The spatial Functional Renormalization Group and Hadamard states on cosmological spacetimes*

Rudrajit (Rudi) Banerjee

Okinawa Institute of Science and Technology

Pheno 2022

*Based on [R. Banerjee, M. Niedermaier, Nucl. Phys. B. (2022)]

Outline

1. Motivation: Lorentzian Asymptotic Safety?
2. Euclidean vs. Lorentzian FRG
3. Ultraviolet flow: state independence
4. Infrared aspects: state-dependence of flow, comparison to non-local heat kernel
5. Outlook

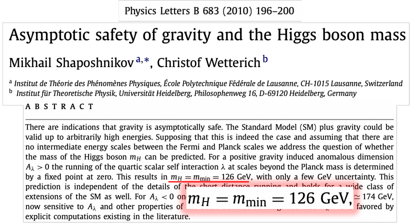
Lorentzian Asymptotic Safety?

The idea: Base ultraviolet (UV) completion of quantum gravity (QGR) as a quantum field theory (QFT) on non-Gaussian fixed point (NGFP) of renormalization group flow.

Main technique: (Euclidean) Functional Renormalization Group (FRG).

- Has produced strong evidence in support of the existence of a NGFP for quantum gravity.
- Associated finite-dimensional UV critical surface potentially leads to enhanced predictivity in the infrared (IR).

Phenomenological implications studied by flowing from UV to IR scales.



Physics Letters B 683 (2010) 196–200

Asymptotic safety of gravity and the Higgs boson mass

Mikhail Shaposhnikov [a,*], Christof Wetterich [b]

[a] Institut de Théorie des Phénomènes Physiques, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland
[b] Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, D-69120 Heidelberg, Germany

A B S T R A C T

There are indications that gravity is asymptotically safe. The Standard Model (SM) plus gravity could be valid up to arbitrarily high energies. Supposing that this is indeed the case and assuming that there are no intermediate energy scales between the Fermi and Planck scales we address the question of whether the mass of the Higgs boson m_H can be predicted. For a positive gravity induced anomalous dimension $A_\lambda > 0$ the running of the quartic scalar self interaction λ at scales beyond the Planck mass is determined by a fixed point at zero. This results in $m_H = m_{\min} = 126$ GeV, with only a few GeV uncertainty. This prediction is independent of the details of ... wide class of extensions of the SM as well. For $A_\lambda < 0$ on ... $\simeq 174$ GeV, now sensitive to A_λ and other properties of ... favored by explicit computations existing in the literature.

$m_H = m_{\min} = 126$ GeV,

- Higgs mass prediction [M. Shaposknikov, C. Wetterich, Phys. Lett. B. 683 (2010)].

A (major) criticism: Results hinge strongly on use of Euclidean signature!

Wick rotation? Quantum state dependence?

[J. F. Donoghue, Front.in Phys. 8 (2020) 56]
[A. Bonanno et al., Front.in Phys (2020) 269]

Euclidean vs. Lorentzian FRG

	Lorentzian	Euclidean
ϵ_g	-1	1
$\sqrt{\epsilon_g}$	i	1

- The FRG technique studies the non-linear response of Legendre Effective Action Γ_k to scale-dependent mode modulation

$$S[\chi; g] \mapsto S[\chi; g] + \Delta S_k[\chi; g]\,, \quad \Delta S_k[\chi; g] = \frac{\epsilon_g}{2} \chi \cdot \mathcal{R}_k(g) \cdot \chi\,.$$

- On a foliated manifold $M = I \times \Sigma$, $ds^2 = \epsilon_g N^2 dt^2 + \mathrm{g}_{ij} dx^i dx^j$, the *Wetterich equation* is

$$k\partial_k \Gamma_k[\varphi] = -\frac{\epsilon_g \sqrt{\epsilon_g}\hbar}{2} \mathrm{Tr}\Big\{ k\partial_k \mathcal{R}_k \cdot G_k[\varphi] \Big\}\,, \quad \left[\frac{\delta^2 \Gamma_k}{\delta\varphi\delta\varphi} + \epsilon_g \mathcal{R}_k \right] \cdot G_k[\varphi] = \epsilon_g \mathbb{1}\,.$$

Euclidean covariant FRG

Ellipticity of the principal part of the Hessian $\mathcal{H}_k = \Gamma_k^{(2)} + \mathcal{R}_k$ ensures the existence of a unique inverse.

Heat kernel techniques for the generalized Hessian are central and account robustly for UV behavior for flows.

Lorentzian signature spatial FRG

Spatial modulation: $\mathcal{R}_k$ modulates only spatial modes, leaves temporal modes unaffected.

Hyperbolicity of $\mathcal{H}_k = \Gamma_k^{(2)} - \mathcal{R}_k$ implies inverse is non-unique. But needs to be of *Hadamard form*!

Universal UV properties follow from Hadamard expansion of $\mathcal{H}_k$'s Green's function (the Hadamard parametrix). The heat-kernel does not exist, but not needed!

The IR properties are state-dependent, and each specific Hadamard state has **its own IR completion**, dictated by whatever principle is used to construct it.

This talk: Self-interacting scalars on spatially flat Friedmann-Lemaître spacetimes.

- UV flow is state-independent.
- Evolution of the FRG flow to small k requires a choice of state. Once such a choice has been made, the flow in the IR can be studied.

Ultraviolet flow

Λ is UV RG scale

Goal: renormalize the bare action $S_\Lambda[\chi_\Lambda] = -\int d^4y\sqrt{-g}\Big\{\frac{1}{2}g^{\mu\nu}\partial_\mu\chi_\Lambda\partial_\nu\chi_\Lambda + \mathcal{U}(\chi, R)\Big\}$.

Compute divergent parts of one-loop Legendre effective action using one-loop FRG: insert $\Gamma_k[\varphi] = S[\varphi] + \hbar\Gamma_{k,1}[\varphi] + O(\hbar^2)$ into Wetterich equation for flow of one-loop correction

$$k\partial_k\Gamma_{k,1}[\varphi] = -\frac{i}{2}\mathrm{Tr}\big\{k\partial_k\mathcal{R}_k G_k[\varphi]\big\}\,, \qquad G_k[\varphi] = \big[S^{(2)}[\varphi] - \mathcal{R}_k\big]^{-1}\,.$$

k-independent

Key points:

- UV divergent part of the one-loop correction arises integrating the flow from $k = \mu$ to $k = \Lambda$ (with $\mu \leq \Lambda$ sufficiently large).
- Large k-regime accessible through closed recursion (generalized resolvent expansion)*. This *avoids* the ill-defined (pseudo-) heat kernel.
- The recursion coefficients are universal (reflecting the universality of the Hadamard property), hence the UV divergent parts are state-independent!

*[R. Banerjee, M. Niedermaier, J. Math. Phys. 61, 103511 (2020)]

Result: $\Gamma_1^{\mathrm{div}}[\varphi] = \frac{1}{(4\pi)^2}\int d\eta dx\, a(\eta)^4 \Big\{ q_0(\Lambda^4 - \mu^4) + g_1(\eta)(\Lambda^2 - \mu^2) + g_2(\eta)\ln(\Lambda/\mu)\Big\}\,.$

$\mathcal{R}_k$ is non-covariant: modulates only *spatial* modes!

$$g_1 = \breve{q}R - q_1(\mathcal{U}'' - R/6)\,,$$

$$g_2 = \frac{1}{2}(\mathcal{U}'' - R/6)^2 - \frac{1}{6}\nabla^2(\mathcal{U}'' - R/6) + B_1\frac{a^{(4)}}{a^5} + B_2\frac{a^{(1)}a^{(3)}}{a^6} + B_3\frac{a^{(2)\,2}}{a^6} + B_4\frac{a^{(1)\,2}a^{(2)}}{a^7} + B_5\frac{a^{(1)\,4}}{a^8}$$
$$+\Big(B_6\frac{R}{6} + B_7\frac{a^{(1)\,2}}{a^4}\Big)(\mathcal{U}'' - R/6) + B_8\frac{a^{(1)}}{a^3}\partial_\eta(\mathcal{U}'' - R/6)$$
$$= \frac{1}{2}(\mathcal{U}'' - R/6)^2 - \frac{1}{6}\nabla^2(\mathcal{U}'' - R/6) + b_1\nabla^2 R + b_2\Big[-R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\Big]$$
$$+ b_3 a^{-4}\partial_\eta\big[aa^{(1)}(\mathcal{U}'' - R/6)\big]\,.$$

Re-covariantizes!

$$\hbar\Gamma_1^{\rm div}[\phi] = \frac{\hbar}{(4\pi)^2}\int d\eta dx\, a^4\Big\{q_0(\Lambda^4-\mu^4) + [\check{q}R - q_1(\mathcal{U}''-R/6)](\Lambda^2-\mu^2) + \frac{1}{2}(\mathcal{U}''-R/6)^2\ln(\Lambda/\mu)\Big\}\,.$$

Summary of ultraviolet properties:

- Although spatial regulator breaks covariance, $\Gamma_1^{\rm div}$ is manifestly covariant (computationally non-trivial property, valid for any smooth $\mathcal{R}_k$ modulator).
- The UV divergent parts, and hence the UV renormalization group flow of the couplings is state-independent.
- Covariant pseudo heat kernel result *almost* identical

$$\hbar\Gamma_1^{\rm div}[\phi] = \frac{\hbar}{(4\pi)^2}\int d\eta dx\, a^4\Big\{q_0^{\rm cov}(\Lambda^4-\mu^4) + q_1^{\rm cov}(\Lambda^2-\mu^2)A_2 + A_4\ln(\Lambda/\mu)\Big\}\,,$$

$$A_2 \simeq -\mathcal{U}'' + R/6\,,\quad A_4 \simeq \frac{1}{2}(\mathcal{U}''-R/6)^2\,.$$

The $\check{q}R$ term is absent. The latter enters through the $a(\eta)$ dependence of modulator $\mathcal{R}_k$. Gives rise to additional contribution to running of dimensionless Newton coupling.

Implications for quantum gravity?

Infrared aspects of RG flow

- The small k regime of the flow equation is of interest: Phenomenological implications of asymptotic safety, physics at large (super-Hubble) scales.
- In the Euclidean ($\epsilon_g = 1$) setting, the small k form of the flow equation

$$k\partial_k \Gamma_k[\varphi] = \frac{\epsilon_g \sqrt{\epsilon_g}\hbar}{2} \mathrm{Tr}\left\{ k\partial_k \mathcal{R}_k \, G_k[\varphi] \right\} ,$$

is accessible for (a) maximally symmetric backgrounds; or (b) via (non-local) heat kernel.

- Conceptually, in Lorentzian setting expect state-dependent effects to appear at super-Hubble scales when $k \ll$ background curvature.
- Technically – how to extract φ dependence of RHS for small k?

Are there effects distinct from (non-local) heat kernel?

Results for *generic(!) FL spacetimes?* State dependence?

Infrared aspects of RG flow

- The small k regime of the flow equation is of interest: Phenomenological implications of asymptotic safety, physics at large (super-Hubble) scales.
- In the Euclidean ($\epsilon_g = 1$) setting, the small k form of the flow equation

$$k\partial_k \Gamma_k[\varphi] = \frac{\epsilon_g \sqrt{\epsilon_g}\hbar}{2} \mathrm{Tr}\{k\partial_k \mathcal{R}_k\, G_k[\varphi]\}\,,$$

is accessible for (a) maximally symmetric backgrounds; or (b) via (non-local) heat kernel.

- Conceptually, in Lorentzian setting expect state-dependent effects to appear at super-Hubble scales when $k \ll$ background curvature.
- Technically – how to extract φ dependence of RHS for small k?

Are there effects distinct from (non-local) heat kernel?

Results for *generic(!) FL spacetimes*? State dependence?

Feasible (only?) for State of Low Energy (SLE) induced FRG (convergent small k expansion).

[H. Olbermann, Class.Quant.Grav. 24 (2007) 5011-5030] Complicated! [R. Banerjee, M. Niedermaier, J. Math. Phys. 61, 103511 (2020)]

SLE induced flow in the deep infrared: beyond the heat kernel

$$\mathcal{V}_k(\varphi_0, t) = \sum_{\ell \geq 0} {}_\ell\mathcal{V}_k(\varphi_0)\mathfrak{p}_\ell(t)$$

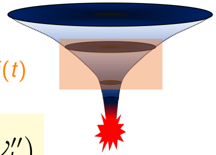
$$k\partial_k \, {}_\ell\mathcal{V}_k(\varphi_0) + (d+1)\, {}_\ell\mathcal{V}_k(\varphi_0) - \frac{d-1}{2}\varphi_0 \frac{\partial}{\partial \varphi_0}\, {}_\ell\mathcal{V}_k(\varphi_0) = {}_\ell Q_0\left(1 \,|\, {}_0\mathcal{V}_k''\right)$$

Autonomous at lowest order

Well-defined infrared fixed-point equation

Analogous structure to flow equation in Minkowski spacetime.

SLE induced flow in the deep infrared: beyond the heat kernel



$$k\partial_k\, {}_\ell\mathcal{V}_k(\varphi_0) + (d+1)\,{}_\ell\mathcal{V}_k(\varphi_0) - \frac{d-1}{2}\varphi_0\frac{\partial}{\partial\varphi_0}\,{}_\ell\mathcal{V}_k(\varphi_0) = {}_\ell Q_0\left(1\,|\,{}_0\mathcal{V}_k''\right)$$

$$-\frac{k^2}{2J_0^2}\,{}_\ell Q_1\left(1\,|\,{}_\ell\mathcal{V}_k''\right)\sum_{\ell_1,\ell_2\ge 0} {}_{\ell_1}\mathcal{V}_k''(\varphi_0)\,{}_{\ell_2}\mathcal{V}_k''(\varphi_0)\, E_{\ell_1,\ell_2}$$

$$+\frac{k^2}{J_0}\sum_{\ell_1\ge 0}{}_{\ell_1}\mathcal{V}_k''(\varphi_0)\left({}_\ell Q_0\left(\jmath_{\ell_1}\,|\,{}_0\mathcal{V}_k''\right) - \frac{1}{2J_0}\,{}_\ell Q_1\left(E_{\ell_1}\,|\,{}_0\mathcal{V}_k''\right)\right)$$

$$+\frac{k^2}{J_0}\left({}_\ell Q_0\left(\jmath\,|\,{}_0\mathcal{V}_k''\right) - \frac{1}{2J_0}\,{}_\ell Q_1\left(E\,|\,{}_0\mathcal{V}_k''\right)\right) + O(k^4)\,.$$

Non-autonomous higher order corrections

Not visible in non-local heat kernel

Autonomous at lowest order

Well-defined infrared fixed-point equation

- Non-autonomous character from temporal averaging in SLE.
- $J_0,\ E_{\ell_1,\ell_2},\ldots$ depend explicitly on SLE window function f – absent in non-local heat kernel!
- Applicable to all FL spacetimes. Specializes to Minkowski correctly.

Outlook

The longterm goal of this project is to study asymptotically safe quantum gravity manifestly in Lorentzian signature.

First step: FRG for scalars on generic cosmological backgrounds.

Next steps:

- Non-perturbative flow of inflationary power-spectrum.
- Infrared FP structure under investigation (specific to SLE?).
- Quantum gravity extension?

Thank you for your attention!

Back-up slides

Hadamard Expansion

$$G_\epsilon^{\text{Had}}(y,y') = H_\epsilon(y,y') + W(y,y')\,,$$

$$H_\epsilon(y,y') = \begin{cases} \dfrac{U(y,y')}{\sigma_\epsilon(y,y')^{\frac{d-1}{2}}} + V(y,y')\ln \mu^2\sigma_\epsilon(y,y')\,, & d \ \text{odd} \\ \dfrac{U(y,y')}{\sigma_\epsilon(y,y')^{\frac{d-1}{2}}}\,, & d \ \text{even} \end{cases}\,.$$

Relation to heat kernel coefficients for $d \geq 3$ odd

$$U_n(y,y') = \frac{((d-3)/2-n)!}{2^n((d-3)/2)!}A_n(y,y')\,, \quad n = 0,1,\ldots(d-3)/2\,,$$

$$V_n(y,y') = \frac{(-)^{n+1}}{2^{n+(d-1)/2}n!((d-3)/2)!}A_{n+(d-1)/2}(y,y')\,, \quad n \in \mathbb{N}\,.$$