W boson mass prediction in the Standard Model: the $\overline{MS}$ way

Stephen P. Martin
Northern Illinois University
spmartin@niu.edu

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Based on: arXiv:2203.05042 (3-loop QCD) and arXiv:1503.03782 (full 2-loop)

Software implementation (with Dave Robertson): `SMDR` arXiv:1907.02500
Standard Model in Dimensional Regularization, available on github.

Renewed interest in the W boson mass due to the recent CDF measurement:

$$M_W = 80.4335 \pm 0.0064_{\text{stat}} \pm 0.0069_{\text{syst}}\ \text{GeV} \quad \text{CDF 2022}$$

compared to

$$M_W = 80.379 \pm 0.012\ \text{GeV} \quad \text{Particle Data Group 2021}$$

In the Standard Model, the W-boson mass is predicted to be closer to

$$M_W = 80.354 \pm 0.004\ \text{GeV} \quad \text{SM prediction}$$

Actually, there are three distinct schemes that differently organize the perturbation theory for the W pole mass prediction in the Standard Model:

- On-shell scheme
- Hybrid scheme
- Pure $\overline{MS}$ scheme

Differences between them give an indication of the theory error.

In this talk I will compare these, highlighting my favorite, the pure $\overline{MS}$ scheme.

An important point:

The "theoretical uncertainty" in a predicted quantity like M_W is a poorly defined concept, especially if you are a frequentist statistician. Certainly not Gaussian!

Some imperfect indications of theoretical uncertainty that we use in practice:

- Try to estimate uncalculated contributions (can be tricky!)
- Reduction of contributions of known $\mathrm{N}^n\mathrm{LO}$ compared to $\mathrm{N}^{n-1}\mathrm{LO}$
- Renormalization scale dependence (gives a lower bound on the error, notoriously underestimates the actual error in many cases)
- Comparison of different schemes = different organizations of perturbation theory, neglect different higher-order contributions

On-shell scheme takes as inputs to the calculation:

$$M_Z,\ G_\mu,\ \alpha,\ \Delta\alpha_{\rm hadronic},\ M_t,\ \alpha_S^{(5)}(M_Z),\ M_h,\ \ldots$$

Then obtain

$$M_W^2 = M_Z^2 \left[\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\pi\alpha}{\sqrt{2} G_\mu M_Z^2}(1+\Delta r)}\right]$$

where $\Delta r(M_Z, M_t, \alpha, \ldots)$ is computed in a loop expansion:

Sirlin, Marciano (1-loop)
Djouadi, Verzegnassi, Halzen, Kniehl, Sirlin, Freitas, Hollik, Walter, Weiglein, Awramik, Czakon, Onischenko, Veretin (2-loop)
Avdeev, Fleischer, Mikhailov, Tarasov, Chetyrkin, Kühn, Steinhauser, Faisst, Seidensticker, Veretin, van der Bij, Jikia, Boughezal, Tausk, Schroder, Maierhofer, Sturm (partial 3-loop, 4-loop)

A convenient interpolating formula (and references to original calculations) for the on-shell M_W can be found in Awramik, Czakon, Freitas, Weiglein, hep-ph/0311148.

With PDG 2021 data for other parameters (including $M_t^{\rm pole} = 172.5$ GeV):

$$M_W = 80.3539 \quad \text{(on-shell scheme prediction)}$$

A reorganized treatment of the mixed QCD/EW 2-loop corrections apparently can lower the predicted M_W by $\sim$2 MeV; see Stål, Weiglein, Zeune, arXiv:1506.07465.

Hybrid scheme: the gauge couplings in loop diagrams are expressed in terms of $\overline{MS}$ running couplings $\alpha_{\overline{MS}}(M_Z)$ and $\sin^2\theta_W^{\overline{MS}}(M_Z)$, while propagator masses are M_Z, M_W, M_h, and M_t.

Avoids certain large logarithms associated with trading g, g' for M_W and M_Z.

Degrassi, Gambino, Giardino arXiv:1411.7040 provides (among other things) an interpolating formula for M_W, including full 2-loop and partial QCD 3-loop and 4-loop contributions. With PDG 2021 input data:

$$M_W = 80.3494 \quad \text{(hybrid scheme prediction)}$$

Note this is about 4.5 MeV lower than the on-shell scheme prediction.

Pure $\overline{MS}$ scheme:

- Inputs are all running Lagrangian parameters:

$$g_3,\ g,\ g',\ v,\ m_H^2,\ \lambda,\ y_t,\ y_b,\ \ldots$$

The only additional input parameter is $\Delta\alpha_{\rm had}$, which in principle is determined by the Lagrangian parameters, but in practice depends on non-perturbative effects.

- The Higgs VEV v is defined as the minimum of the full effective potential in Landau gauge (currently known to full 3-loop order, with leading QCD 4-loop contributions).
- Although G_μ and $\alpha = 1/137.035999139\ldots$ are extremely well known experimentally, they are output parameters in the pure $\overline{MS}$ scheme.
- Pole masses are also outputs, notably:

$$M_Z, M_W, M_t, M_h.$$

- Calculated outputs, including pole masses, have residual Q dependence ($\overline{MS}$ renormalization scale) due to finite order in perturbation theory. Gives a lower bound on error.
- Using $Q = 160$ GeV, find for the same 2021 PDG inputs: $M_W = 80.3525$ GeV, between the results obtained from on-shell and hybrid schemes.

To find pole mass, first calculate the 1-particle irreducible (1PI) transverse self-energy function:

$$\Pi_W(s) = \frac{1}{(16\pi^2)}\Pi_W^{(1)}(s) + \frac{1}{(16\pi^2)^2}\Pi_W^{(2)}(s) + \ldots$$

and then solve for the pole mass using

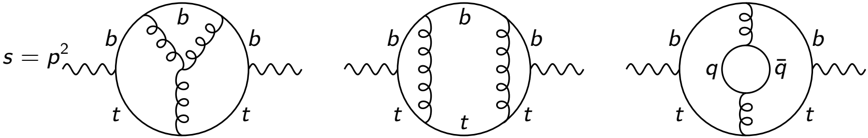
$$s_W^{\rm pole} = m_W^2 + \frac{1}{(16\pi^2)}\Pi^{(1)}(m_W^2) + \frac{1}{(16\pi^2)^2}\left[\Pi^{(2)}(m_W^2) + \Pi^{(1)}(m_W^2)\Pi^{(1)\prime}(m_W^2)\right] + \ldots,$$

where $m_W^2 = g^2 v^2/4$.

Note: the complex pole squared mass $s_W^{\rm pole}$ is not the same thing as the "variable-width Breit-Wigner" squared mass M_W^2 in the PDG and as reported by experimentalists. The real part of $s_W^{\rm pole}$ is slightly lower than M_W^2; see Scott Willenbrock's talk later in the session.

Below, I do the correction and report the PDG/experimentalist M_W.

Sample 3-loop QCD corrections to W-boson self-energy, pole mass:

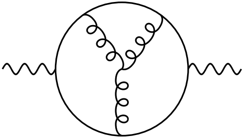


Reduce to renormalized ϵ-finite master integrals, keeping full $s = p^2 = M_W^2$ dependence. See arXiv:2112.07694 for details of master integrals.

Expand in $r = M_W^2/M_t^2$. Series includes powers of r and $\ln(M_t^2/Q^2)$, and $\ln(-M_W^2/Q^2)$ when (t, b) are replaced by massless quarks. Here Q is the $\overline{MS}$ renormalization scale. Converges rapidly for the physically relevant values $r < 1$.

Same strategy was used to find 3-loop QCD corrections to Z and Higgs self-energies and masses.

Although the series in powers of $r = M_W^2/M_t^2$ converges rapidly, the **sub-leading** term in the expansion is larger than the leading term. This is mostly due to:



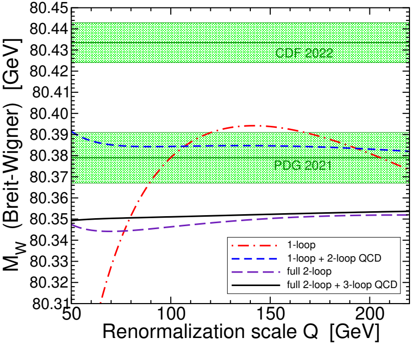
$$\Delta M_W^2 \quad = \quad -30.667 \frac{g^2 g_3^4}{(16\pi^2)^3} M_W^2 \left[\ln\left(-\frac{M_W^2}{Q^2}\right)\right]^2$$

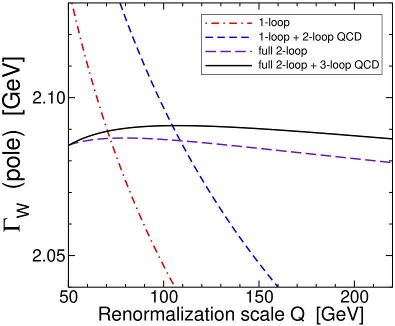
Note large coefficient, and $\left[\ln\left(-\frac{M_W^2}{Q^2}\right)\right]^2 = \left[-i\pi + \ln(M_W^2/Q^2)\right]^2 \sim -10.$

Imaginary part of logarithm from massless virtual particles, gets squared.

This is numerically larger than the $\frac{g^2 g_3^4}{(16\pi^2)^3} M_t^2$ contribution from (t, b) loops.

Results from pure $\overline{MS}$ scheme, with PDG 2021 data for M_t, M_Z, $\Delta\alpha_{\rm had}$, etc.

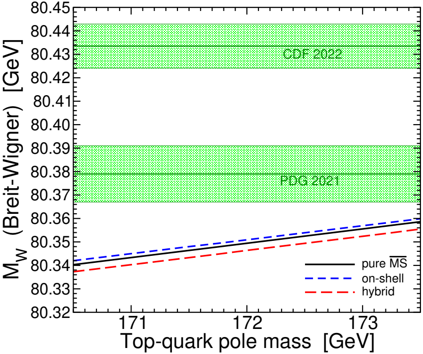




Parametric dependence is at least comparable to renormalization scale dependence:

$$M_W = M_W^{\rm ref} + 6.1 \text{ MeV} \left(\frac{M_t - M_t^{\rm ref}}{\text{GeV}}\right) + 1.3 \text{ MeV} \left(\frac{M_Z - M_Z^{\rm ref}}{\text{MeV}}\right) - 1.8 \text{ MeV} \left(\frac{\Delta\alpha_{\rm had} - \Delta\alpha_{\rm had}^{\rm ref}}{0.0001}\right) - 0.7 \text{ MeV} \left(\frac{\alpha_S - \alpha_S^{\rm ref}}{0.001}\right).$$

Comparison of the state-of-the-art computations of M_W in the three schemes:



on-shell = Awramik, Czakon, Freitas, Weiglein, hep-ph/0311148

hybrid = Degrassi, Gambino, Giardino arXiv:1411.7040

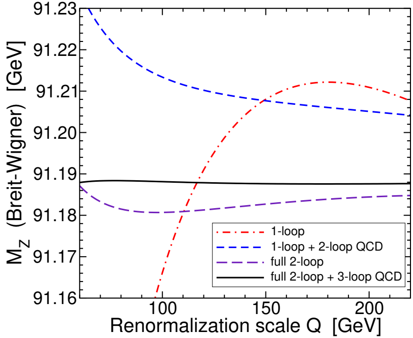
pure $\overline{MS}$ = arXiv:2203.05042, arXiv:1503.03782, `SMDR` arXiv:1907.02500 (w/ D. Robertson) available from github from command line: `./calc_fit -int`

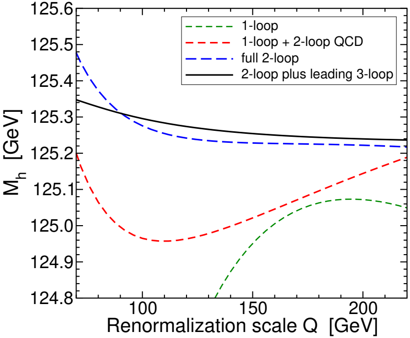
Input/fit data for M_Z, G_μ, α, $\Delta\alpha_{\rm had}$, α_S, . . . are from 2021 PDG.

Differences between schemes are consistent with $\pm$4 MeV theoretical error, excluding parameteric uncertainties.

4-loop QCD effects, neglected so far in pure $\overline{MS}$ scheme, are tiny.

In the pure $\overline{MS}$ scheme, as part of the same process that gives M_W, also get M_Z, M_h, M_t. Renormalization scale (Q) dependence of Z and Higgs masses:





Scale dependence of M_Z is extremely small. (Almost certainly a lucky accident.)

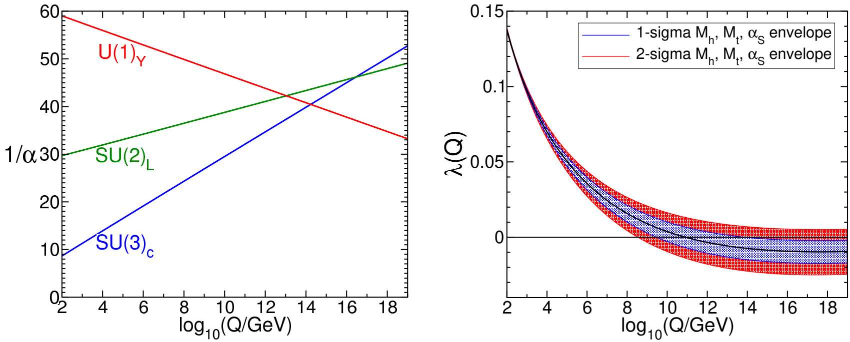
Scale dependence of M_h is much smaller than current experimental uncertainty, but this will change in the future.

Question: Why use the pure $\overline{MS}$ scheme, if it just agrees with the other schemes?

Answer: By fitting to data, provides the running renormalized Standard Model Lagrangian parameters $g_3, g, g', v, m_H^2, \lambda, y_t, y_b, \ldots$, appropriate for matching to your favorite UV completion model, or running to very high energy scales.

The public code `SMDR` provides automated state-of-the-art evaluation of multiloop effects in the $\overline{MS}$ scheme, including: effective potential, renormalization group running, decoupling at thresholds, matching between running masses and pole masses, fits to data.

What happens if we take the Standard Model seriously as an exact theory up to very high energy scales?



I claim that this determination and extrapolation of the Higgs self-coupling λ to high scales is the state-of-the-art now, since it is based on the most advanced calculation of the Higgs mass. (This whole business is dominated by parametric uncertainties anyway, for the foreseeable future.)

Outlook:

- Standard Model predictions for W boson mass show good agreement between 3 schemes: on-shell, hybrid, pure $\overline{MS}$. Consistent with previously claimed theoretical error ± 4 MeV.
- Next improvements in pure $\overline{MS}$ case: mixed QCD-EW 3-loop, non-QCD 3-loop, QCD 4-loop.
- To take advantage of these, need more precise experimental determinations of M_t (or y_t), M_W, M_Z, α_S, $\Delta\alpha_{\rm had}$.
- The code SMDR provides fits to the Standard Model $\overline{MS}$ Lagrangian parameters.
- Clearly, if CDF 2022 result for M_W is correct, the Standard Model is thoroughly dead! As they say on the internet:

"Big, if true."