Systematics of U-spin Amplitude Sum Rules

MARGARITA GAVRILOVA
Cornell University
PHENO 22

with YUVAL GROSSMAN and STEFAN SCHACHT

Motivation

- probing **EW physics** with hadrons in the presence of **non-perturbative QCD** is **challenging**
- **U-spin symmetry** can be utilized to write approximate relations between decay amplitudes (and observables) → **reduce the number of unknown hadronic parameters**

Example: $\bar{D} \to P^+P^-$

$$\epsilon^0: \quad \frac{\mathcal{A}(\bar{D}^0 \to \pi^+ K^-)}{V_{cd}V_{us}^*} = \frac{\mathcal{A}(\bar{D}^0 \to K^+ \pi^-)}{-V_{cs}V_{ud}^*} = \frac{\mathcal{A}(\bar{D}^0 \to \pi^+ \pi^-)}{V_{cs}V_{us}^*} = \frac{\mathcal{A}(\bar{D}^0 \to K^+ K^-)}{V_{cs}V_{us}^*}$$

$$\epsilon^1: \quad \frac{\mathcal{A}(\bar{D}^0 \to \pi^+ K^-)}{V_{cd}V_{us}^*} + \frac{\mathcal{A}(\bar{D}^0 \to K^+ \pi^-)}{-V_{cs}V_{ud}^*} = \frac{\mathcal{A}(\bar{D}^0 \to \pi^+ \pi^-)}{V_{cs}V_{us}^*} + \frac{\mathcal{A}(\bar{D}^0 \to K^+ K^-)}{V_{cs}V_{us}^*}$$

Brod, Grossman, Kagan, Zupan, arXiv:1203.6659 [hep-ph]

U-spin

- approximate symmetry of QCD
- $SU(2)$ subgroup of $SU(3)$ flavour
- fundamental doublets are

$$\begin{bmatrix} d \\ s \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}, \qquad \begin{bmatrix} \bar{s} \\ -\bar{d} \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{bmatrix}$$

U-spin symmetry → algebraic relations between decay amplitudes ≡ sum rules

U-spin set

- **U-spin set** is a set of amplitudes (processes) that are related by U-spin
- U-spin set is defined via U-spin properties of:
  - initial/final state
  - and the Hamiltonian
- U-spin limit Hamiltonian:

$$\mathcal{H}_{\text{eff}}^{(0)} = \sum_m f_{u,m} H_m^u$$

Example: $\bar{D}^0 \to P^+P^-$

Initial and final state multiplets:

$$\bar{D}^0 = |u\bar{c}\rangle = |0,0\rangle\,, \qquad P^+ = \begin{bmatrix} K^+ \\ \pi^+ \end{bmatrix} = \begin{bmatrix} |u\bar{s}\rangle \\ -|u\bar{d}\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2},+\frac{1}{2}\right\rangle \\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle \end{bmatrix}, \qquad P^- = \begin{bmatrix} \pi^- \\ K^- \end{bmatrix} = \begin{bmatrix} |d\bar{u}\rangle \\ |s\bar{u}\rangle \end{bmatrix} = \begin{bmatrix} \left|\frac{1}{2},+\frac{1}{2}\right\rangle \\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle \end{bmatrix}$$

Hamiltonian:

$$\mathcal{H}_{\text{eff}}^{(0)} = \sum_{m=-1}^{1} f_{1,m} H_m^1$$

$$f_{1,1} = V_{cd}^* V_{us}, \qquad f_{1,-1} = -V_{cs}^* V_{ud}, \qquad f_{1,0} = \frac{V_{cs}^* V_{us} - V_{cd}^* V_{ud}}{\sqrt{2}} \approx \sqrt{2}\,(V_{cs}^* V_{us})$$

U-spin set of processes:

$$\bar{D}^0 \to \pi^+K^-, \qquad \bar{D}^0 \to K^+\pi^-, \qquad \bar{D}^0 \to \pi^+\pi^-, \qquad \bar{D}^0 \to K^+K^-$$

Expansion in the U-spin breaking

- On the fundamental level the U-spin breaking comes from the **mass difference between strange and down quarks**
- The small parameter is $\epsilon = \frac{m_s - m_d}{\Lambda_{QCD}} \sim 0.3$
- The breaking is realized via $(1,0)$-operator H_ϵ

$$\mathcal{H}_{\text{eff}} = \sum_{m,b} f_{u,m} \left(H_m^u \otimes H_\epsilon^{\otimes b} \right)$$

$$H_\varepsilon^{\otimes b} \equiv \underbrace{H_\varepsilon \otimes \cdots \otimes H_\varepsilon}_{b}$$

$$\mathcal{A}_j = f_{u,m} \sum_{\alpha} C_{j\alpha} X_{\alpha}$$

Reduced matrix element (RME)

- Below is the matrix $C_{j\alpha}$ up to $b = 2$
- To find the **sum rules** one needs to find the **null space of the matrix**

Example: $C_b \to L_b P^+ P^-$

Decay amplitude	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
$A\left(\Lambda_c^+ \to \Sigma^+ K^- K^+\right)$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{\sqrt{2}}{3}$	0	0	0
$A\left(\Xi_c^+ \to p\pi^- \pi^+\right)$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$-\frac{\sqrt{2}}{3}$	0	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	0	0
$A\left(\Xi_c^+ \to p K^- K^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- K^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	$\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$
$A\left(\Xi_c^+ \to p K^- \pi^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$
$A\left(\Lambda_c^+ \to p K^- \pi^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	0	0
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- K^+\right)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	0	0
$A\left(\Lambda_c^+ \to p K^- K^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	$\frac{\sqrt{2}}{3}$	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$-\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{1}{3\sqrt{2}}$	$\frac{1}{\sqrt{6}}$
$A\left(\Lambda_c^+ \to p\pi^- \pi^+\right)$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	0	$\frac{2}{3\sqrt{5}}$	0	0	0	$-\frac{\sqrt{2}}{3}$	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- K^+\right)$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	0	$-\frac{2}{3\sqrt{5}}$	0	0	0	$\frac{\sqrt{2}}{3}$	0
$A\left(\Lambda_c^+ \to p\pi^- K^+\right)$	1	0	0	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{2}}$	0	0	0	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- \pi^+\right)$	1	0	0	$-\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{2}}$	0	0	0	0

Decay amplitude	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}
$A\left(\Lambda_c^+ \to \Sigma^+ K^- K^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	0	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$-\frac{1}{3}$	0	0	0	0
$A\left(\Xi_c^+ \to p\pi^- \pi^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{\sqrt{15}}$	0	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$-\frac{1}{3}$	0	0	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Xi_c^+ \to p K^- K^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Lambda_c^+ \to \Sigma^+ \pi^- K^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$
$A\left(\Xi_c^+ \to p K^- \pi^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{2}}$
$A\left(\Lambda_c^+ \to p K^- \pi^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- K^+\right)$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{15}}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{3}}$	0	0	0
$A\left(\Lambda_c^+ \to p K^- K^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$-\frac{1}{3\sqrt{2}}$
$A\left(\Xi_c^+ \to \Sigma^+ \pi^- \pi^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$\frac{1}{3}\sqrt{\frac{2}{15}}$	$-\frac{1}{3}\sqrt{\frac{2}{5}}$	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$\frac{1}{3\sqrt{6}}$	$-\frac{1}{3\sqrt{2}}$
$A\left(\Lambda_c^+ \to p\pi^- \pi^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	0	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- K^+\right)$	0	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	$-\frac{2}{3}\sqrt{\frac{2}{15}}$	0	0	0	0	0	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	$-\frac{1}{3}\sqrt{\frac{2}{3}}$	0
$A\left(\Lambda_c^+ \to p\pi^- K^+\right)$	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{2}\sqrt{\frac{3}{5}}$	0	0	$\frac{1}{2\sqrt{5}}$	$\frac{1}{2}$	0	0	0	0	0
$A\left(\Xi_c^+ \to \Sigma^+ K^- \pi^+\right)$	$\frac{1}{2\sqrt{15}}$	$-\frac{1}{2}\sqrt{\frac{3}{5}}$	0	0	$\frac{1}{2\sqrt{5}}$	$\frac{1}{2}$	0	0	0	0	0

$b = 0$ (X_1–X_3) $b = 1$ (X_4–X_9) $b = 2$ (X_{10}–X_{20})

Systematics of U-spin sum rules

1) Any U-spin system can be constructed from doublets
2) The movement of irreps between initial/final state and the Hamiltonian doesn't change the structure of sum rules

Without loss of generality, we consider a system with the following U-spin structure

$$0 \to \left(\frac{1}{2}\right)^{\otimes n}, \qquad u = 0$$

U-spin pairs

$$\pm\frac{1}{2} \to \pm$$

Amplitude $A_j :$ $\underbrace{(-,-,+,-,+,\ldots,+)}_{n}$ $A_j = \sum_\alpha C_{j\alpha} X_\alpha$

U-spin conjugate $\bar{A}_j :$ $\underbrace{(+,+,-,+,-,\ldots,-)}_{n}$ $\bar{A}_j = (-1)^p \sum_\alpha (-1)^b C_{j\alpha} X_\alpha$

a- and s-type amplitudes

$$A_j = \sum_\alpha C_{j\alpha} X_\alpha$$

$$\bar{A}_j = (-1)^p \sum_\alpha (-1)^b C_{j\alpha} X_\alpha$$

$$a_j \equiv A_j - (-1)^p \bar{A}_j, \qquad s_j \equiv A_j + (-1)^p \bar{A}_j$$

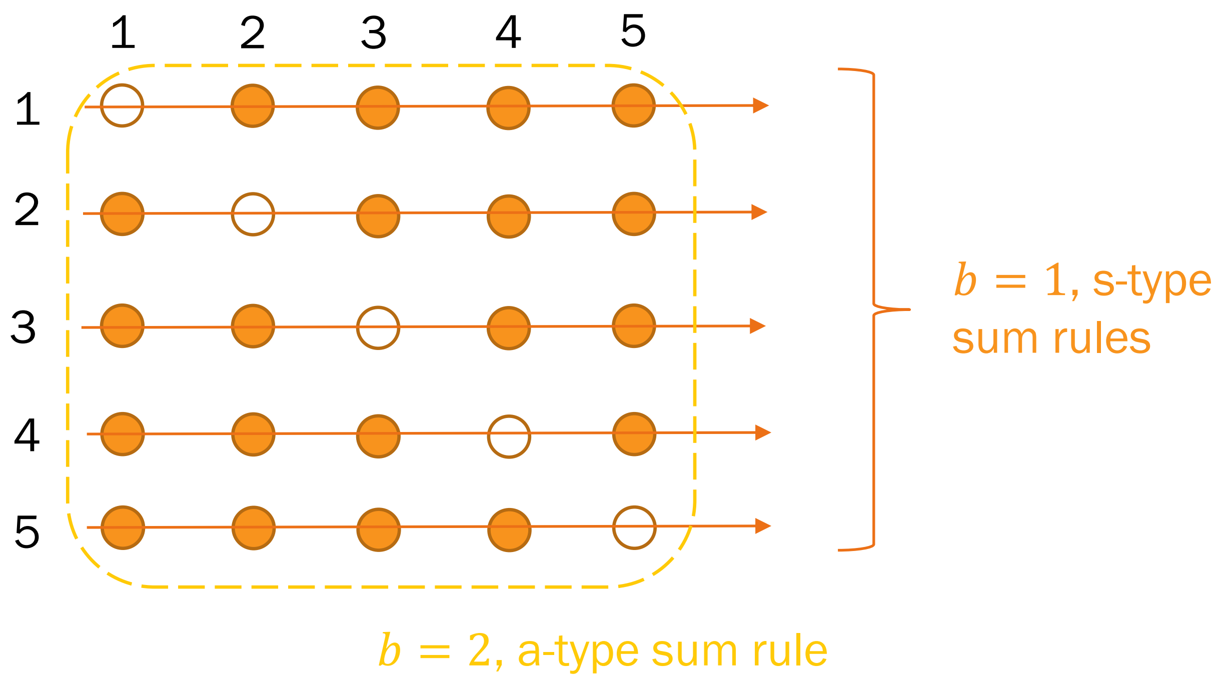
- all sum rules of the system can be written in terms of a- and s-type amplitudes
- $\boldsymbol{a_j}$ contain only the terms that are **odd in breaking $\boldsymbol{b}$**
- $\boldsymbol{s_j}$ contain only the terms that are **even in breaking $\boldsymbol{b}$**
- **a-type** sum rules that are **valid up to odd order $\boldsymbol{b}$ also hold at $\boldsymbol{b+1}$**
- **s-type** sum rules that are **valid up to even order $\boldsymbol{b}$ also hold at $\boldsymbol{b+1}$**
- for any system there are $n/2$ **trivial a-type sum rules** at $b = 0$: $a_j = 0$
- all sum rules at any order b have the form:

$$\sum a_j = 0 \quad \text{and} \quad \sum s_j = 0$$

Diagrammatic approach: $n = 6$ example

$$d = \frac{n}{2} - 1 = 2$$

- each node corresponds to a U-spin pair
- each node is a trivial a-type sum rule valid up to $b = 0$
- the sums of nodes in lines are s-type sum rules valid up to $b = 1$
- the sum of all nodes of the lattice is an a-type sum rule valid up to $b = 2$



Backup slides

Outline

- U-spin symmetry
- U-spin set of processes
- U-spin breaking
- Standard approach to U-spin sum rules
- Systematics of U-spin sum rules

Standard approach to writing sum rules

1) Basis rotation: from physical to U-spin basis

2) Wigner-Eckart theorem

Amplitude in the **physical basis** (states and the Hamiltonian are given by tensor products):

$$\mathcal{A}_j = \langle \text{out}|\mathcal{H}_{\text{eff}}|\text{in}\rangle_j$$

Wigner-Eckart theorem:

$$\langle u_2; m_2|O(u,m)|u_1; m_1\rangle = C^{u_2,m_2}_{u_1,m_1 \atop u,m} \langle u_2|O(u)|u_1\rangle$$

Reduced matrix element (RME)

$$\mathcal{A}_j = f_{u,m} \sum_\alpha C_{j\alpha} X_\alpha$$