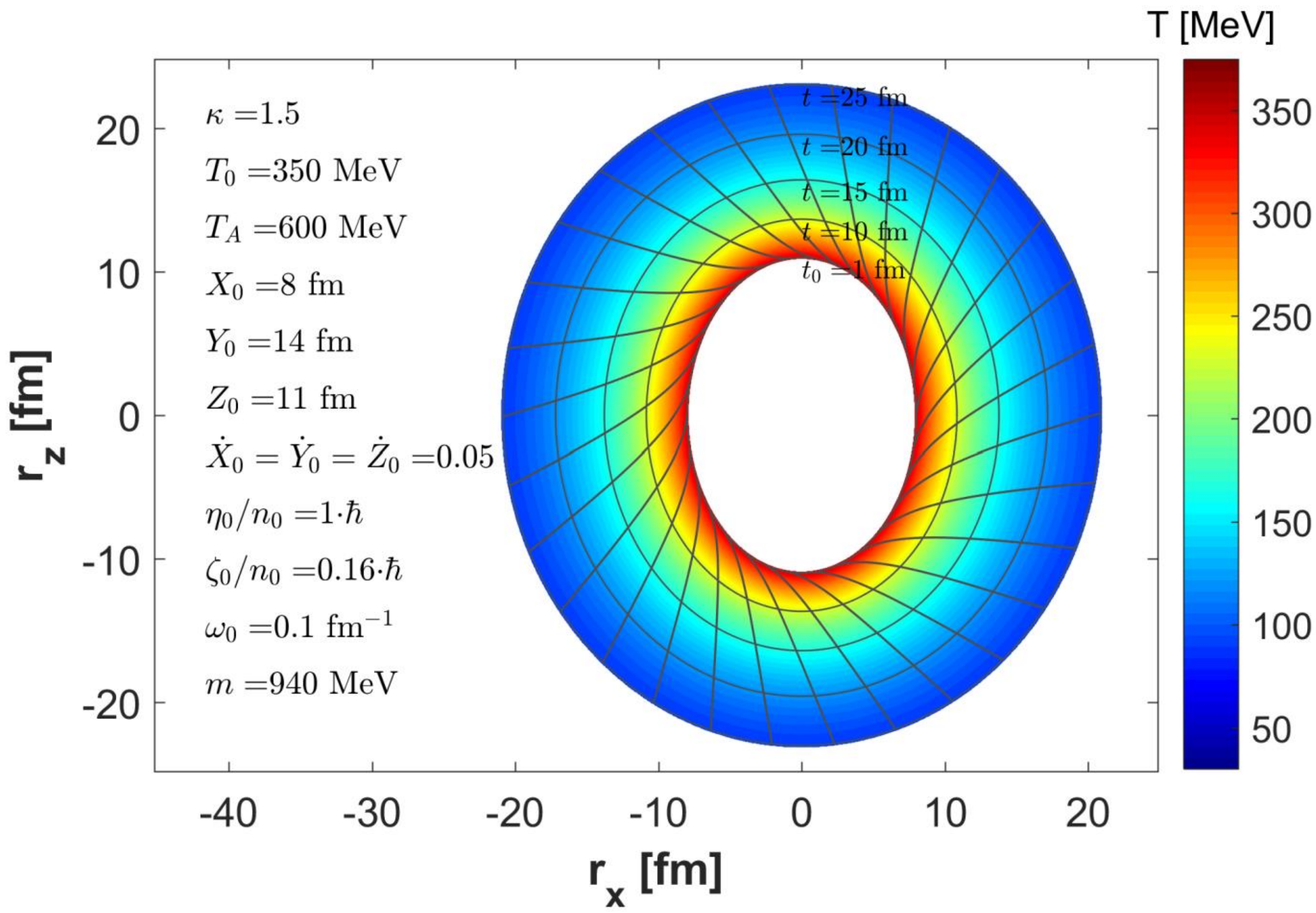


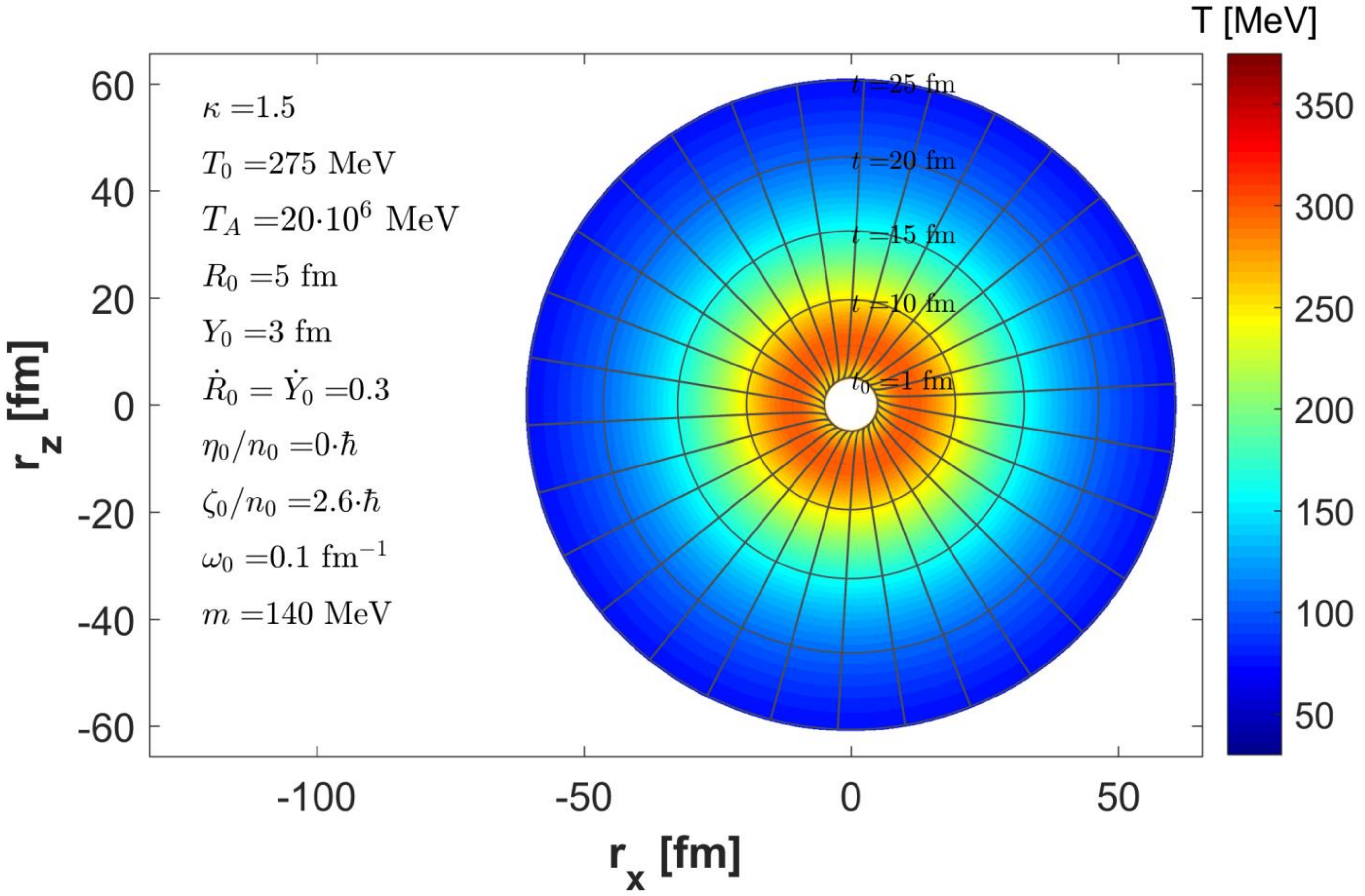
Trajectory and evolution

- Evolution of the edge of fireball
- From 1 to 25 fm
- Ellipsoidal symmetry
- **Proton** mass
- *η*, *ζ* and rotation are included
- Greater initial size



Trajectory and evolution

- Evolution of the edge of fireball
- From 1 to 25 fm
- **Spheroidal** symmetry
- Pion mass
- ζ and rotation are included
- ***η is not included***
- **Smaller** initial size
- ***Reheating effect!***
  → T_A is not realistic



Summary

Another application of out new relativistic, viscous solutions:

→ ***New, analytic, exact solutions of non relativistic Navier-Stokes equations with Hubble-flow***

Only academic results, *not plan to describe measurements*

The effects of viscosities and rotation are vanishing for late times

The solutions are asymptotically perfect both for a finite and vanishing μ

These exact solutions tend to perfect fluid solutions

Solutions with κ=κ(T) can be easily derived (see my previous presentation)

Thank you for your attention!

(again)