University of CINCINNATI
ARTS AND SCIENCES

# Higher-Order Electroweak Contributions to Indirect *CP* Violation

## Pheno 2022

**Zachary Polonsky**

**Based on 2108.00017 with J. Brod and S. Kvedaraitė**

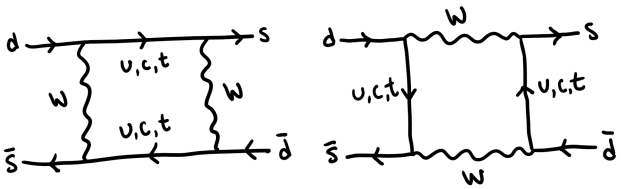
**and 2205.XXXXX with J. Brod, S. Kvedaraitė, and A. Youssef**

May 09, 2022

University of Cincinnati Department of Physics

# *CP* Violation in Neutral Kaons

- *CP* violation first discovered in decays of kaons (1964)
- $K^0 - \bar{K}^0$ mixing can lead to (indirect) *CP*-violation $\longrightarrow$ parameterized by $\epsilon_K$
- Sensitive probe of new physics, input for global CKM fit



# Diagonalizing the Hamiltonian

- Time evolution of $K^0 - \bar{K}^0$ system described by

$$i\frac{d}{dt}\begin{pmatrix} |K^0\rangle \\ |\bar{K}^0\rangle \end{pmatrix} = \left(M - \frac{i}{2}\Gamma\right)\begin{pmatrix} |K^0\rangle \\ |\bar{K}^0\rangle \end{pmatrix}$$

- Diagonalized by linear combinations

$$|K_L\rangle = p\,|K^0\rangle - q\,|\bar{K}^0\rangle\,, \qquad |K_S\rangle = p\,|K^0\rangle + q\,|\bar{K}^0\rangle$$

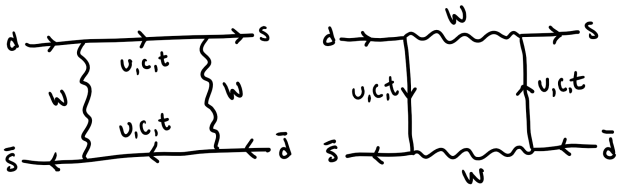
- If $|p/q| = 1$, $K_L$ and $K_S$ $CP$ eigenstates $\Rightarrow$ $K_L$ ($CP$ odd) $\rightarrow \pi\pi$ ($CP$ even) forbidden

# Definition of $\epsilon_K$

- Indirect *CP* violation suppressed by GIM $\rightarrow$ sensitive probe of high energies

$$\epsilon_K \equiv \frac{\langle (\pi\pi)_{I=0} | K_L \rangle}{\langle (\pi\pi)_{I=0} | K_S \rangle}$$

- Experimentally[1]: $\left|\epsilon_K\right|_{\text{ex}} = 2.228 \pm 0.011 \times 10^{-3}$




[1] **PDG 2020**.

# Why Electroweak?

- Re-parameterize Hamiltonian using CKM unitarity

$$\mathcal{H}_{n_f=3}^{|\Delta S|=2} = \frac{G_F^2 M_W^2}{4\pi^2}\left[\lambda_c^2 C_{S2}''^{cc}(\mu) + \lambda_t^2 C_{S2}''^{tt}(\mu) + \lambda_c\lambda_t C_{S2}''^{ct}(\mu)\right] Q_{S2}'' + h.c. + \ldots$$

$$\mathcal{H}_{n_f=3}^{|\Delta S|=2} = \frac{G_F^2 M_W^2}{4\pi^2}\left[\lambda_u^2 C_{S2}''^{uu}(\mu) + \lambda_t^2 C_{S2}''^{tt}(\mu) + \lambda_u\lambda_t C_{S2}''^{ut}(\mu)\right] Q_{S2}'' + h.c. + \ldots$$

- Drastic reduction of perturbative theory errors[2][3]

$$|\epsilon_K|_{\text{th}} = 1.81(16)_{\text{pert}}(5)_{\text{non-pert}}(23)_{\text{param}} \times 10^{-3} \quad \rightarrow \quad |\epsilon_K|_{\text{th}} = 2.16(6)_{\text{pert}}(7)_{\text{non-pert}}(15)_{\text{param}} \times 10^{-3}$$

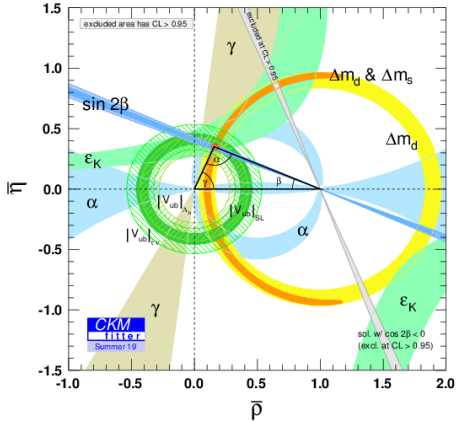
- Perturbative errors $\lesssim$ expected e/w effects $\rightarrow$ relevant!

[2] **Brod, Gorbahn, Stamou, Phys.Rev.Lett. 125 (2020) 17, 171803, 1911.06822 [hep-ph].**

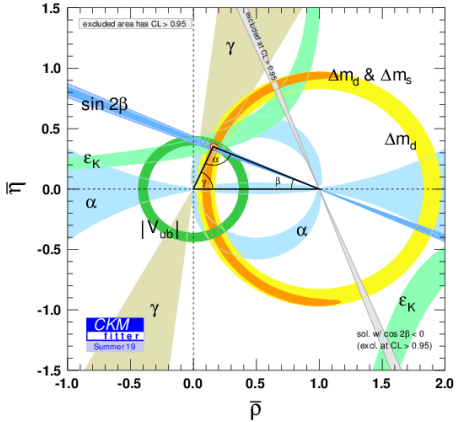
[3] **Brod, Gorbahn, Phys.Rev.Lett. 108 (2012) 121801, 1108.2036 [hep-th].**

# Reduction of Perturbative Error

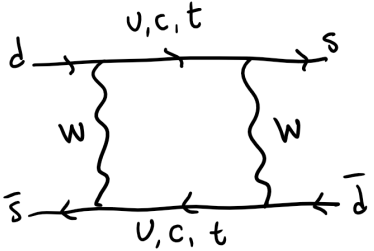
## Before

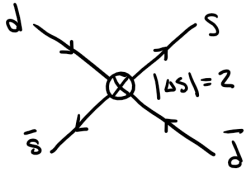


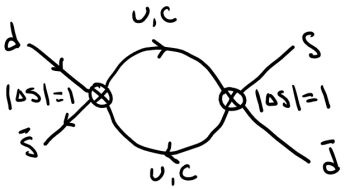

## After



# Different Contributions in EFT

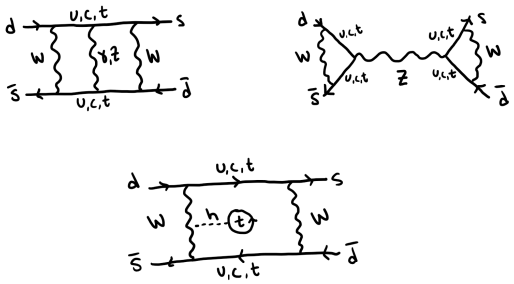


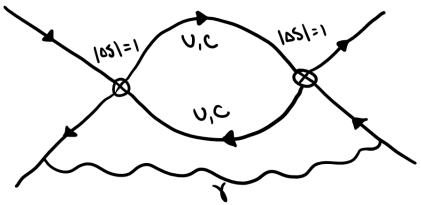
# Top Quark Contribution: E/W Matching

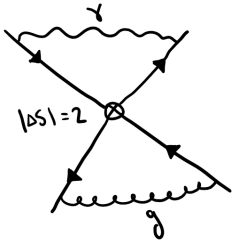
- Calculate all $O(30,000)$ two-loop Feynman diagrams including Goldstones, ghosts, tadpoles and "$Z$-bridges"
- Diagrams generated by `qgraf` and calculated using self-written `FORM` routines
- Separately calculated and verified agreement



# Charm-Top Contribution: Exercise in EFTs

- Generated via mixing of 2 $\times$ $|\Delta S| = 1 \rightarrow |\Delta S| = 2$ below E/W scale
- Threshold corrections when decoupling at quark mass scales, $\mu_b$ and $\mu_c$



# Contribution to $\epsilon_K$

- Top contribution gets $-1.0 \pm 0.04\%$ correction
- E/W charm-top contributions small

$$|\epsilon_K|_{\text{th}} = 2.15(6)_{\text{pert}}(7)_{\text{non-pert}}(15)_{\text{param}} \times 10^{-3}, \quad |\epsilon_K|_{\text{ex}} = 2.228 \pm 0.011 \times 10^{-3}$$

- Previous theory value[4]

$$|\epsilon_K| = 2.16(6)_{\text{pert}}(7)_{\text{non-pert}}(15)_{\text{param}} \times 10^{-3}$$

[4] **Brod, Gorbahn, Stamou, Phys.Rev.Lett. 125 (2020) 17, 171803, 1911.06822 [hep-ph].**

# EFT Scheme Dependence

- **Any physical observable must be independent of renormalization scheme!!!**
- Both top and charm contributions are scheme-dependent $\longrightarrow$ cancel with $\langle Q_{S2} \rangle$ (lattice)
- May ignore dependence in top contribution: $\mathcal{O}(\alpha)$ vs. $\mathcal{O}(y_t^2/4\pi)$
- Charm-top-contribution scheme dependence may play larger role

# Conclusions

- Presented NLO e/w corrections to $\epsilon_K$
- Analogous top calculation: $B^0 - \bar{B}^0$ system[5] (First independent re-calculation, reproduced numerics)
- See $\sim -1.0\%$ shift in $\left|\epsilon_K\right|_{\text{th}}$
- Upcoming three-loop QCD top contributions[6] and possible updated lattice calculations will reduce theory uncertainty of $\epsilon_K$

[5] **Gambino**, **Kwiatkowski**, **Pott**, **Nucl.Phys. B544 (1999) 532-556**, **9810400 [hep-th]**.

[6] **Brod**, **Gorbahn**, **Stamou**, **Yu**, **WIP**.

# Evanescent Operators

- Higher orders $\longrightarrow$ Dirac structures like $\gamma^\mu\gamma^\nu\gamma^\rho \otimes \gamma_\mu\gamma_\nu\gamma_\rho$, etc.
- Use dimensional regularization ($d = 4 - 2\epsilon$) $\Rightarrow$ can't use $\gamma_5$ relations!
- Introduce *evanescent operators* (unphysical) which vanish when $d \longrightarrow 4$, e.g.

$$E = \gamma^\mu\gamma^\nu\gamma^\rho \otimes \gamma_\mu\gamma_\nu\gamma_\rho - \left(16 - a_{11}\epsilon - a_{12}\epsilon^2 - \dots\right)\gamma^\mu \otimes \gamma_\mu$$

- Subtlety: After renormalization, $Z_{iE}E$, $1/\epsilon^n$ terms in $Z_{iE}$ cancel $\epsilon^n$ term in $E \longrightarrow$ values of $a_{ij}$ define renormalization scheme (arbitrary)

# EFT Normalization

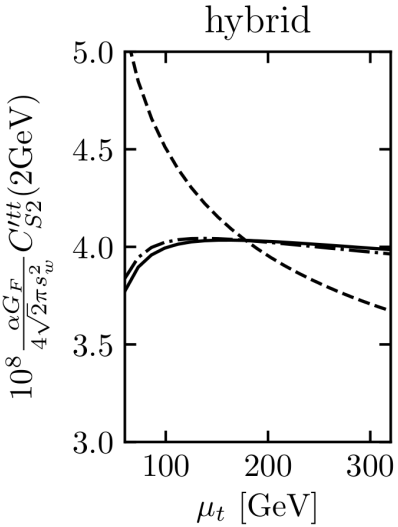
- Including e/w effects $\Rightarrow$ EFT normalizations no longer equivalent
- Different normalizations have different dependence on matching scale

$$\frac{\alpha(\mu)^2}{8M_W(\mu)^2 s_w(\mu)^4} \neq \frac{\alpha(\mu) G_F}{4\sqrt{2}\pi s_w(\mu)^2} \neq \frac{G_F^2 M_W(\mu)}{4\pi^2}$$

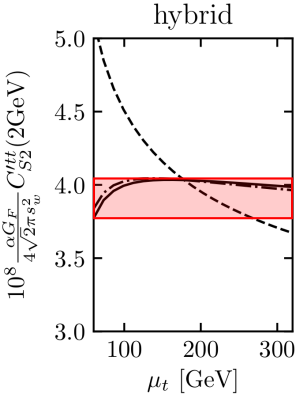
$$(\text{A2}) \qquad (\text{GF}) \qquad (\text{GF2})$$

- Relate e/w parameters and $G_F$ through NLO muon decay

# NLO QCD + E/W Contribution



# Perturbative Theory Error




- Higher order effects “leak in” (e.g. solving RGEs)
- Residual scale variation give estimation of size of these effects $\longrightarrow$ Perturbative error

# Sensitivity to $V_{cb}$

- $|\epsilon_K|$ *highly* sensitive to $V_{cb}$ ($\sim |V_{cb}|^4$ for top contribution)
- Different values for exclusive vs. inclusive[7]

$$|V_{cb}^{\text{incl.}}| = (42.2 \pm 0.8) \times 10^{-3}, \quad |V_{cb}^{\text{excl.}}| = (39.5 \pm 0.9) \times 10^{-3}, \quad |V_{cb}^{\text{avg.}}| = (41.0 \pm 1.4) \times 10^{-3}$$

$$|\epsilon_K|_{\text{th}}^{\text{incl.}} = (2.15 \pm 0.2) \times 10^{-3}, \quad |\epsilon_K|_{\text{th}}^{\text{excl.}} = (1.71 \pm 0.2) \times 10^{-3}, \quad |\epsilon_K|_{\text{th}}^{\text{avg.}} = (1.94 \pm 0.3) \times 10^{-3}$$

- $\sim 3\sigma$ tension for exclusive! (SM prefers inclusive)

[7] **PDG 2021**.

# A “Small” Correction

$$
\begin{aligned}
C_{S2}^{tt,(sc)} = {} & [128y - 896yz - 384yz^2 + 288xz^2 + 2752yz^2 + 1504yz^3 + 448z^2y^2 \\
& - 1080z^2z^3 - 7536xyz^3 + 872z^2y^3 - 256z^2y^3 - 576y^2z^3 + 144xz^4 \\
& - 864z^3z^4 + 608yz^4 - 1240xyz^4 + 1564z^2y^4 - 1664z^3y^4 + 64z^4y^4 \\
& + 2016xy^2z^4 + 288y^3z^4 - 540z^2z^5 + 6408z^4z^5 - 1632xyz^5 + 4740z^2y^5 \\
& - 10694z^3y^5 + 184z^4y^5 - 3438z^3z^6 - 720xyz^5 + 432z^3z^6 \\
& - 5184z^5z^6 + 2117z^4y^6 - 1668z^3y^6 - 4050z^4y^6 + 1872z^5y^6 \\
& + 765z^4z^7 - 36xz^7 + 432z^6z^7 - 373z^5y^7 + 364z^6yz^7 \\
& + 1080z^5yz^7 - 198z^5y^2z^7 - 225z^5y^3z^7]/[288(z-1)z(xz-1)^3] \\
& + \pi^2 \big[ -16x + 128xz + 16z^2z - 404xz^2 - 24z^2z^2 + 8z^3z^2 + 472xz^3 - 156z^2z^3 \\
& - 8z^3z^3 + 32z^4 - 32xz^4 + 148z^2z^4 - 134z^3z^4 + 48z^4z^4 + 8z^5z^4 - 16z^6 \\
& + 64xz^5 - 24z^2z^5 + 44z^3z^5 - 24z^4z^5 + 8z^5z^5 + 2z^6 - 14xz^6 + 2z^2z^6 \\
& - 96z^4z^6 + 2z^5z^6 - 18z^3y^2z^6 - 6z^2z^7 - 108z^5z^7 - z^7z^8]/[216z^2(z-1)z^3] \\
& + \Phi\left(\frac{1}{4x}\right) \big[ -256z + 2816zz - 10624z^2z + 15232z^3z - 8192z^4z - 2048z^5z \\
& + 128z^2 - 512z^2z - 3584z^3z^2 + 21888z^4z^2 - 27520z^5z^2 + 11264z^6z^2 \\
& + 10240y^6z^2 - 16z^3 - 32z^3z + 1304z^2z^3 - 4752z^3z^3 + 2928z^4z^3 \\
& - 12704z^5z^3 + 1024xz^3 - 14848z^2z^4 + 32xz^4 - 1056z^2z^4 + 5936z^3z^4 \\
& - 4640z^4z^4 - 27952z^5z^4 + 72256z^6z^4 - 3840z^7z^4 + 7168z^8z^4 \\
& - 8z^2z^5 + 2304z^3z^5 - 19252z^4z^5 + 50320z^5z^5 - 35164z^6z^5 \\
& - 40352z^7z^5 + 19456z^8z^5 - 512z^9z^5 - 1532z^4z^6 + 11756z^5z^6 \\
& - 23860z^6z^6 + 4184z^7z^6 + 13696z^8z^6 - 2816z^9z^6 + 41z^4z^7 \\
& - 198z^5z^7 - 2506z^6z^7 + 1612z^7z^7 - 252z^8z^7 - 1280z^9z^7] \\
& \quad /[576z^2(4x-1)(z-1)(xz-1)^4] \\
& + \Phi\left(\frac{y}{4x}\right) \big[ -176z^2z^3 + 256xyz^3 - 56y^2z^3 + 656z^3z^4 - 1088z^2yz^4 + 304xy^2z^4 \\
& - 16y^3z^4 - 1164z^4z^5 + 1740z^3yz^5 - 564z^2y^2z^5 + 48xy^3z^5 + 728z^5z^6 \\
& - 948z^4yz^6 + 324z^3y^2z^6 - 32z^2y^3z^6 - 44z^6z^7 + 112z^5yz^7 \\
& - 66z^4y^2z^7 + 14z^3y^3z^7 - z^2y^4z^7]/[64(z-1)(xz-1)^4] \\
& + \Phi\left(\frac{z}{4}\right) \big[ 256z^2z + 7872z^2z^2 - 512z^3z^2 - 23072z^2z^3 - 17216z^3z^3 + 256z^4z^3 \\
& + 5664z^2z^4 - 1680z^3z^4 + 2048z^4z^4 + 5688z^2z^5 - 6240z^3z^5 \\
& - 2768z^4z^5 - 1222z^2z^6 - 1636z^3z^6 + 2032z^4z^6 - 73z^2z^7 \\
& + 900z^3z^7 + 322z^4z^7 + 32z^3z^8 - 208z^4z^8 - 8z^4z^9] \\
& \quad /[576(z-1)(xz-1)^4] \\
& + \Phi\left(\frac{yz}{4}\right) [144z^2z^3 + 48x^2z^4 - 224z^2yz^4 - 192z^4z^5 + 336z^3yz^5 + 132z^2y^2z^5 \\
& - 88z^4yz^6 - 228z^3y^2z^6 - 18z^2y^3z^6 + 98z^4y^2z^7 + 36z^3y^3z^7 \\
& - x^2y^4z^7 - 16z^4y^3z^8]/[64(z-1)(xz-1)^4] \\
& + \Phi\left(\frac{1}{xz}, \frac{1}{z}\right) \big[ -64 + 576z + 384xz - 2128z^2 - 2848xz^2 - 928x^2z^2 + 3824z^3 \\
& + 8704xz^3 + 5664x^2z^3 + 1120z^3z^3 - 3312z^4 - 12752xz^4 \\
& - 14056x^2z^4 - 5816z^3z^4 - 672x^4z^4 + 992z^5 + 9472xz^5 \\
& + 14904x^2z^5 + 11672z^3z^5 + 3544x^4z^5 + 224z^5z^5 + 272z^6 \\
& - 3032xz^6 - 6904x^2z^6 - 9866z^3z^6 - 6800x^4z^6 - 1840z^5z^6 \\
& - 224z^8z^6 - 144z^7 - 424xz^7 + 2232x^2z^7 + 4678z^3z^7 \\
& + 6914x^4z^7 + 4536z^5z^7 + 1104x^6z^7 + 288z^7z^7 - 16z^8 \\
& + 456xz^8 - 592x^2z^8 - 607z^3z^8 + 6444x^4z^8 + 368z^5z^8 \\
& - 2664x^6z^8 - 408z^7z^8 - 160z^8z^8 + 40xz^9 - 304x^2z^9 \\
& - 1787z^3z^9 + 2891x^4z^9 + 2268z^5z^9 - 2720x^6z^9 \\
& + 1000z^7z^9 - 8z^8z^9 + 32x^9z^9 - 16x^2z^{10} + 565z^3z^{10} \\
& - 1766x^4z^{10} + 2963x^5z^{10} - 2524x^6z^{10} + 1018x^7z^{10} \\
& - 272x^8z^{10} + 32x^9z^{10} + 41z^3z^{11} - 403x^4z^{11} \\
& + 1093z^5z^{11} - 1279x^6z^{11} + 694x^7z^{11} - 154x^8z^{11} + 8x^9z^{11}] \\
& \quad /[288x^2(z-1)z^4(xz-1)^4] \\
& + \Phi\left(\frac{1}{xz}, \frac{y}{x}\right) \big[ -16 + 104xz + 64yz - 296x^2z^2 - 296xyz^2 - 96y^2z^2 \\
& + 476x^2z^3 + 544x^2yz^3 + 344xy^2z^3 + 64y^3z^3 - 436x^4z^4 \\
& - 568x^3yz^4 - 416x^2y^2z^4 - 216xy^3z^4 - 16y^4z^4 + 160z^5z^5 \\
& + 326x^4yz^5 + 271x^3y^2z^5 + 192x^2y^3z^5 + 64xy^4z^5 + 80z^6z^6 \\
& - 106x^5yz^6 - 185x^4y^2z^6 + 19x^3y^3z^6 - 88x^2y^4z^6 - 100z^7z^7 \\
& + 114x^5yz^7 - 31x^5y^2z^7 - 24x^4y^3z^7 + 41x^3y^4z^7 + 28x^2y^5z^7 \\
& - 78x^5yz^8 + 65x^6y^2z^8 - 7x^5y^3z^8 - 9x^4y^4z^8 + x^5y^5z^8] \\
& \quad /[32x^2(z-1)z(xz-1)^4] \\
& + \mathrm{Li}_2\left(1 - \frac{1}{z}\right) \big[ 16 - 112z - 48xz + 292z^2 + 232xz^2 + 40x^2z^2 - 260z^3 \\
& - 420xz^3 - 128x^2z^3 + 16z^4 + 236xz^4 + 114x^2z^4 - 40x^3z^4 \\
& + 44z^5 - 64xz^5 + 186x^2z^5 + 72x^3z^5 + 4z^6 - 42xz^6 + 8x^2z^6 \\
& - 155x^3z^6 - 2xz^7 - 54x^2z^7 + 31x^3z^7 - 4x^2z^8 + 36x^3z^8 + 2x^3z^9] \\
& \quad /[36xz^3(xz-1)^2] \\
& + \mathrm{Li}_2(1-yz) \big[ 4 - 14xz - 12yz + 20x^2z^2 + 24xyz^2 + 12y^2z^2 - 13x^3z^3 \\
& - 12x^2yz^3 - 18xy^2z^3 - 4y^3z^3 + 6x^3yz^4 - 6x^2y^2z^4 + 8xy^3z^4 \\
& + 9x^3y^2z^5 - 2x^2y^3z^5 - 2x^3y^3z^6]/[4x(z-1)(xz-1)^2] \\
& + \mathrm{Li}_2(1-xz) \big[ -32z + 256z + 128xz - 808z^2 - 816xz^2 - 176x^2z^2 + 944z^3 \\
& + 2112xz^3 + 896x^2z^3 + 80x^3z^3 - 2504xz^4 - 1788x^2z^4 \\
& - 396x^3z^4 + 1968z^2z^5 + 616x^3z^5 + 164x^4z^5 + 32yz^5 \\
& - 3z^3z^6 - 644x^4z^6 - 164x^5z^6 - 48x^6z^6 - 375x^4z^7 - 332x^5z^7 \\
& + 44x^6z^7 + 16z^7z^7 + 25x^7z^8 - 112x^6z^8 + 16x^7z^8 + 449x^6z^9 \\
& + 4x^7z^9 - 216z^7z^{10}]/[72x(z-1)z^3(xz-1)^3] \\
& + \mathrm{Li}_2\left(1 - \frac{x}{y}\right) [9x^2z^3 - 18xyz^3 + 9y^2z^3 - 20x^3z^4 + 42x^2yz^4 - 24xy^2z^4 \\
& + 2y^3z^4 + 8x^4z^5 - 18x^3yz^5 + 12x^2y^2z^5 - 2xy^3z^5] \\
& \quad /[4(z-1)(xz-1)^2] \\
& + \mathrm{Li}_2(1-x) \big[ -64z + 192xz - 160z^2z + 16x^3z + 16x^5z + 32z^2 - 32xz^2 \\
& - 72x^2z^2 - 16x^3z^2 + 200x^4z^2 - 96x^5z^2 - 16z^6z^2 - 4x^3 \\
& - 2xz^3 + 141x^2z^3 - 256x^3z^3 + 165x^4z^3 - 84x^5z^3 + 32z^6z^3 \\
& + 8x^7z^3 + 4xz^4 - 98x^2z^4 + 136x^3z^4 - 6x^4z^4 - 16x^5z^4 \\
& - 28x^6z^4 + 8x^7z^4 + 84x^3z^5 - 188x^4z^5 + 126x^5z^5 \\
& - 24x^6z^5 + 2x^7z^5]/[36x^2(z-1)(xz-1)^2] \\
& + \log^2(x)[256yz - 1280xyz + 1408x^2yz - 128x^3yz - 128z^2yz - 128yz^2 \\
& - 512xyz^2 + 4096x^2yz^2 - 4352x^3yz^2 - 1216x^4yz^2 + 768z^5yz^2 \\
& + 512x^6yz^2 + 16yz^3 + 256xyz^3 - 376x^2yz^3 - 2688x^3yz^3 \\
& + 3776x^4yz^3 + 8160x^5yz^3 - 2560x^6yz^3 - 832x^7yz^3 + 1296x^3y^2z^3 \\
& - 504x^2y^3z^3 - 48xyz^4 + 736x^2yz^4 + 72x^3yz^4 - 3904x^4yz^4 \\
& - 7296z^5yz^4 - 12608x^6yz^4 + 3008x^7yz^4 + 704x^8yz^4 - 6480x^4y^2z^4 \\
& + 2952x^3y^3z^4 - 144x^2y^4z^4 - 5184x^7z^5 + 40x^2yz^5 - 3120x^3yz^5 \\
& + 6228x^4yz^5 + 968x^5yz^5 + 13272x^6yz^5 + 6000x^7yz^5 - 1344x^8yz^5 \\
& - 320x^9yz^5 + 13284x^5y^2z^5 - 6660x^4y^3z^5 + 576x^3y^4z^5 \\
& + 10368x^8z^6 - 8x^3yz^6 + 3788x^4yz^6 - 8232x^5yz^6 + 2092x^6yz^6 \\
& - 2760x^7yz^6 + 48x^8yz^6 + 64x^9yz^6 + 64x^{10}yz^6 - 12528x^6y^2z^6 \\
& + 6552x^5y^3z^6 - 720x^4y^4z^6 - 5184x^9z^7 - 41x^4yz^7 - 1580x^5yz^7 \\
& + 2772x^6yz^7 + 804x^7yz^7 - 5584x^8yz^7 - 272x^9yz^7 + 64x^{10}yz^7 \\
& + 4824x^7y^2z^7 - 2736x^6y^3z^7 + 396x^5y^4z^7 - 9x^4y^5z^7 + 41x^5yz^8
\end{aligned}
$$

# A “Small” Correction

$$
\begin{aligned}
&+ 48x^6y^2z^8 - 208x^7y^2z^8 - 88x^8y^2z^8 + 3696x^9y^2z^8 + 16x^{10}y^2z^8 \\
&- 396x^4y^3z^8 + 396x^5y^3z^8 - 108x^6y^4z^8 + 9x^7y^5z^8 - 864x^{10}y^2z^7] \\
&/[576x^2y(z-1)(xz-1)^7] \\
&+ \log^2(y)[9x^7z^4 - 18xyz^4 + 7y^2z^4 - 20x^3z^4 + 42x^2yz^4 - 20xy^2z^4 + 2y^3z^4 \\
&+ 8x^5z^5 - 18x^4yz^5 + 10x^3y^2z^5 - 2xy^3z^5]/[8(z-1)(xz-1)^2] \\
&+ \log(z)[128y - 832yz - 448xyz + 1776yz^2 + 2976xyz^2 + 512x^2yz^2 - 192yz^3 \\
&- 6256xyz^3 - 4784x^2yz^3 - 160y^2z^3 - 144y^2z^3 + 216xz^4 \\
&- 648x^2z^4 - 16yz^4 + 232xyz^4 + 10336x^2yz^4 + 4032x^3yz^4 \\
&- 64z^4y^2z^4 + 684xy^2z^4 - 1026x^2z^5 + 3888x^4z^5 + 40xyz^5 \\
&+ 268x^2yz^5 - 5374x^3yz^5 - 1576x^4yz^5 + 32x^5yz^5 - 1134x^2y^2z^5 \\
&+ 1566x^3y^2z^5 - 7128x^4y^2z^5 - 491x^2yz^6 - 1668x^3yz^6 + 9057x^4yz^6 \\
&+ 184x^5yz^6 + 1377x^3y^2z^6 - 171x^4y^2z^6 - 810x^5z^7 + 4104x^6z^7 \\
&+ 203x^3yz^7 + 340x^4yz^7 - 1938x^5yz^7 - 648x^4y^2z^7 + 189x^5y^2z^7 \\
&+ 54x^5z^8 - 216x^7z^8 - 96x^8yz^8 - 78x^7yz^8 + 567x^6y^2z^8 \\
&+ 27x^5y^2z^8 - 72x^4y^3z^8]/[144y(z-1)z(xz-1)^4] \\
&+ \log^3(z)[-288y + 1872xyz - 560x^2yz + 864y^2z - 5328x^2yz^2 + 1424x^3yz^2 \\
&+ 2096x^4yz^2 + 128x^2y^2z^2 - 4320xy^2z^2 - 864y^3z^2 + 2348x^2y^2z^3 \\
&- 2672x^3y^2z^3 - 3168x^2y^3z^3 - 448x^3y^3z^3 + 8640xy^3z^3 + 3888xy^2z^3 \\
&+ 288y^4z^3 + 8248x^2yz^4 - 5916x^3yz^4 + 2400x^4yz^4 + 2064y^3z^4 \\
&+ 576x^2y^3z^4 - 9864x^3y^2z^4 - 6048x^2y^3z^4 - 1440xy^4z^4 + 1296x^2z^5 \\
&- 5184x^5z^3 + 2806x^4yz^5 - 16040x^3yz^5 + 20288x^2yz^5 - 2928x^4y^2z^5 \\
&- 944x^3y^2z^5 - 320x^2y^3z^5 + 7920x^4y^2z^5 + 3618x^3y^3z^5 + 2736x^2y^4z^5 \\
&- 2592z^5 + 10368x^7z^6 - 1368x^3y^4z^5 - 2450x^5yz^6 + 9708x^4yz^6 \\
&- 11280x^3yz^6 + 2240x^2yz^6 + 48x^6yz^6 + 64x^5y^2z^6 - 4212x^4y^2z^6 \\
&- 270x^3y^3z^6 - 2484x^2y^4z^6 + 1296z^6z^7 - 5184x^6z^7 - 72x^5yz^7 \\
&+ 2332x^4yz^7 - 498x^3yz^7 - 19164x^2yz^7 - 3060xyz^7 - 480x^5y^2z^7 \\
&+ 64x^4y^2z^7 + 972x^3y^2z^7 - 378x^2y^3z^7 + 1080xy^4z^7 - 9y^5y^2z^7 \\
&+ 105x^4yz^8 - 1188x^3yz^8 + 142x^2yz^8 + 3524x^3yz^8 + 16x^5yz^8 \\
&+ 54x^5y^2z^8 - 180x^2y^4z^8 + 9x^4y^2z^8 - 40x^3yz^8 + 224x^2y^3z^8 \\
&- 864x^8yz^9 + 8x^6yz^{10}]/[576xy(z-1)(xz-1)^5] \\
&+ \log\left(\frac{\mu}{M_H}\right)\Big\{[432xz - 1408xyz - 1620x^2z^2 + 1072xyz^2 + 5280x^2yz^2 \\
&+ 216xz^3 + 432x^3z^3 - 312xyz^3 - 3732x^2yz^3 - 4224x^3yz^3 \\
&+ 108xy^2z^3 - 810x^2z^4 + 3132x^3z^4 + 1323x^2yz^4 - 312x^2y^2z^4 \\
&+ 352x^4yz^4 - 405x^2yz^5 + 648x^3z^5 - 2592x^3z^5 - 171x^2yz^5 \\
&+ 137x^4yz^5 + 324x^3y^2z^5 - 54x^4z^6 + 216x^4z^6 + 78x^5yz^6 \\
&- 135x^5y^2z^6 - 27x^4y^2z^6]/[72y(z-1)(xz-1)^3]
\end{aligned}
$$

$$
\begin{aligned}
&+ [108z^3z^3 - 352x^3yz^3 - 108x^4z^4 + 436x^2yz^4 + 352x^3yz^4 \\
&+ 54x^2z^5 - 216x^3z^5 - 180x^2yz^5 + 113x^3yz^5 + 27x^2y^2z^5 \\
&- 54x^4z^6 + 216x^4z^6 + 27x^3yz^6 - 54x^2yz^6 - 27x^3y^2z^6] \\
&\times \log(xz)/[12y(z-1)(xz-1)^4]\Big\} \\
&+ \log(y)[-16yz^2 + 40xyz^2 + 16y^2z^2 - 30x^2yz^2 - 40xy^2z^4 + 7x^3yz^3 \\
&+ 43x^2y^2z^3 - x^4yz^6 - 13x^3y^2z^5]/[16(z-1)(xz-1)^3] \\
&+ [16 - 88xz - 48yz + 208x^2z^2 + 192xyz^2 + 48y^2z^2 - 268x^3z^3 \\
&- 288x^2yz^3 - 168xy^2z^3 - 16y^3z^3 + 168x^2z^4 + 284x^3yz^4 \\
&+ 168x^2y^2z^4 + 64xy^3z^4 + 8x^5z^5 - 210x^4yz^5 - 39x^3y^2z^5 \\
&- 88x^3y^3z^5 - 72x^4z^6 + 120x^3yz^6 - 48x^4y^2z^6 + 58x^3y^3z^6 \\
&+ 28x^7z^7 - 50x^6yz^7 + 21x^5y^2z^7 - 12x^4y^3z^7] \\
&\times \log(z)[32z(z-1)(xz-1)^4] \\
&+ \log(x)\Big\{[-64y + 256xyz + 448yz - 1600xyz - 768x^2yz - 1248yz^2 + 2528xyz^2 \\
&+ 6688x^2yz^2 + 384x^3yz^2 - 64yz^3 + 5360xyz^3 - 10944x^2yz^3 \\
&- 13504x^3yz^3 + 768x^4yz^3 + 288y^2z^3 - 1152xy^2z^3 + 648x^2z^4 \\
&- 2592x^4z^4 - 296yz^4 + 1496xyz^4 - 11792x^2yz^4 + 19904x^3yz^4 \\
&+ 14048x^4yz^4 - 896x^5yz^4 - 1188y^2z^4 + 4752x^2y^2z^4 - 144y^3z^4 \\
&+ 576xy^3z^4 - 3888x^2z^5 + 15552x^3z^5 + 1104xyz^5 - 4900x^2yz^5 \\
&+ 11174x^3yz^5 - 7328x^4yz^5 - 6432x^5yz^5 + 256x^6yz^5 + 1764x^2y^2z^5 \\
&- 7056x^3y^2z^5 + 504xy^3z^5 - 2016x^2y^3z^5 - 108x^2z^6 + 432x^4z^6 \\
&+ 7128x^5z^6 - 28512x^6z^6 - 1432x^3yz^6 + 7409x^4yz^6 - 20447x^5yz^6 \\
&+ 33124x^6yz^6 + 864x^7yz^6 - 1710x^3y^2z^6 + 6840x^4y^2z^6 - 576x^2y^3z^6 \\
&+ 2304x^3y^3z^6 + 108x^4z^7 - 432x^5z^7 - 4104x^6z^7 + 16416x^7z^7 \\
&+ 1098x^3yz^7 - 4722x^4yz^7 + 5226x^5yz^7 - 7528x^6yz^7 + 720x^4y^2z^7 \\
&- 2880x^5y^2z^7 + 315x^3y^3z^7 - 1260x^4y^3z^7 + 216x^2z^8 - 864x^8z^8 \\
&- 115x^4yz^8 + 439x^5yz^8 - 783x^6yz^8 + 2208x^7yz^8 - 36x^5y^2z^8 \\
&+ 144x^6y^2z^8 - 45x^4y^3z^8 + 180x^5y^3z^8] \\
&/[144(4x-1)y(z-1)z(xz-1)^4] \\
&+ [16 - 88xz - 48yz + 208x^2z^2 + 192xyz^2 + 48y^2z^2 - 340x^3z^3 \\
&- 144x^2yz^3 - 224xy^2z^3 - 16y^3z^3 + 472x^2z^4 - 340x^3yz^4 \\
&+ 440x^2y^2z^4 + 48xy^3z^4 - 448x^3z^5 + 756x^4yz^5 - 495x^3y^2z^5 \\
&- 40x^3y^3z^5 + 216x^6z^6 - 504x^5yz^6 + 272x^4y^2z^6 + 10x^3y^3z^6 \\
&- 36x^7z^7 + 94y^6yz^7 - 59x^5y^2z^7 + 4x^4y^3z^7] \\
&\times \log(y)/[32z(z-1)(xz-1)^4]
\end{aligned}
$$

$$
\begin{aligned}
&+ \big[-64y + 512yz + 384xyz - 1616yz^2 - 2656xyz^2 - 928x^2yz^2 \\
&+ 2064yz^3 + 7456xyz^3 + 5568x^2yz^3 + 1120x^3yz^3 - 1104yz^4 \\
&- 8312xyz^4 - 13680x^2yz^4 - 6008x^3yz^4 - 672x^4yz^4 + 432y^2z^4 \\
&- 112yz^5 + 4512xyz^5 + 11544x^2yz^5 + 12608x^3yz^5 + 3704x^4yz^5 \\
&+ 224x^5yz^5 - 2160xy^2z^5 - 432y^2z^5 + 160y^3z^5 + 248xyz^6 \\
&- 6200x^2yz^6 - 6630x^3yz^6 - 3968x^4yz^6 - 1776x^5yz^6 - 224x^6yz^6 \\
&+ 4320x^2y^2z^6 + 1944xy^2z^6 + 144y^4z^6 + 16yz^7 - 640xyz^7 \\
&+ 176x^2yz^7 + 5732x^3yz^7 - 10218x^4yz^7 - 2248x^5yz^7 + 1072x^6yz^7 \\
&+ 288x^7yz^7 - 4716xy^2z^7 - 3024x^3y^2z^7 - 720xy^4z^7 + 648x^4z^8 \\
&- 5184x^6z^8 - 56xyz^8 + 760x^2yz^8 + 1197x^3yz^8 - 6256x^4yz^8 \\
&+ 25032x^5yz^8 + 520x^6yz^8 - 472x^7yz^8 - 160x^8yz^8 + 3474x^4y^2z^8 \\
&+ 1755x^3y^2z^8 + 1368x^2y^4z^8 - 1296z^5z^9 + 10368x^7z^9 + 56x^2yz^9 \\
&- 870x^3yz^9 - 53x^4yz^9 + 2044x^5yz^9 - 11776x^6yz^9 + 1120x^7yz^9 \\
&+ 24x^8yz^9 + 32x^9yz^9 - 1242x^5y^2z^9 - 351x^3y^3z^9 - 1170x^3y^4z^9 \\
&+ 648x^8z^{10} - 5184x^8z^{10} - 57x^3yz^{10} + 952x^4yz^{10} - 2187x^5yz^{10} \\
&+ 1820x^6yz^{10} - 2962x^7yz^{10} - 240x^8yz^{10} + 32x^9yz^{10} \\
&- 558x^6y^2z^{10} + 243x^5y^2z^{10} + 450x^4y^4z^{10} - 9x^7y^2z^{10} \\
&+ 41x^4yz^{11} - 362x^5yz^{11} + 731x^6yz^{11} - 548x^7yz^{11} \\
&+ 3418x^8yz^{11} + 8x^9yz^{11} + 450x^7y^2z^{11} - 135x^6y^3z^{11} \\
&- 72x^5y^4z^{11} + 9x^7y^2z^{11} - 864x^9yz^{12}] \\
&\times \log(z)\big/[288xy(z-1)z^2(xz-1)^5]\Big\}.
\end{aligned}
$$