Adventures in Perturbation Theory

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University of Pittsburgh

The Niels Bohr International Academy

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Roadmap

✦*Spiritus Movens*

- the *shocking simplicity* of QFT
- what do amplitudes *look like*—**functionally**?
- why is perturbation theory **so hard**?

—and how can we make it *easier*?

✦Improving Integration with Better Integrand Bases

- what makes for a *good* basis of Feynman integrals?
- *stratifying* theories and stratified integrand bases
- generalized vs *prescriptive unitarity*

Explosions of Complexity

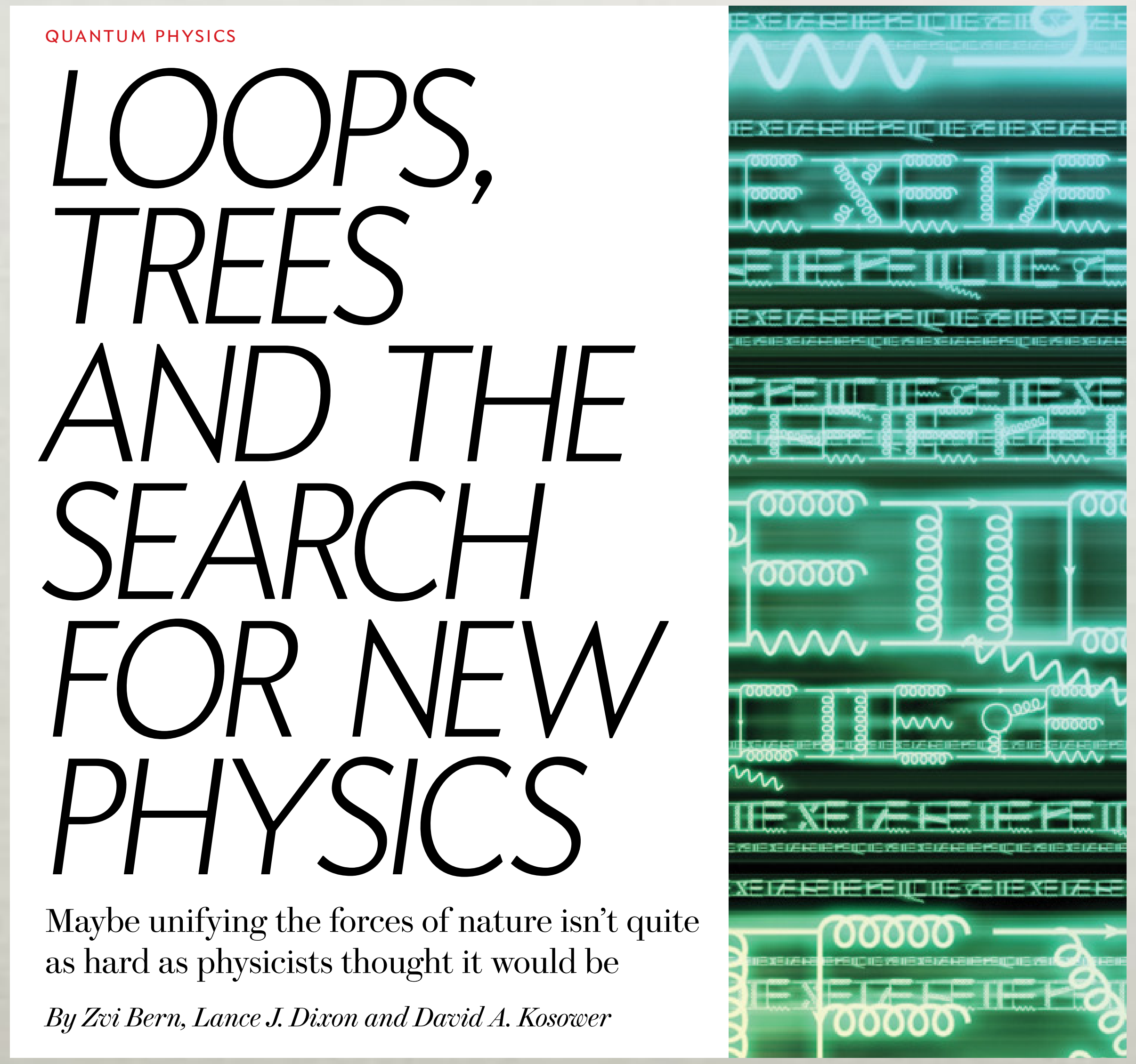
- While ultimately correct, the Feynman expansion renders *all but the* ***most trivial*** predictions—

  involving the fewest particles, at the

  lowest orders of perturbation—

  computationally ***intractable***

  or theoretically ***inscrutable***

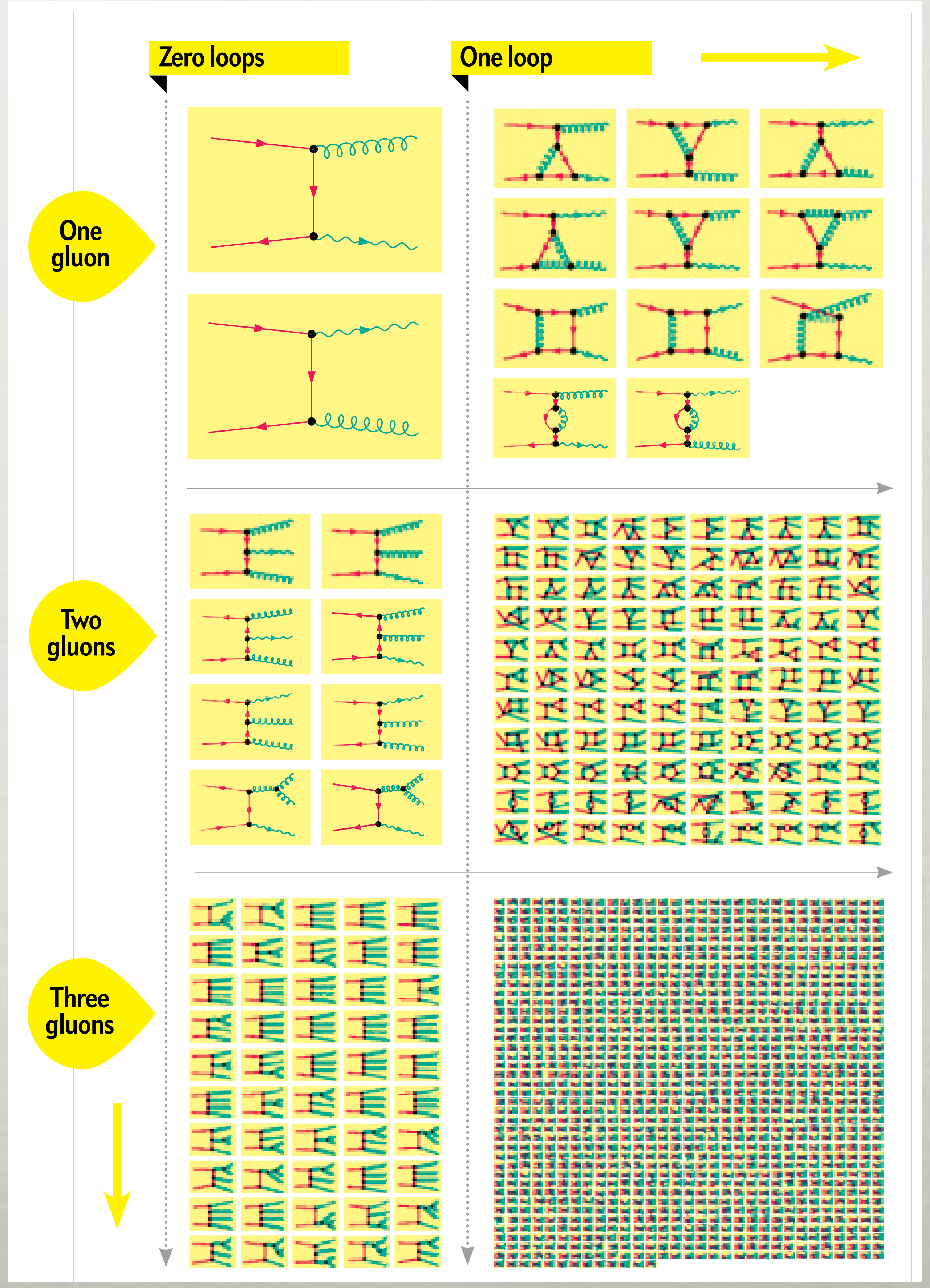

QUANTUM PHYSICS

LOOPS, TREES AND THE SEARCH FOR NEW PHYSICS

Maybe unifying the forces of nature isn't quite as hard as physicists thought it would be

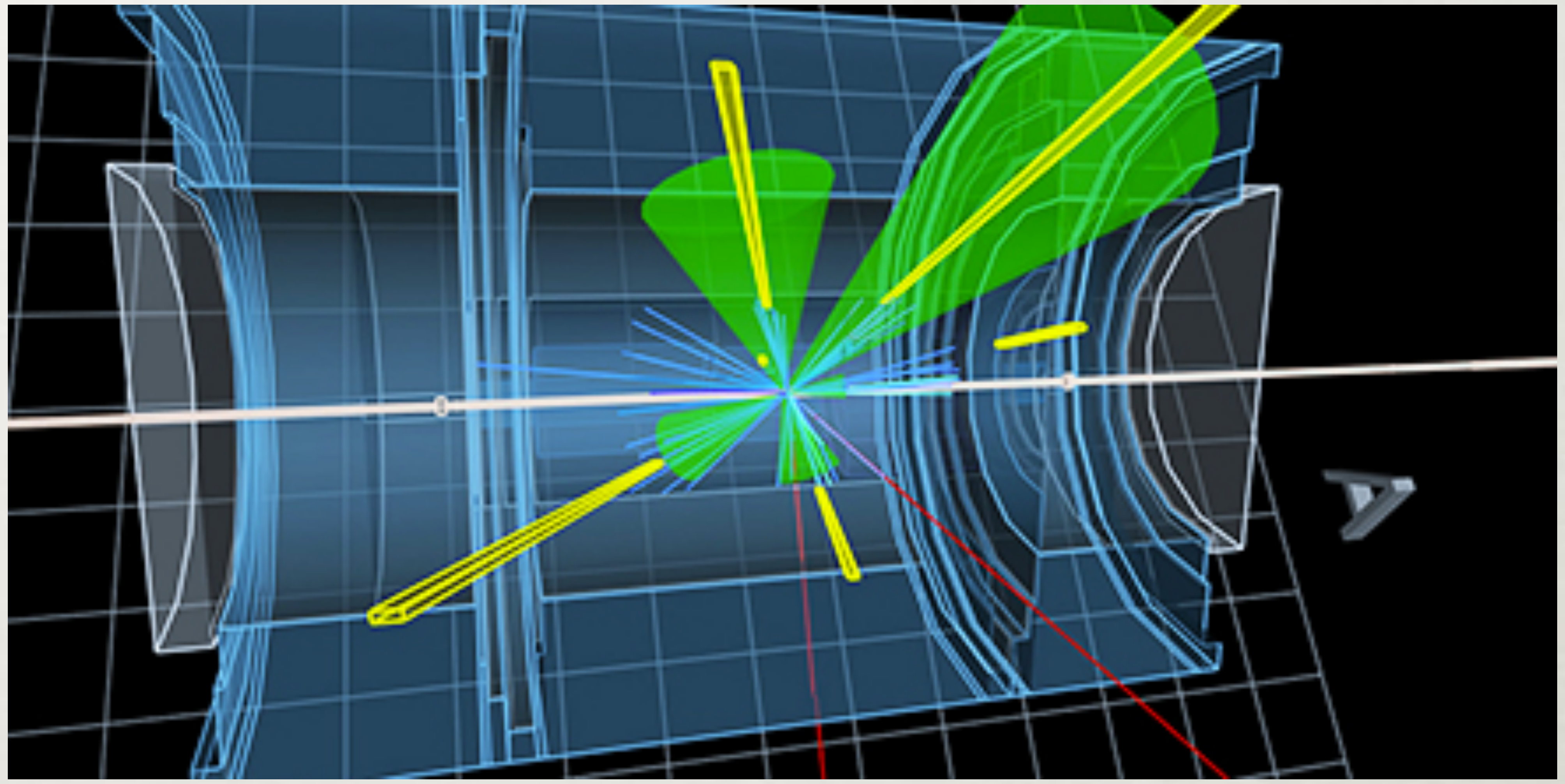
By Zvi Bern, Lance J. Dixon and David A. Kosower

[Bern, Dixon, Kosower, *Scientific American* (2012)]



Needs (Once) Beyond Our Reach

- Background amplitudes **crucial** for *e.g.* colliders



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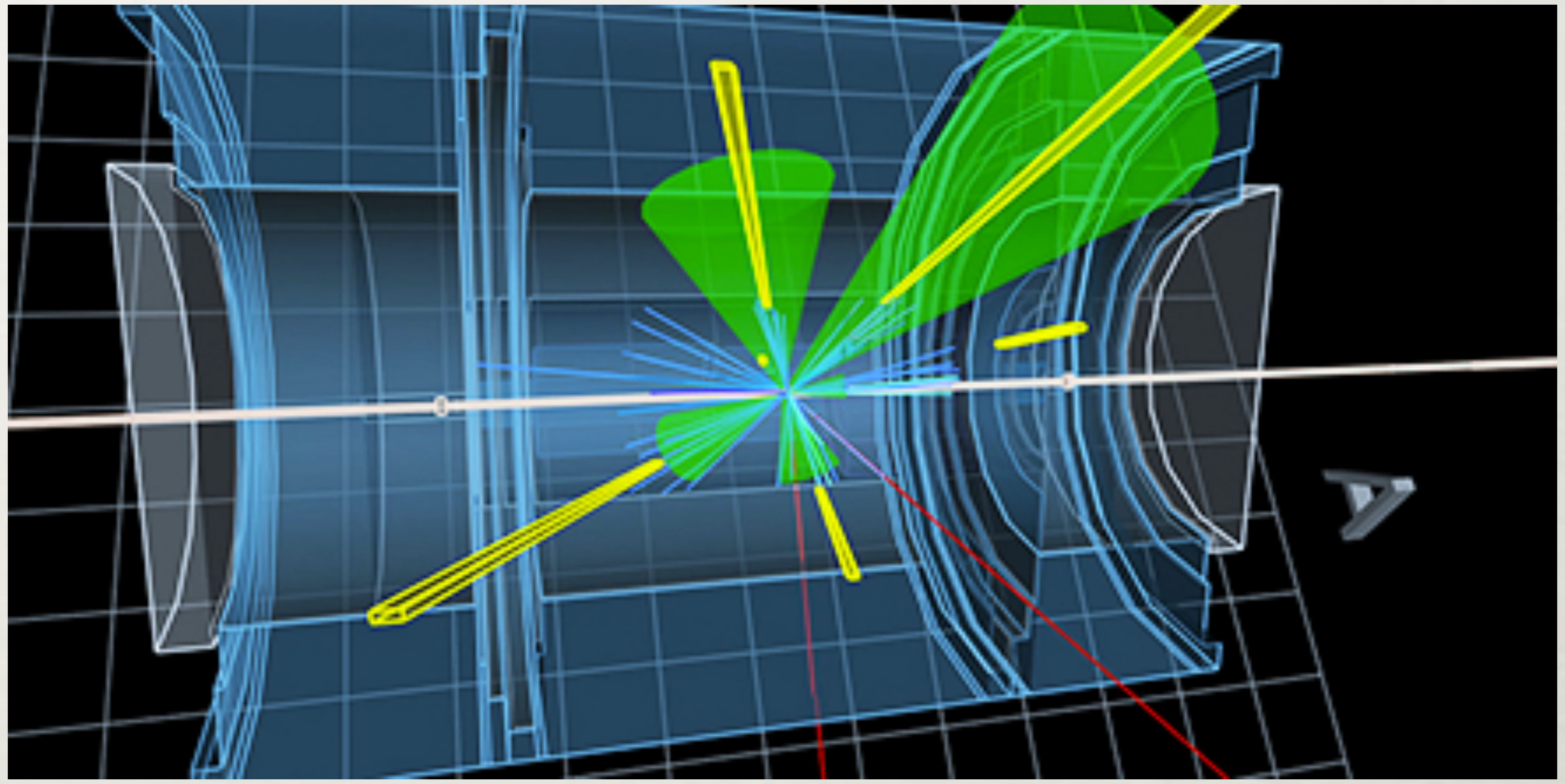

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Supercollider physics [*Rev.Mod.Phys*. **56** (1984)]

E. Eichten

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe

Lawrence Berkeley Laboratory, Berkeley, California 94720

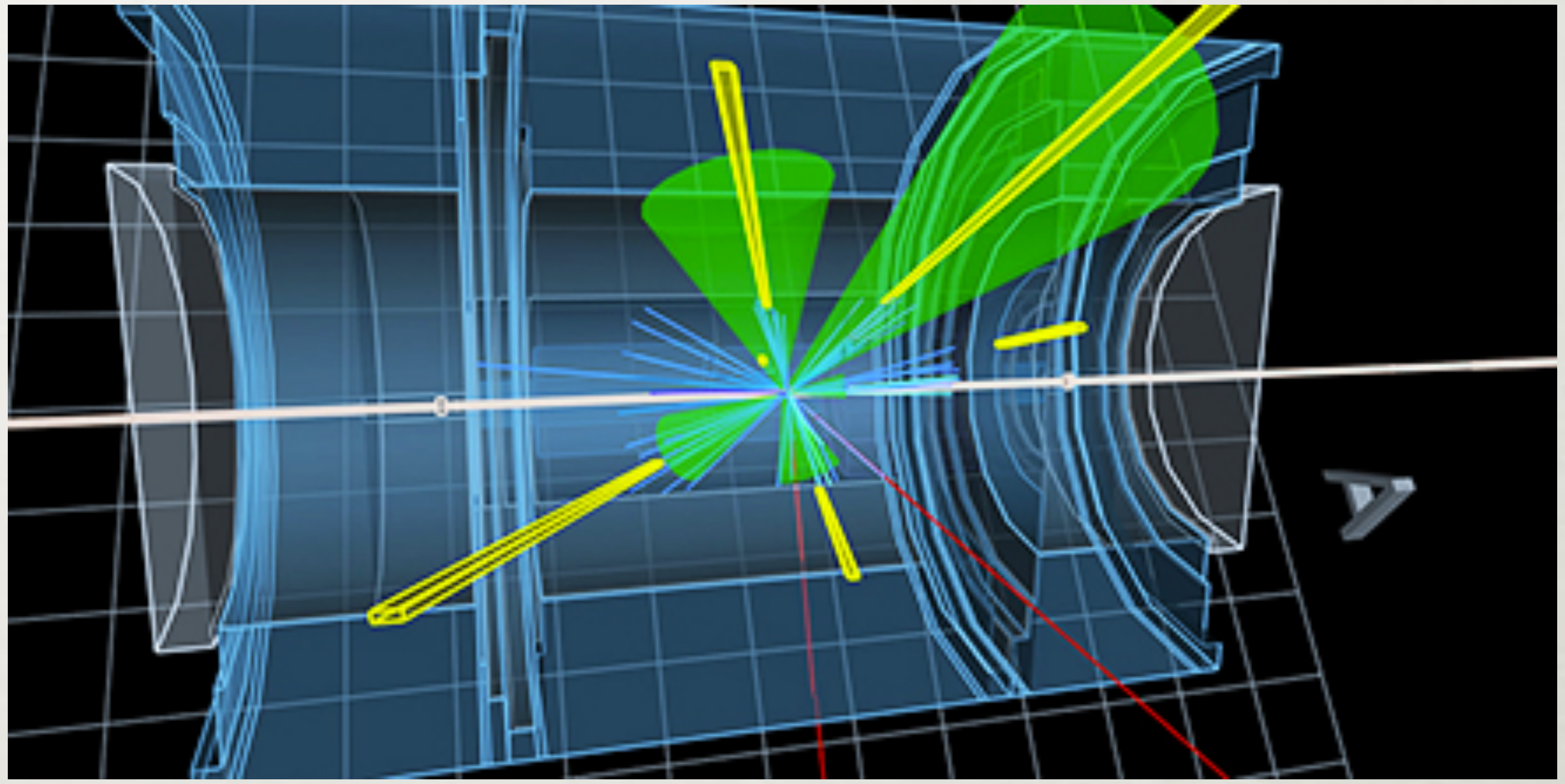
K. Lane

The Ohio State University, Columbus, Ohio 43210

C. Quigg

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.



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For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

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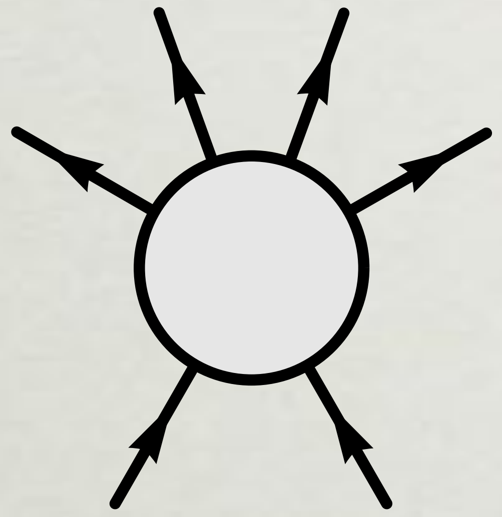

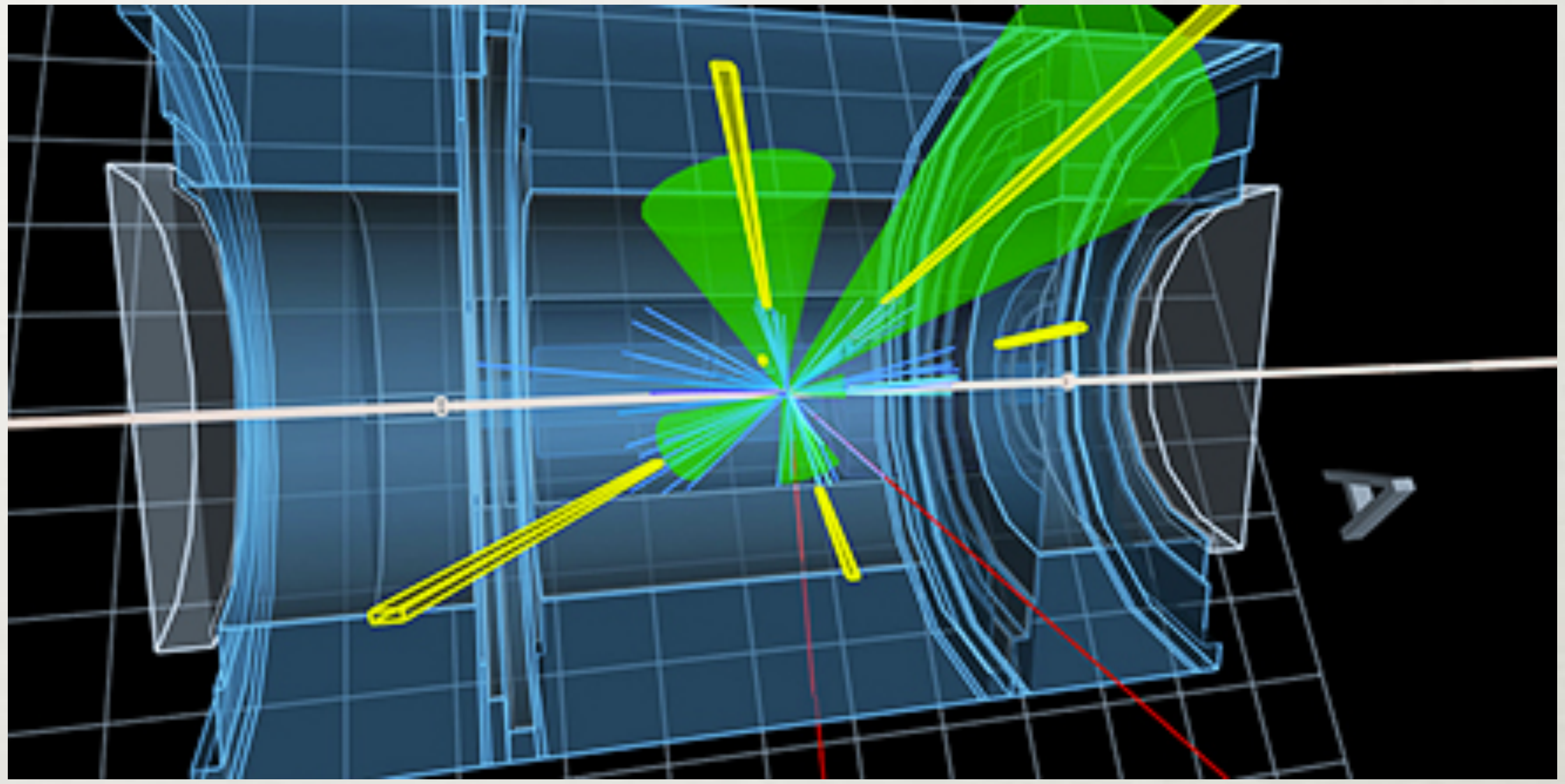

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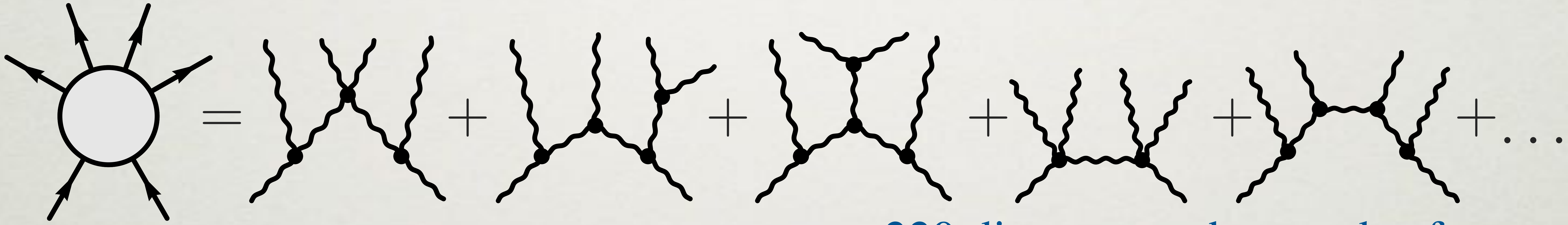

220 diagrams — thousands of terms

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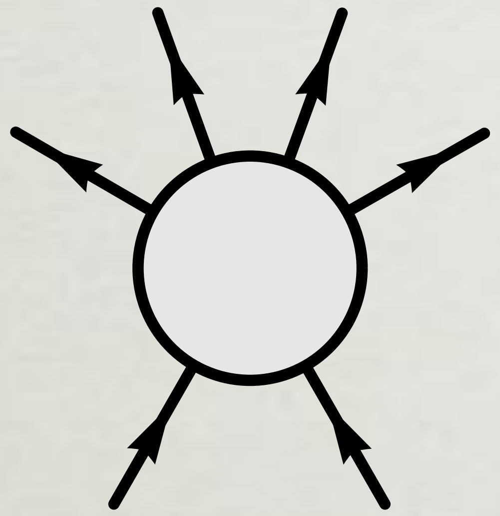

THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON–GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

[*Nucl.Phys.* **B269** (1985)]

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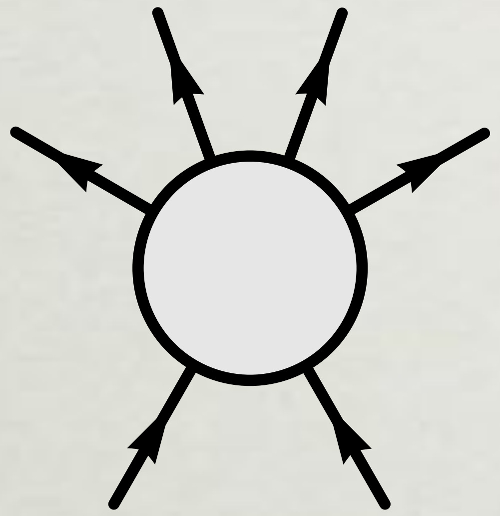
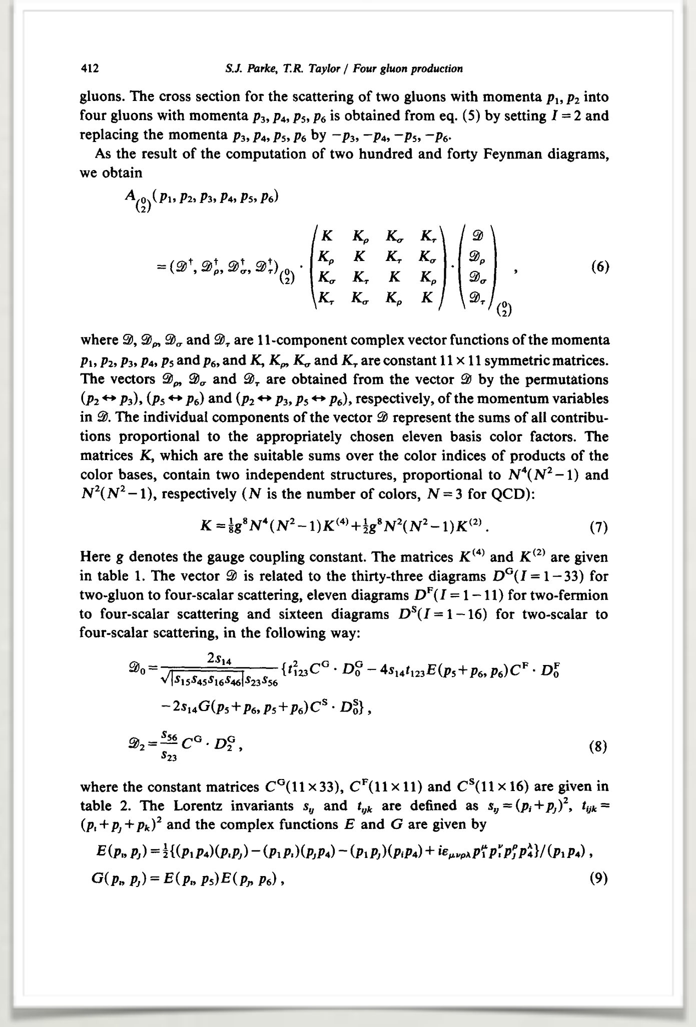
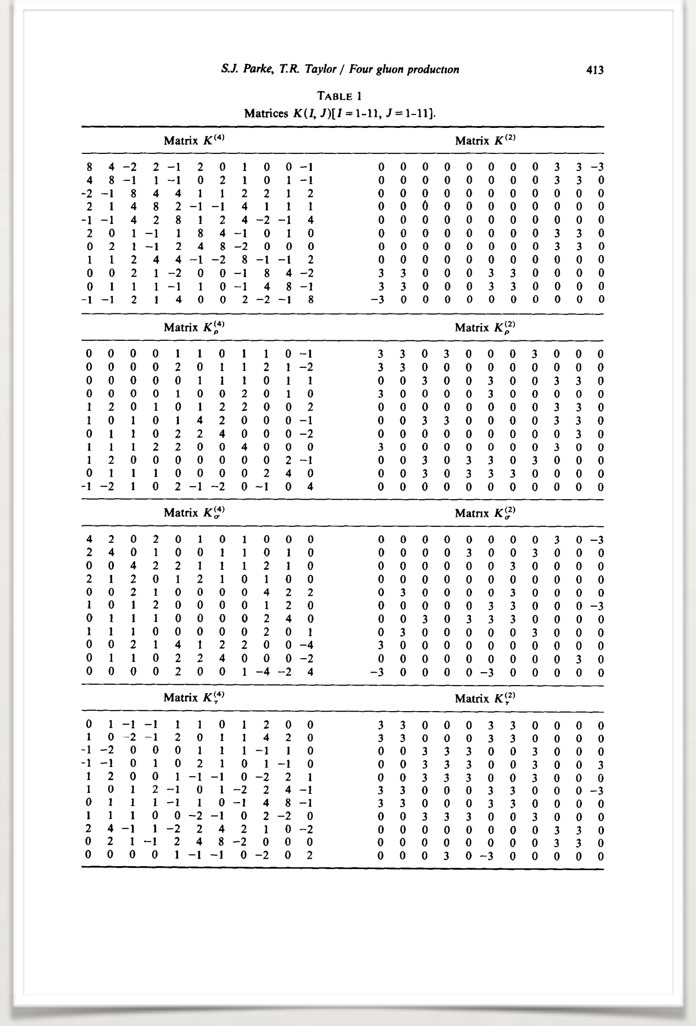



412 S.J. Parke, T.R. Taylor / Four gluon production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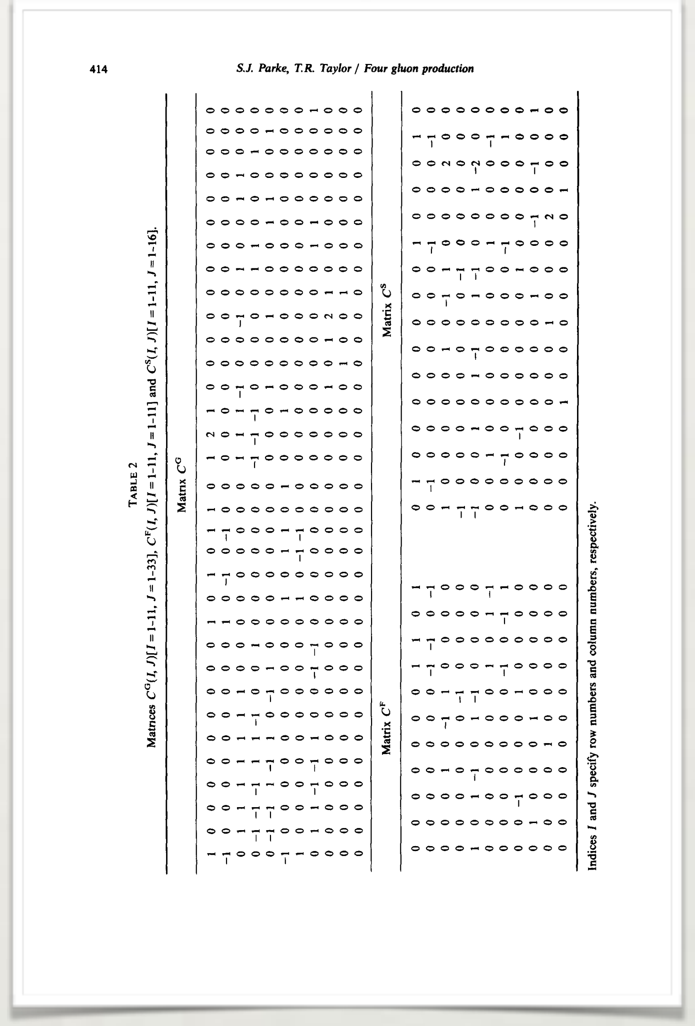
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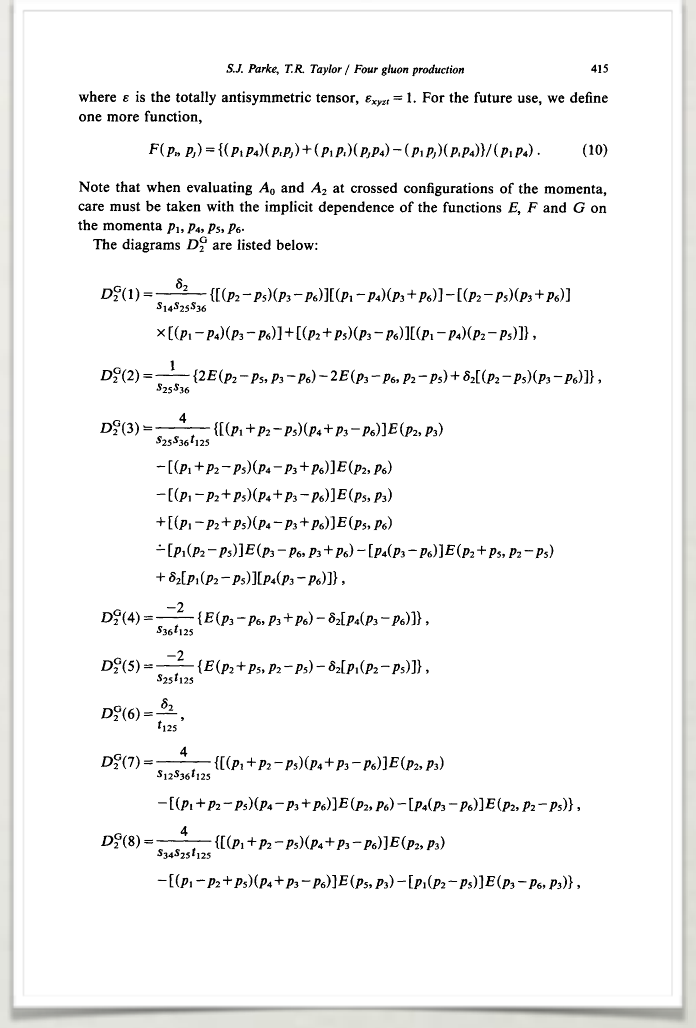
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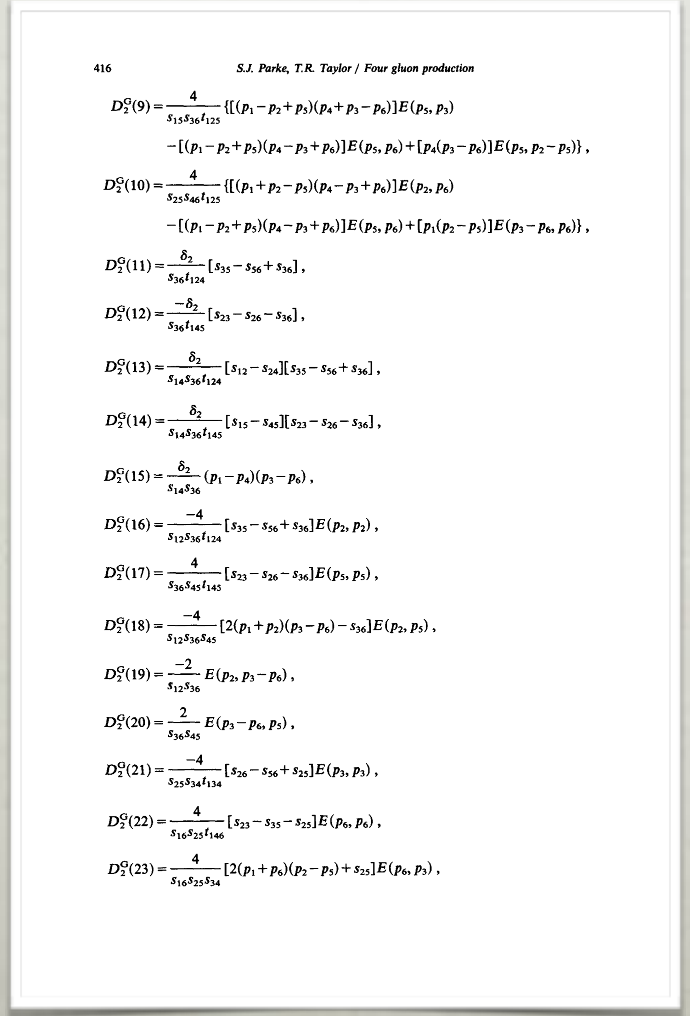
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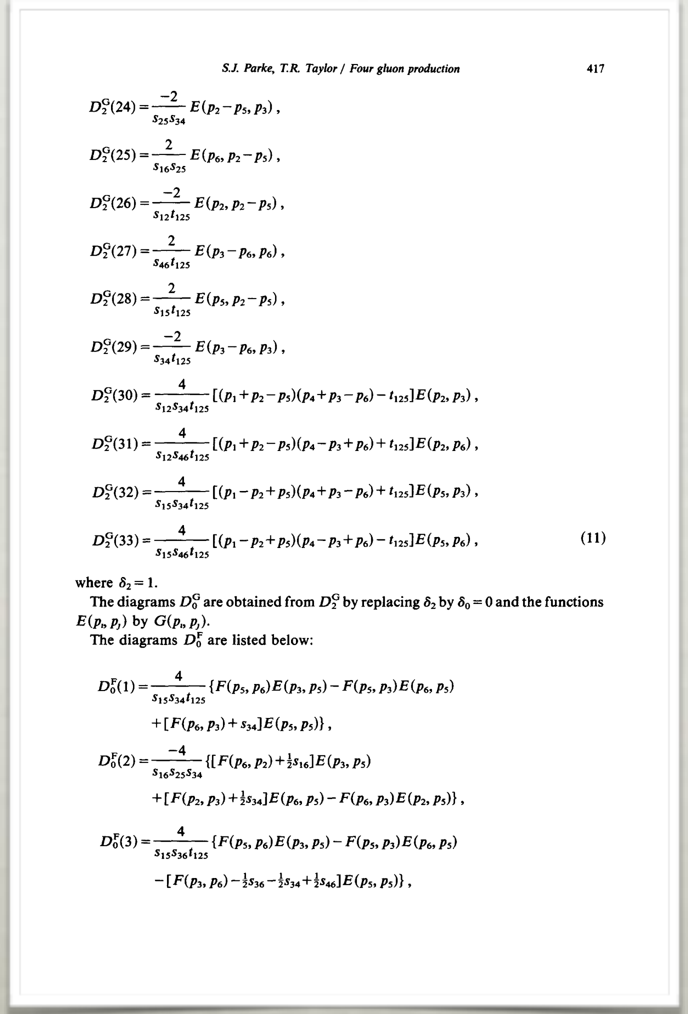
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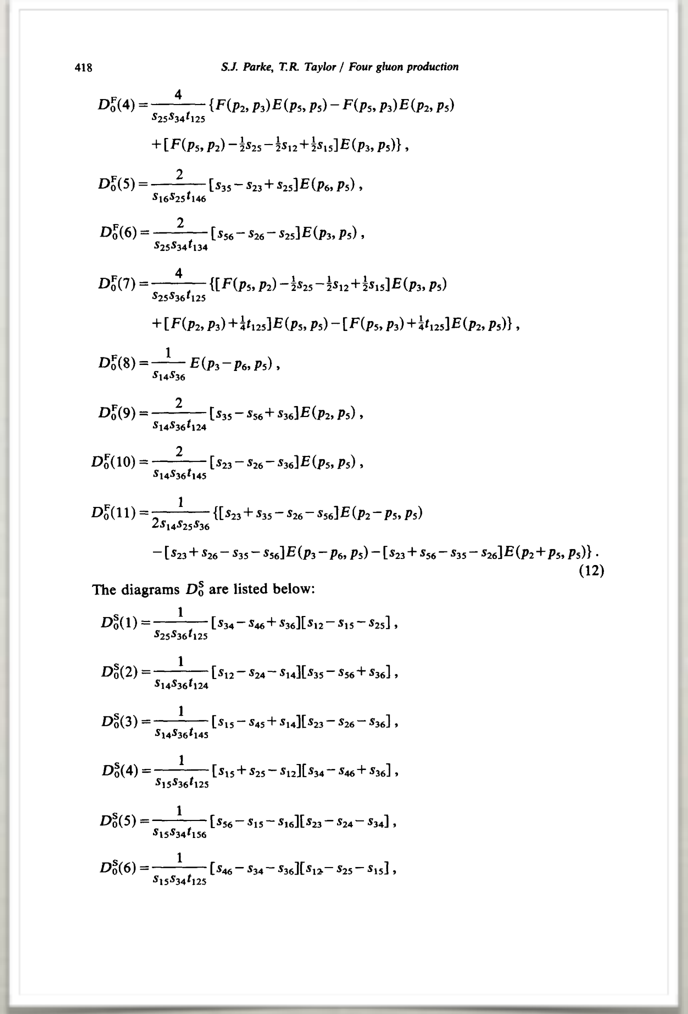
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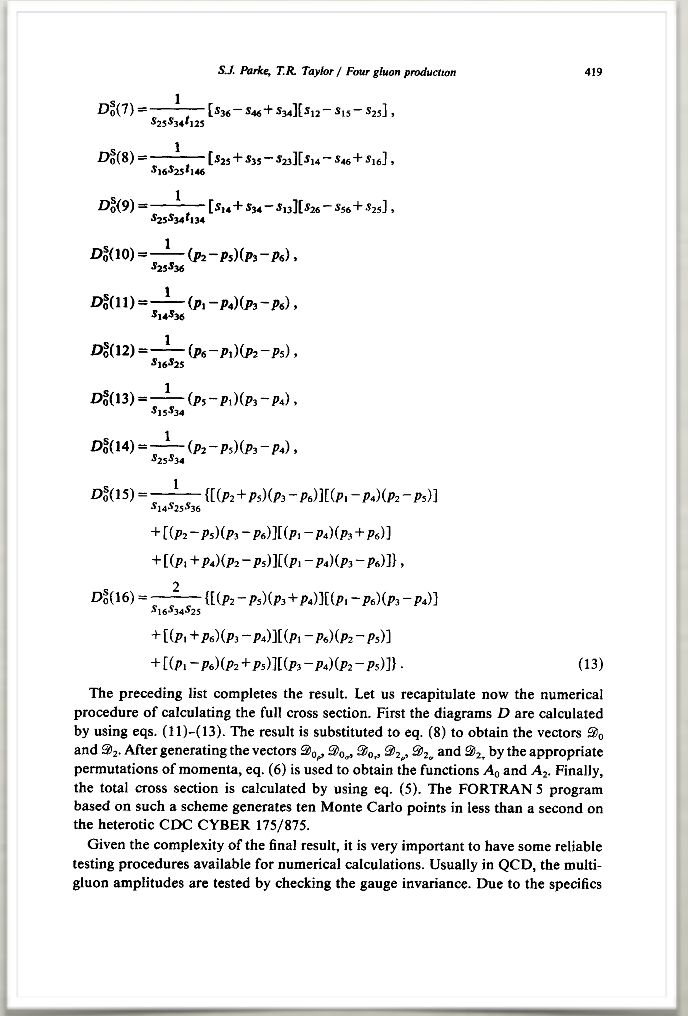
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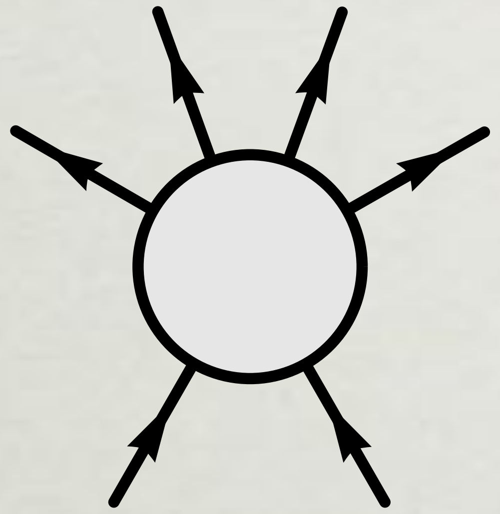

[3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge the hospitality of Aspen Center for Physics, where this work was being completed

Discovery of Shocking Simplicity

- Within six months, Parke-Taylor stumbled on a simple guess —unquestionably a *theorist's delight*:

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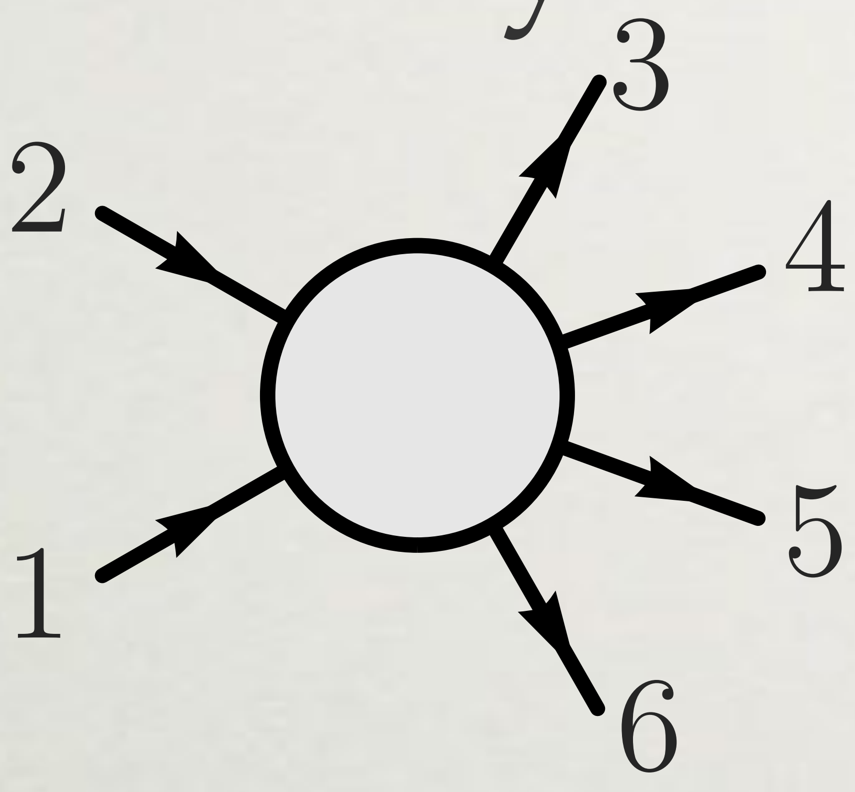

$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle}$$

$$p_a^\mu \equiv \sigma^\mu_{\alpha\dot{\alpha}}\lambda_a^\alpha\widetilde{\lambda}_a^{\dot{\alpha}}$$

$$\langle a\,b\rangle \equiv \det(\lambda_a, \lambda_b)$$

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[van der Waerden (1929)]

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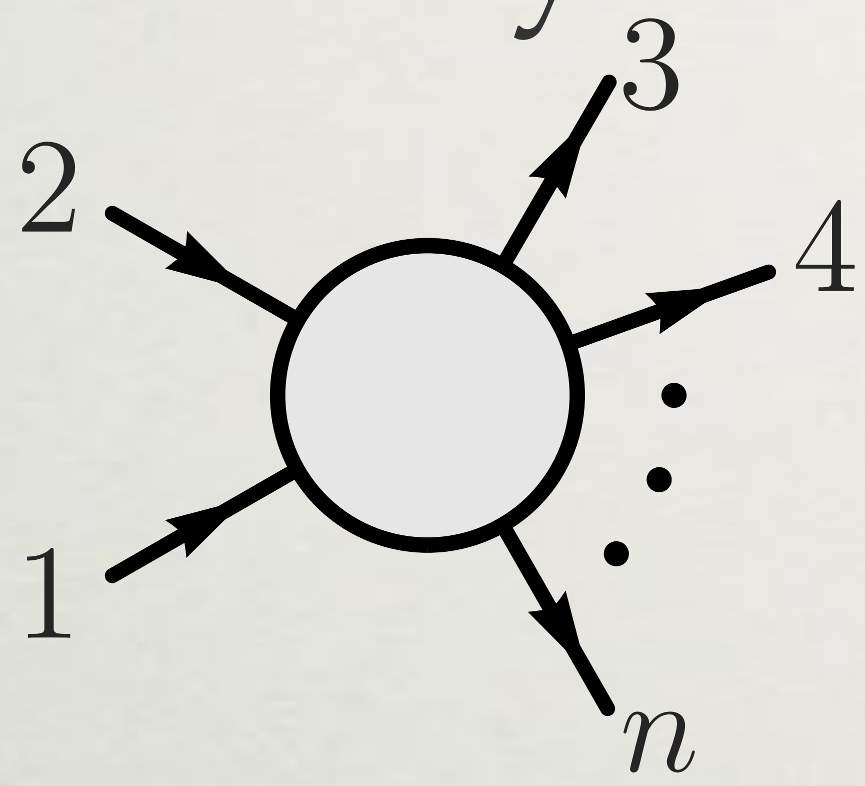

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Amplitude for *n*-Gluon Scattering [*PRL* **56** (1986)]

Stephen J. Parke and T. R. Taylor
Fermi National Accelerator Laboratory, Batavia, Illinois 60510

(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

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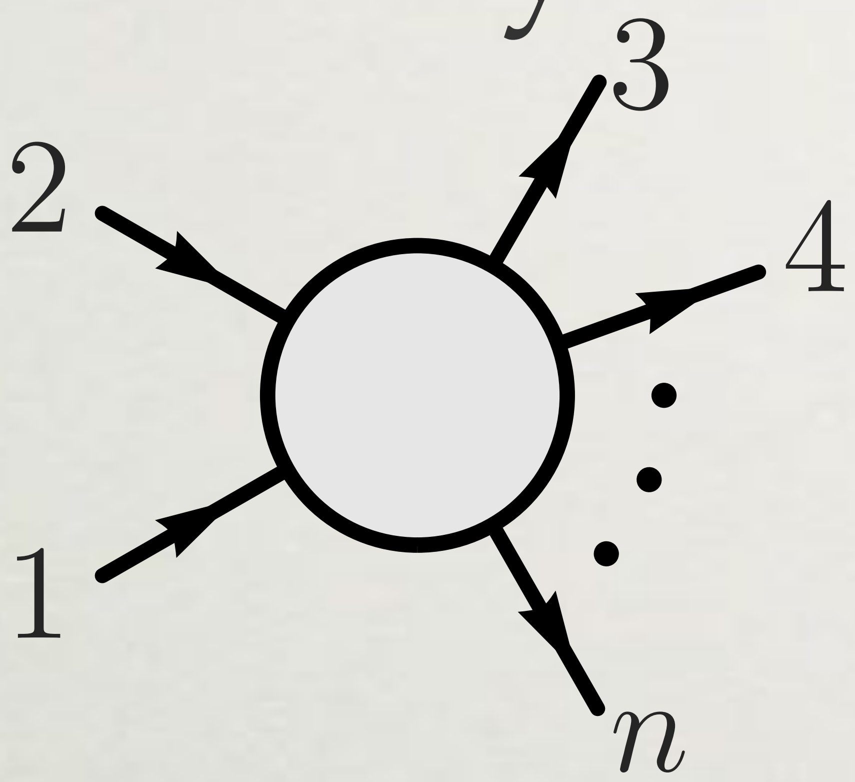

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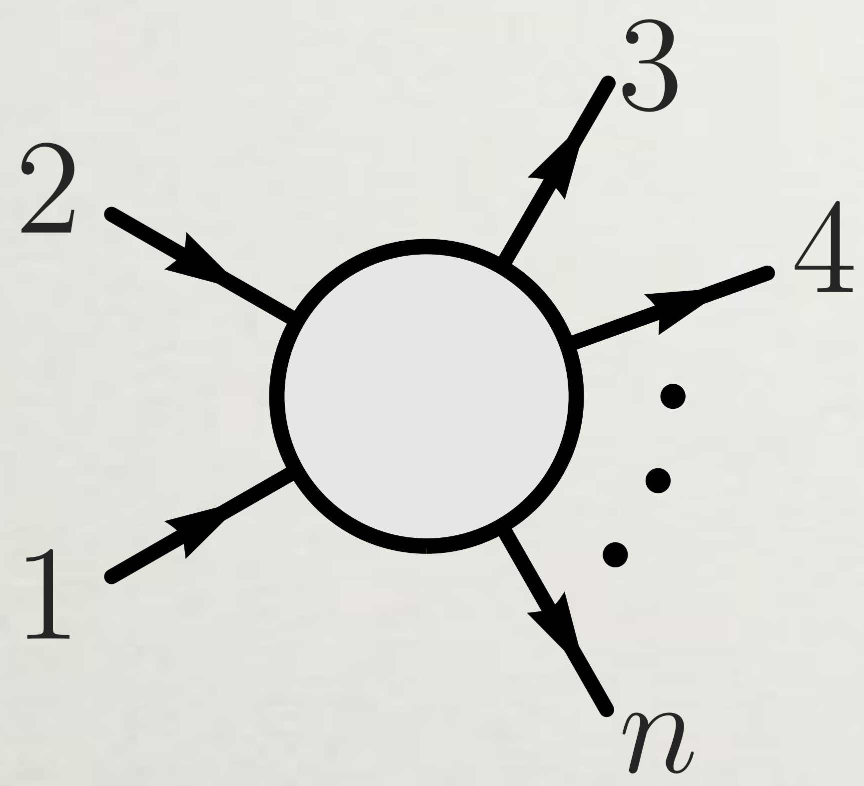
Goal: make the simplicity of amplitudes **manifest** in the way we compute them, *dramatically* extending the reach of the predictions we can make for experiment

Perturbations of Parke/Taylor's Guess

- What about beyond the leading order of approximation?



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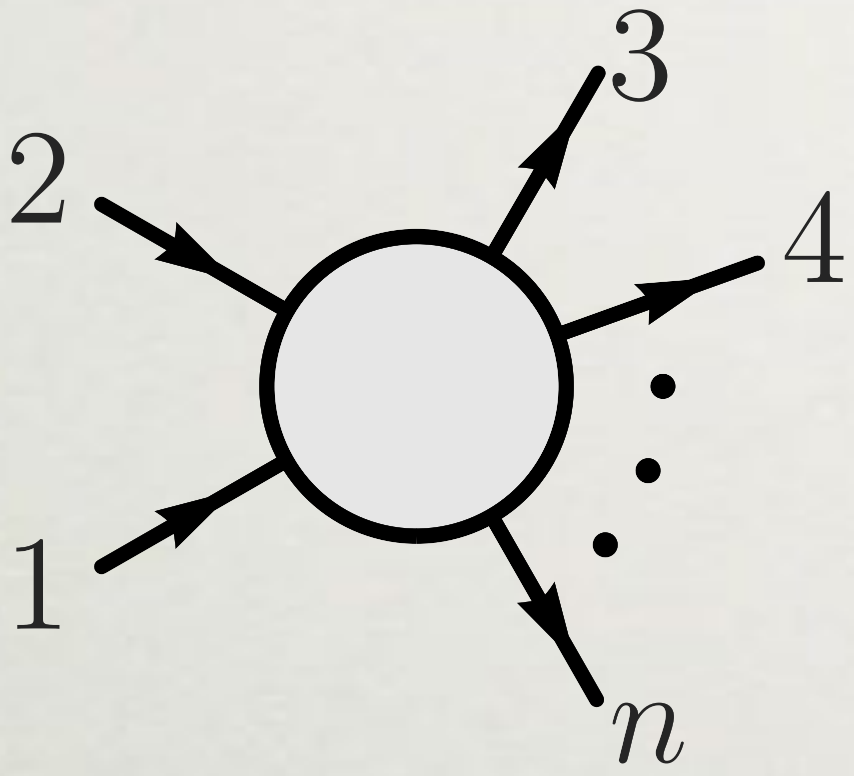

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$$\Big\{ 1 + \ldots \qquad\qquad\qquad\qquad \Big\}$$

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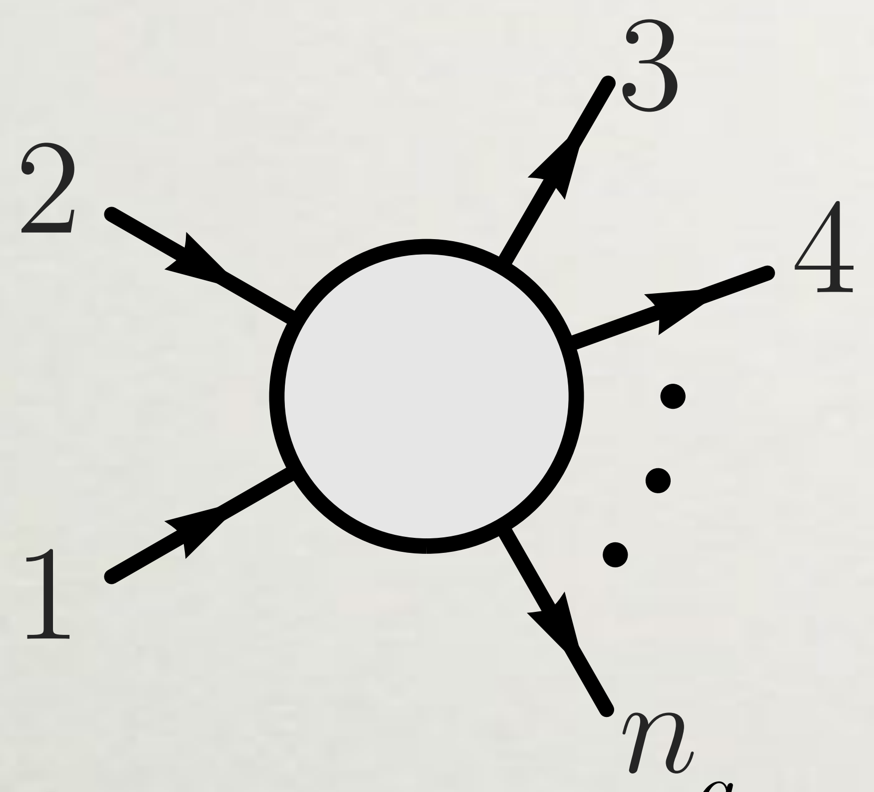
[Bern, Dixon, Dunbar, Kosower (1994)]



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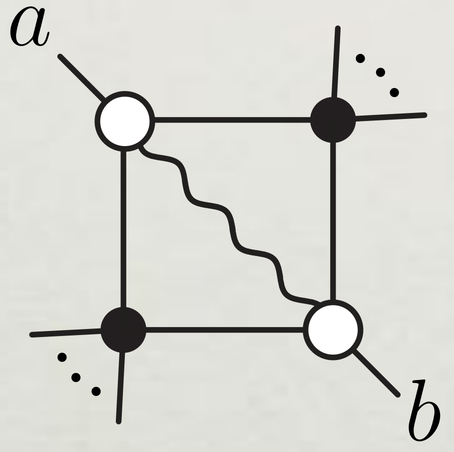
$$\left\{ 1 + \sum_{a<b} \right.$$  $$\left. + \ldots \right\}$$

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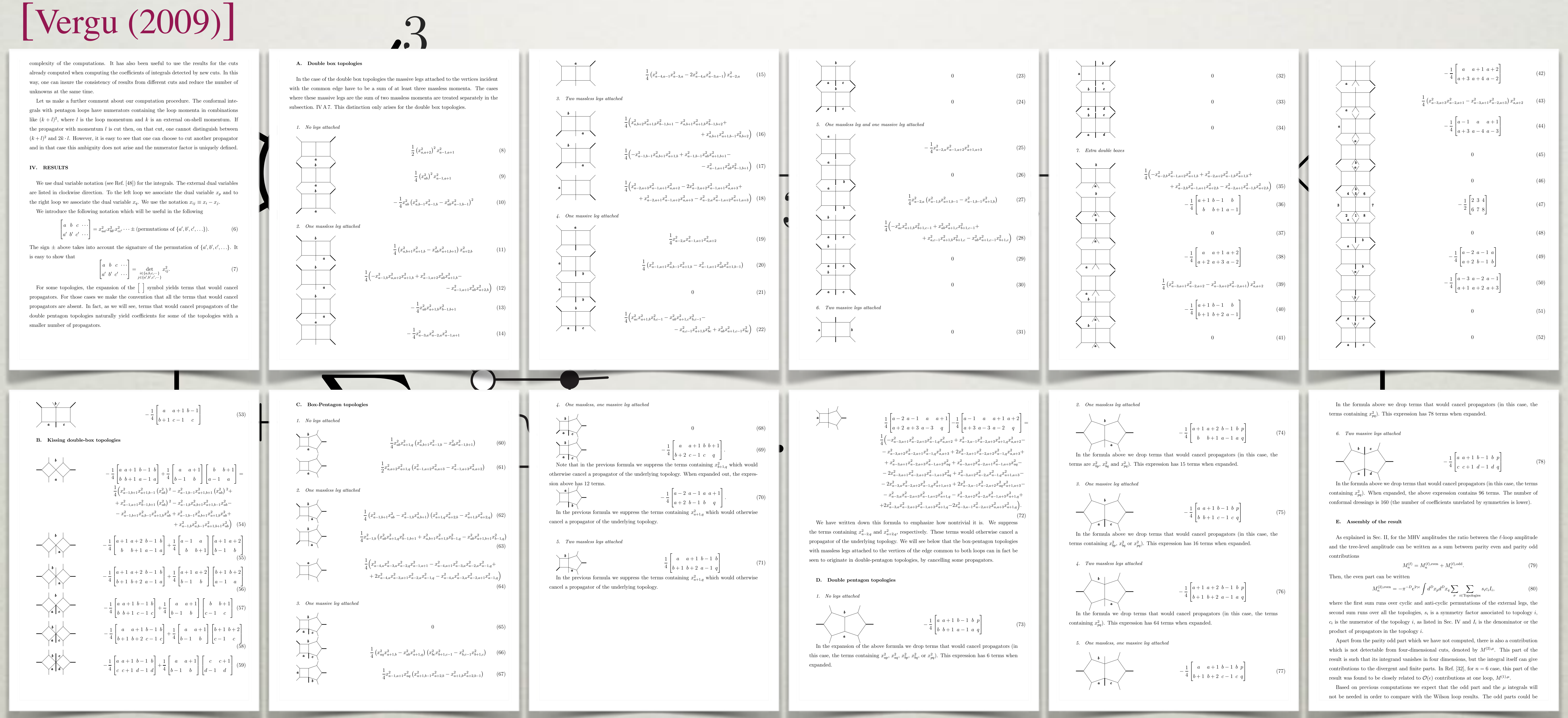
[Vergu (2009)]



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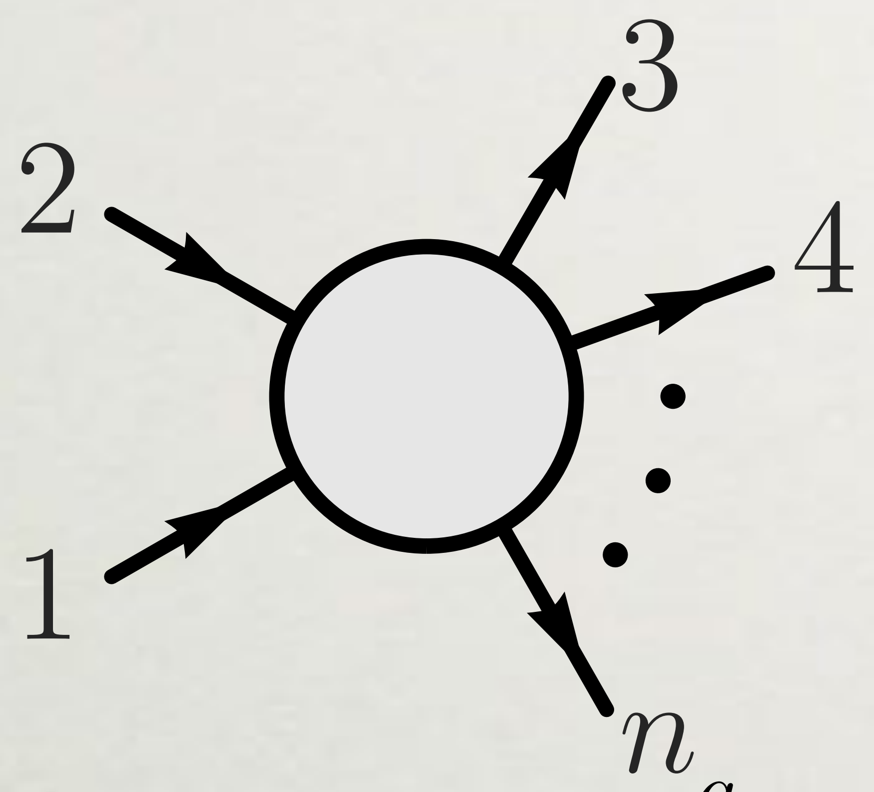
[Arkani-Hamed, **JB**, Cachazo, Trnka (2010)]



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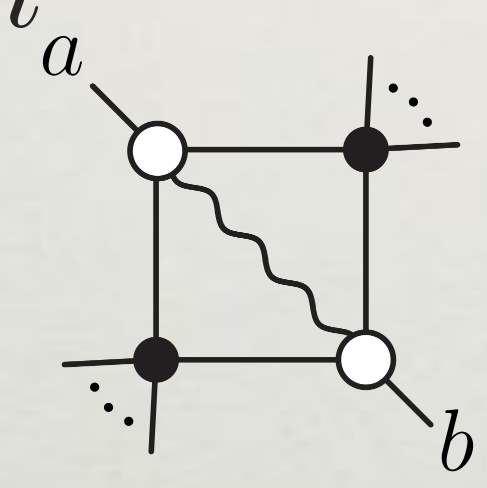
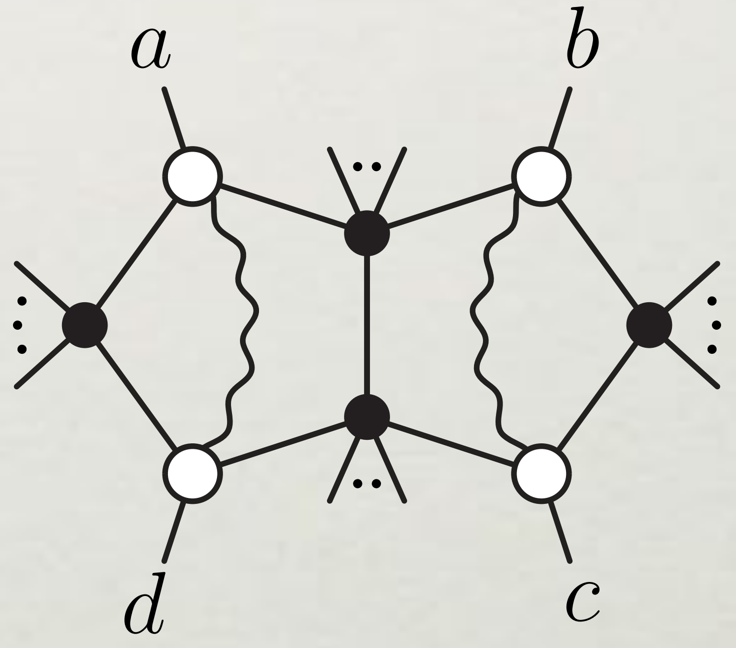
$$\Big\{1 + \sum_{a<b}$$ 

$$+ \sum_{a<b<c<d}$$ 

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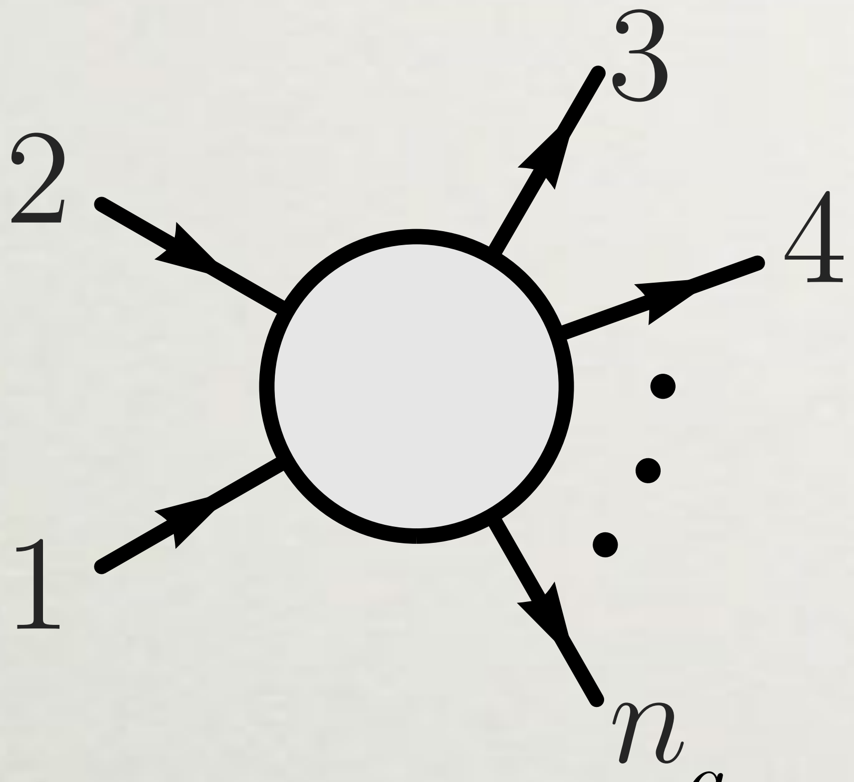

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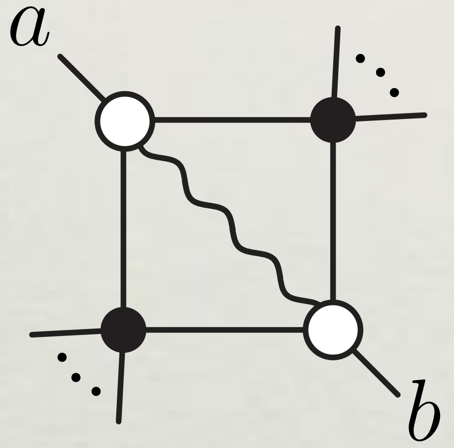

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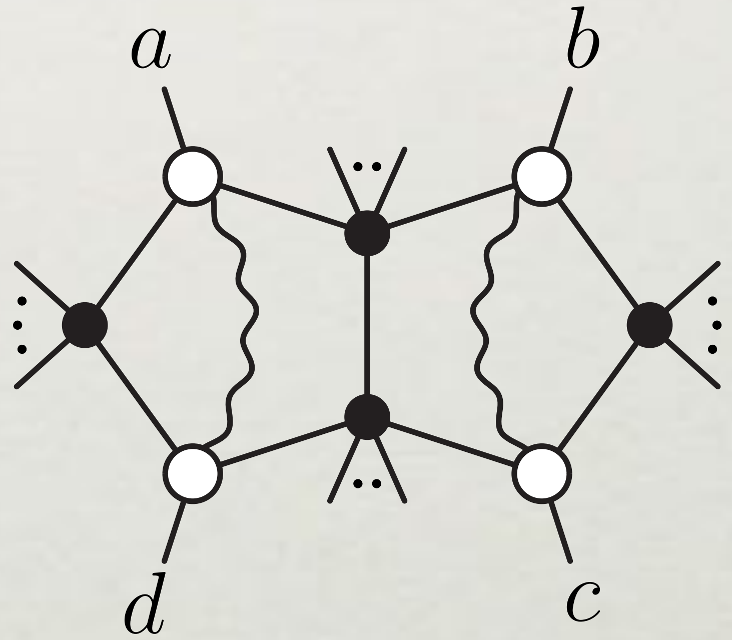
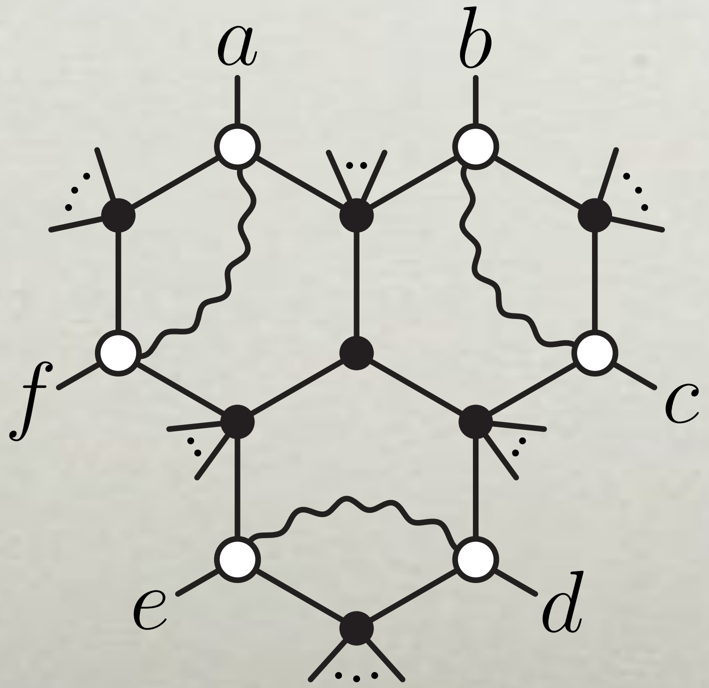


$$+ \sum_{\substack{a<b\leq c<\\<d\leq e<f}}$$



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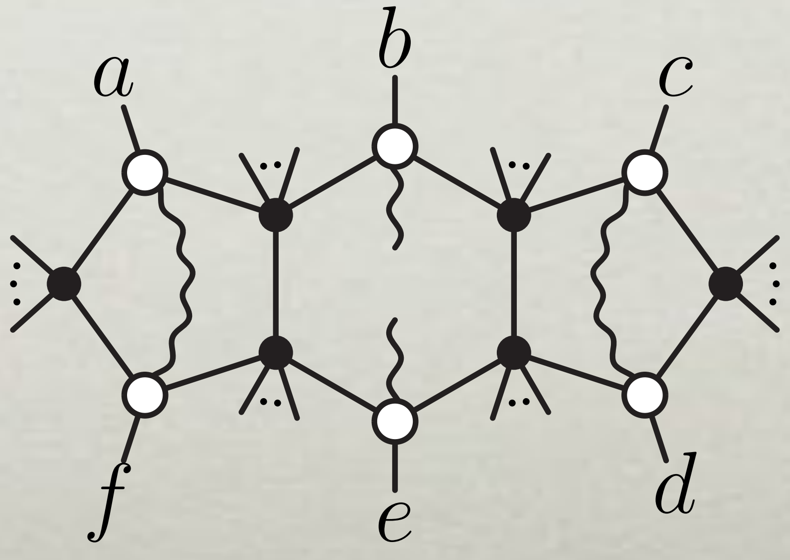

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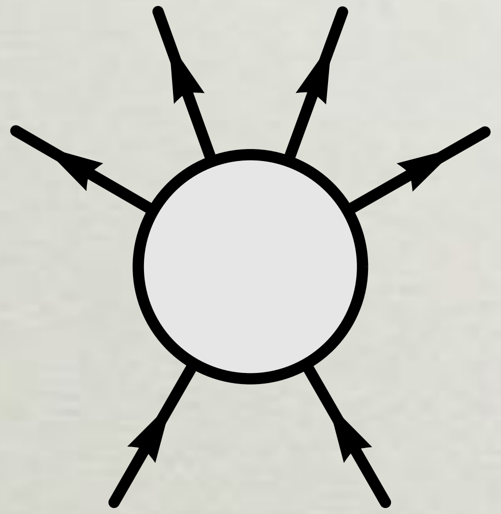

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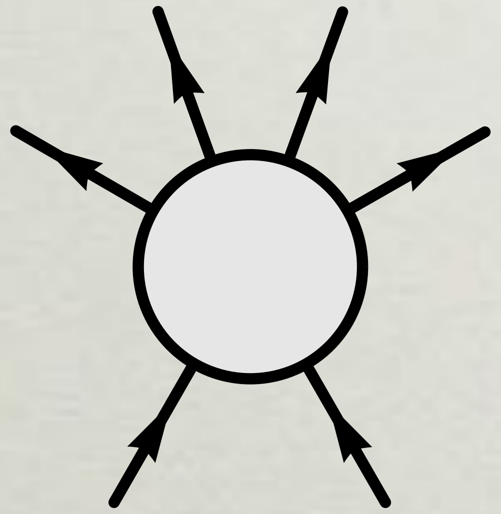

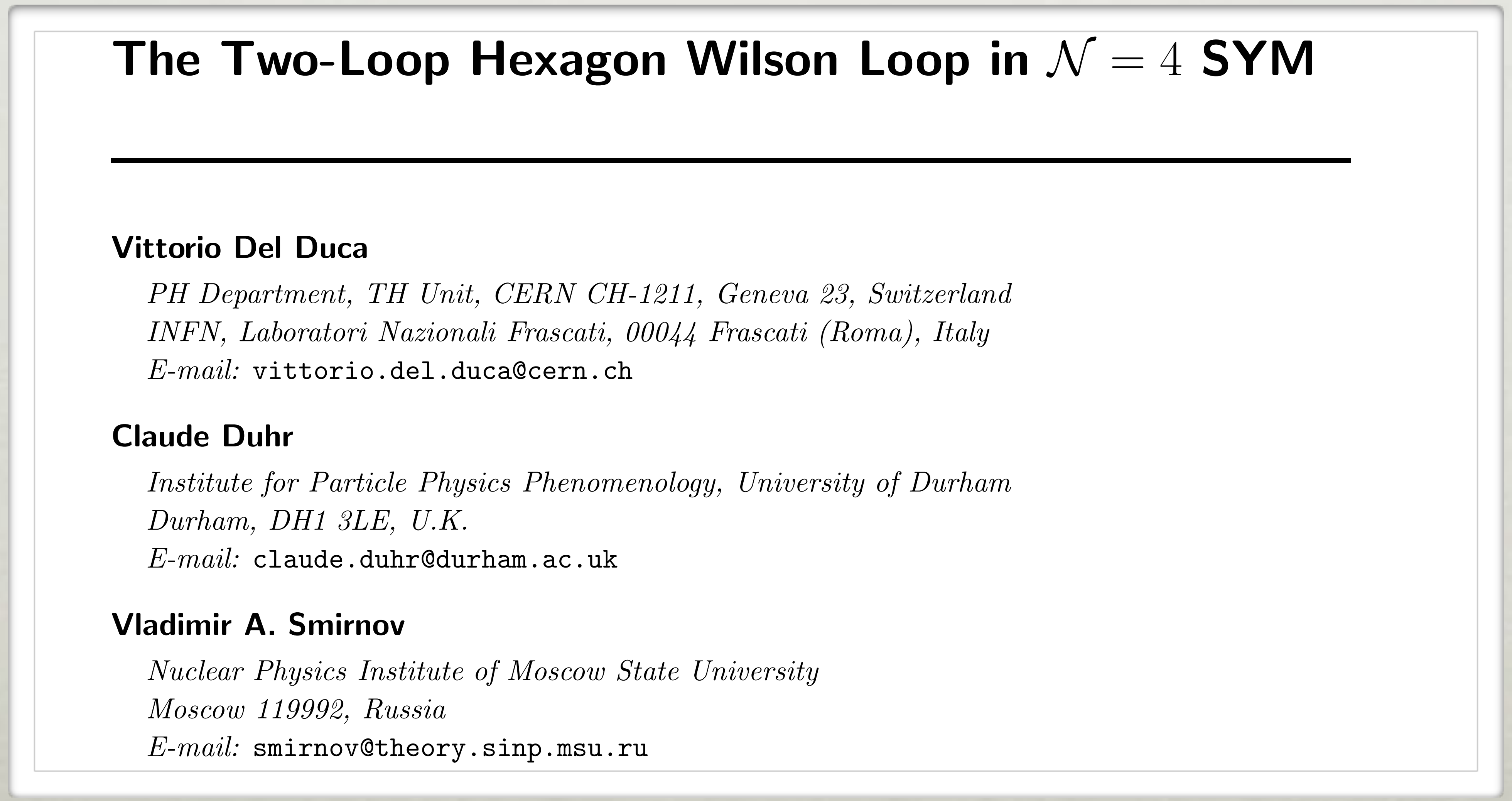

The Two-Loop Hexagon Wilson Loop in $\mathcal{N}=4$ SYM

Vittorio Del Duca
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E-mail: smirnov@theory.sinp.msu.ru

[Del Duca, Duhr, Smirnov (2010)]

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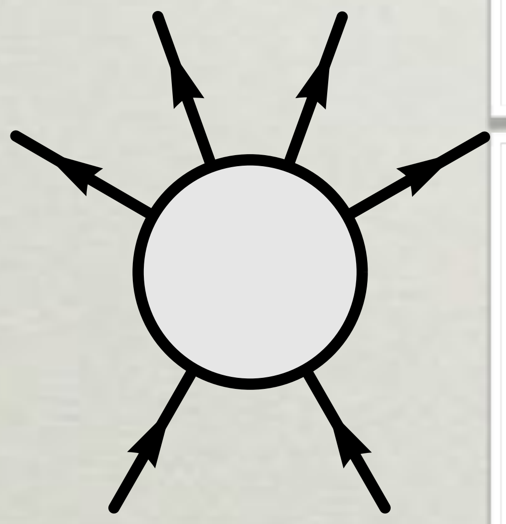

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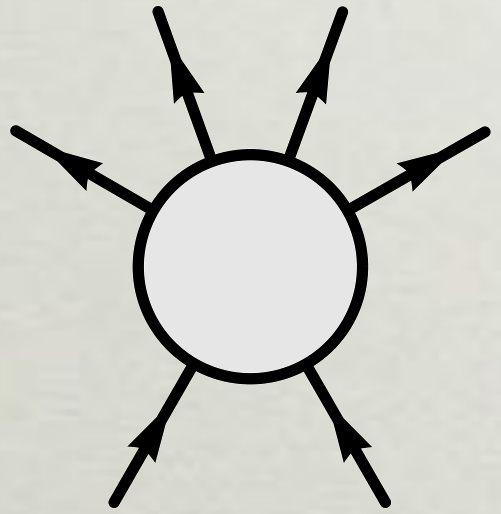

Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,[1] M. Spradlin,[2] C. Vergu,[2] and A. Volovich[2]

[1] *Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*
[2] *Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

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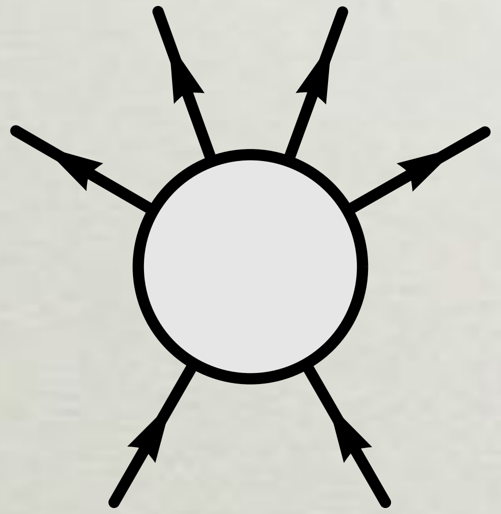

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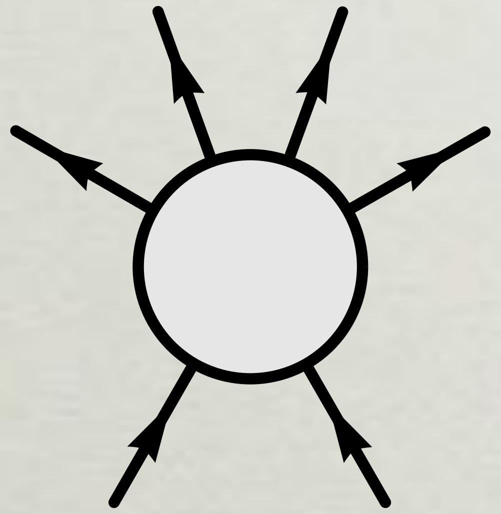
[2] *Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

$$R(u_1,u_2,u_3)=\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{J^4}{24}+\frac{1}{2}\zeta_2(J^2+\zeta_2)$$

What Form do Observables Take?

- But what about after **regularization** and ***loop integration***?
  What is the ***mathematical form*** of the predictions made by QFT?



State of the art:

6-point (N)MHV @ (6) 7 loops(!!!)

7-point (N)MHV @ 4 loops (symbol-level) [Dixon, *et al* (2019);…]

Amplitudes: a Virtuous Cycle

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beyond the reach of
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Discover Simplicity
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Understand Why
study it, understand it,
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Exploit Simplicity
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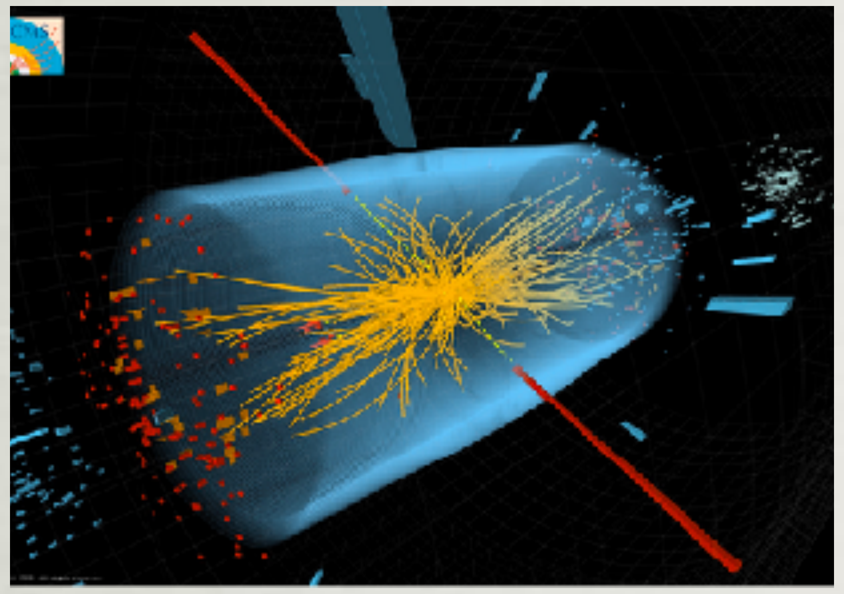

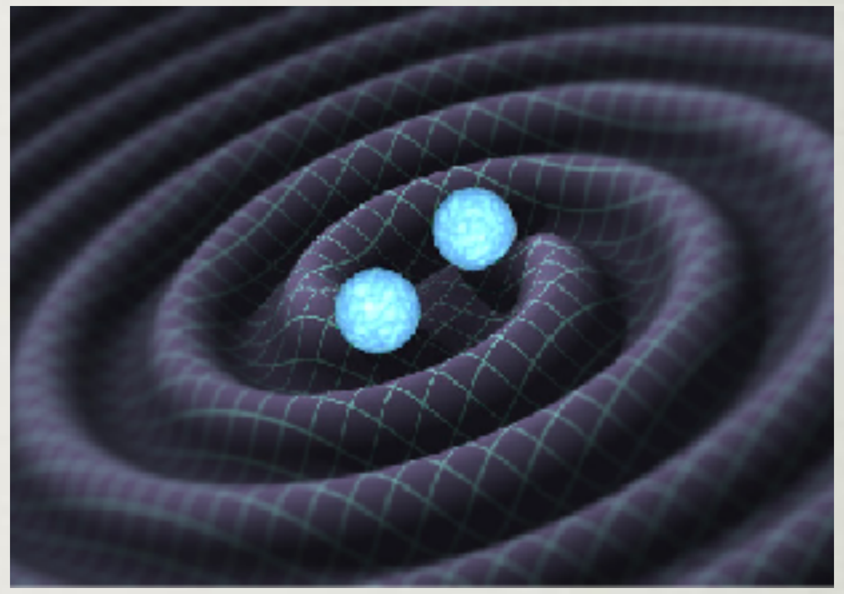

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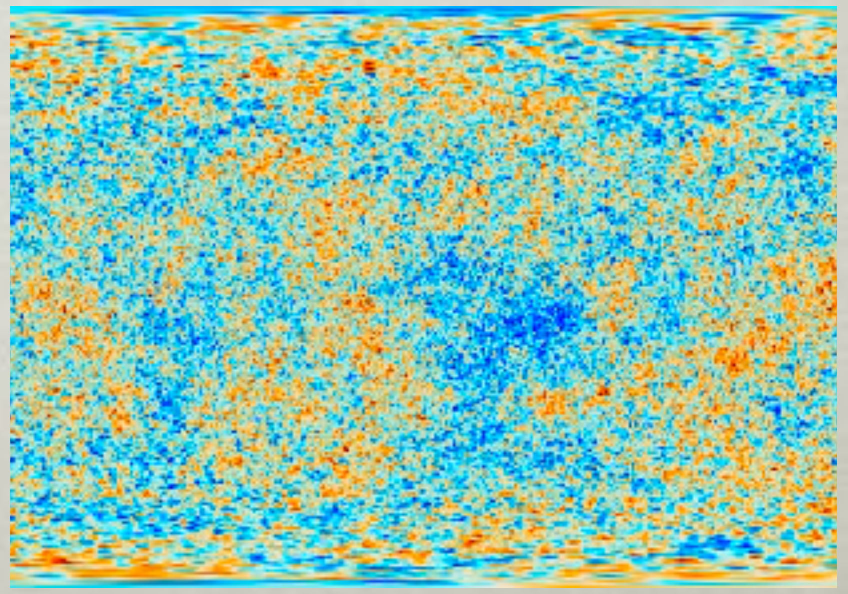

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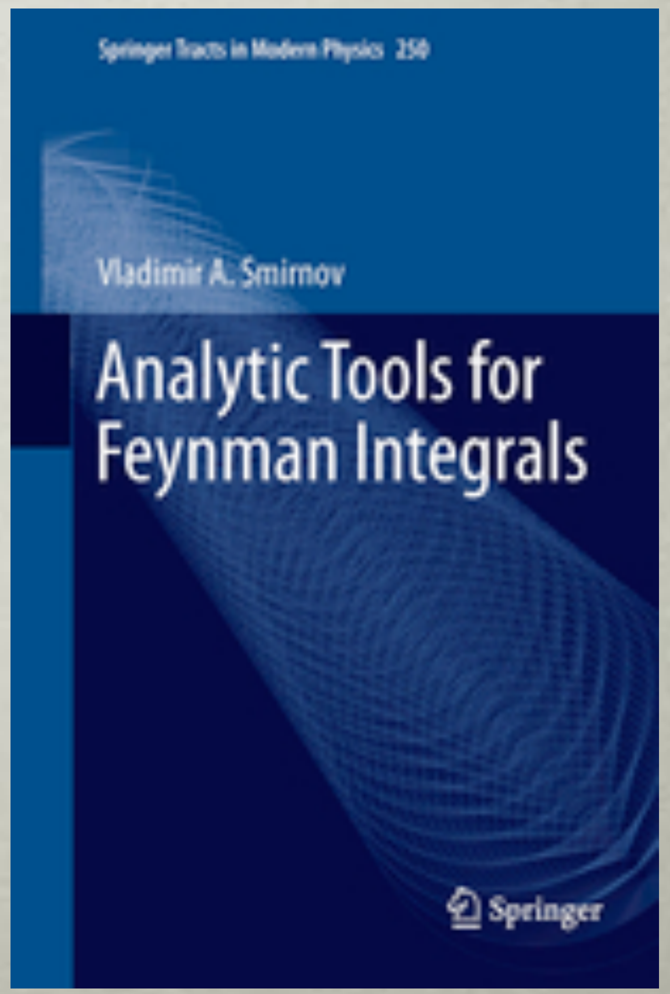

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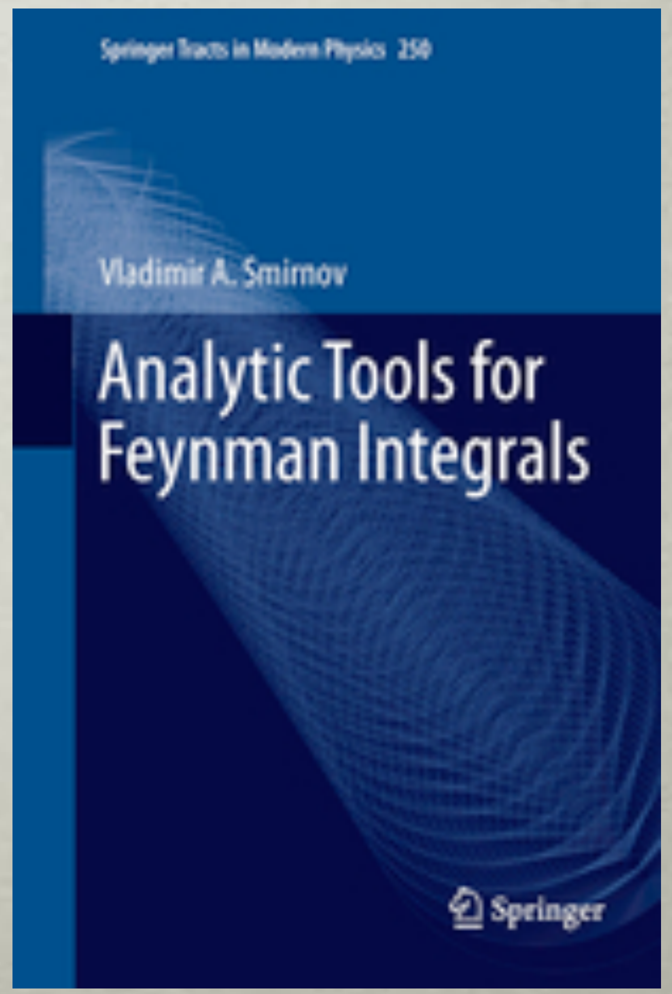

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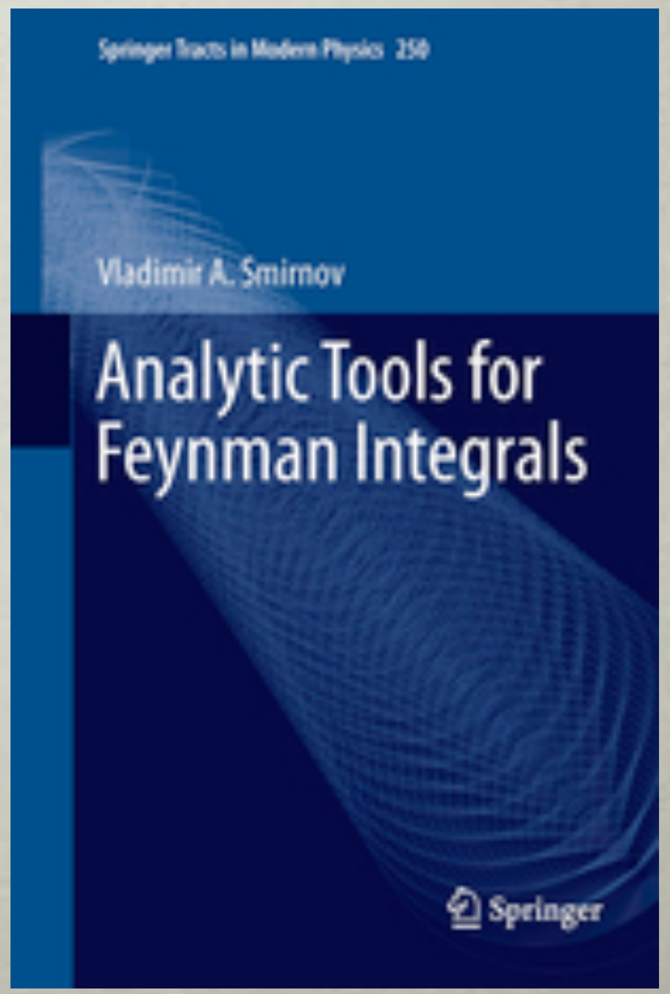

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$$\frac{\partial}{\partial s}\Big[g(s)\log\big(f(s)\big)\Big]=g'(s)\log\big(f(s)\big)+g(s)f'(s)/f(s)$$

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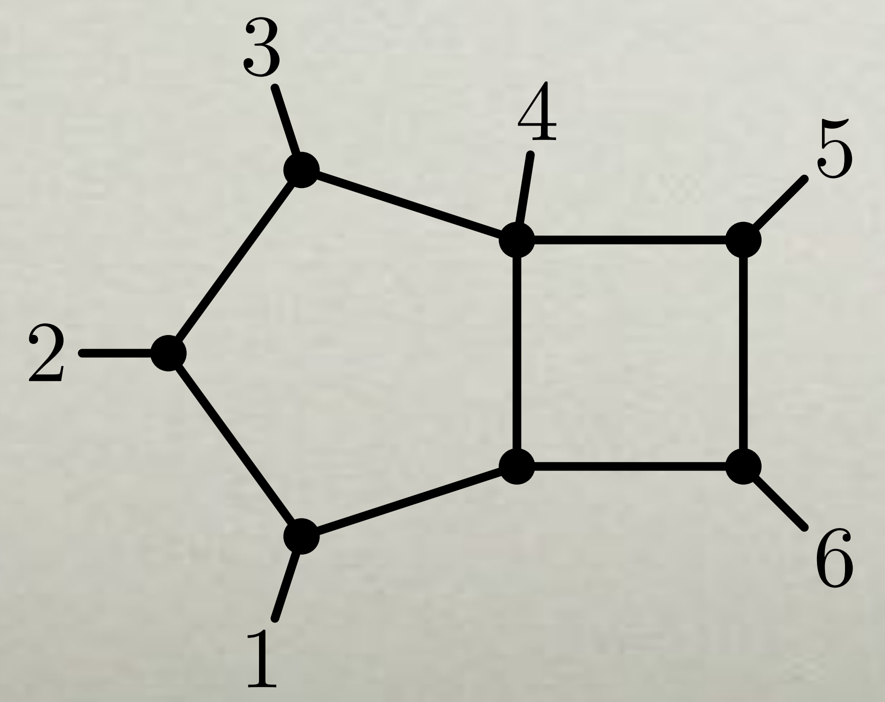

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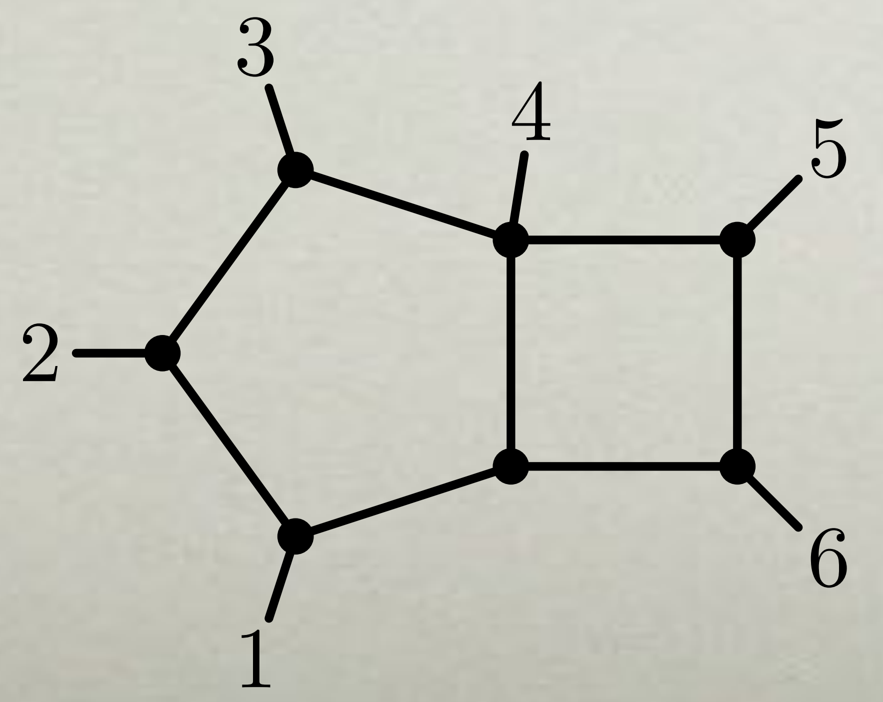

$$\Rightarrow \frac{s_{13}}{s_{12}s_{23}s_{56}}\left(\begin{array}{r}\mathrm{tr}_{+}[p_3,p_{12},p_6,p_1]\big(\mathrm{Li}_4(\cdots)\text{'s}+\ldots\big)\\+\mathrm{tr}_{+}[p_{12},p_6,p_1,p_3]\big(\mathrm{Li}_4(\cdots)\text{'s}+\ldots\big)\end{array}\right)$$

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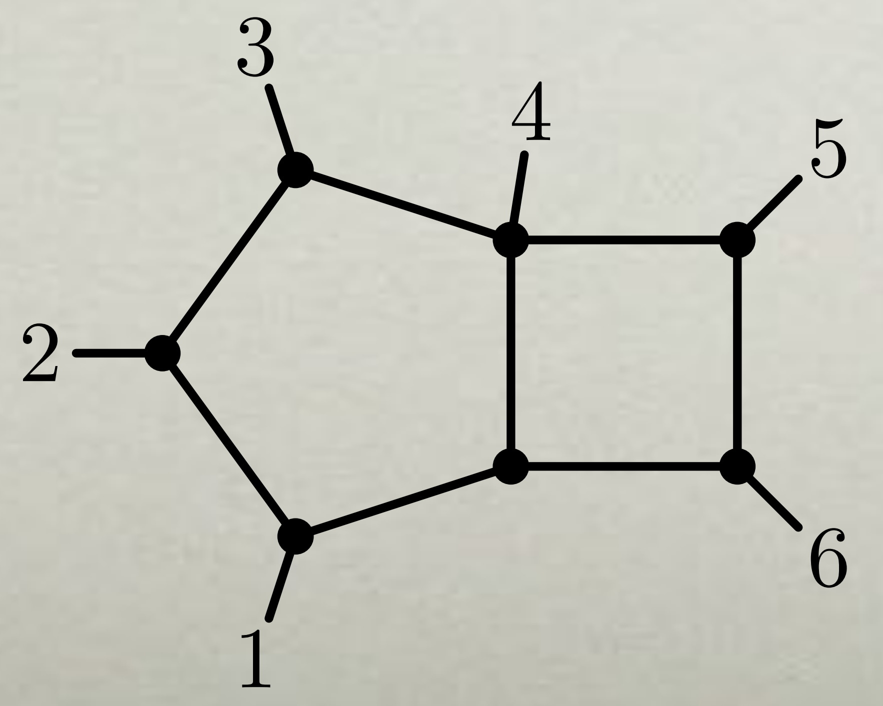

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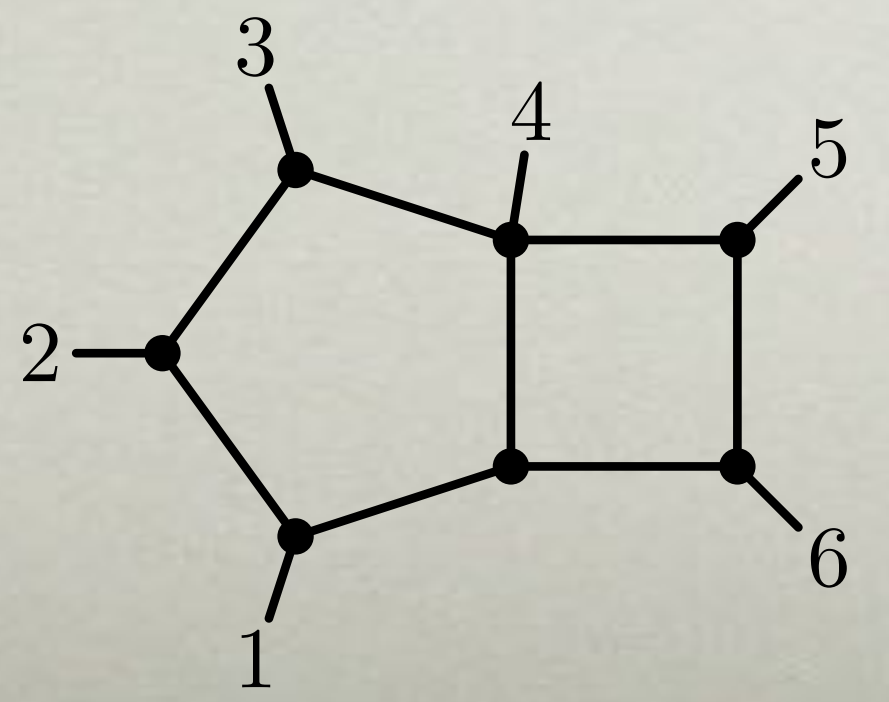
- **Avoid regularization** when**ever** possible:
  - can all(?) ***finite*** quantities be computed *without regularization*?
    —without expanding them in terms of divergent integrals?

(**Answer**: sometimes)

[**JB**, Langer, Patatoukos (2021); ...]



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Improving Loop Integration by Building Better Bases

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  - Why not try to find the *best/easiest* integrands—and use these?

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$$\mathfrak{B}^{\text{SM}} \supset \mathfrak{B}^{\mathcal{N}=2} \supset \mathfrak{B}^{\mathcal{N}=4}$$

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  - recently, we gave an intrinsically graph-theoretic definition of power-counting for ***non-planar*** integrand bases

[**JB**, Herrmann, Langer, Trnka (2020)]

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- ¿Can we further stratify each part by ***transcendental*** *structure*?

$$\overbrace{\big\{\text{max-weight}\big\} \oplus \big\{\text{next-to-max-weight}\big\}}^{\{\text{finite}\}} \oplus \cdots \oplus \big\{\text{rational}\big\}$$

$$\underbrace{\big\{\text{polylogs}\big\} \oplus \big\{\text{elliptic-polylogs}\big\}}_{\text{(max-weight)}} \oplus \big\{\text{K3-polylogs}\big\} \oplus \cdots$$

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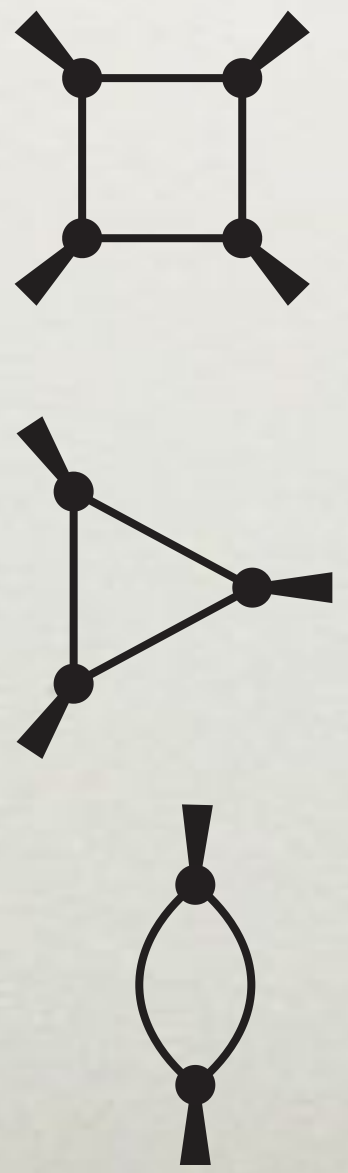

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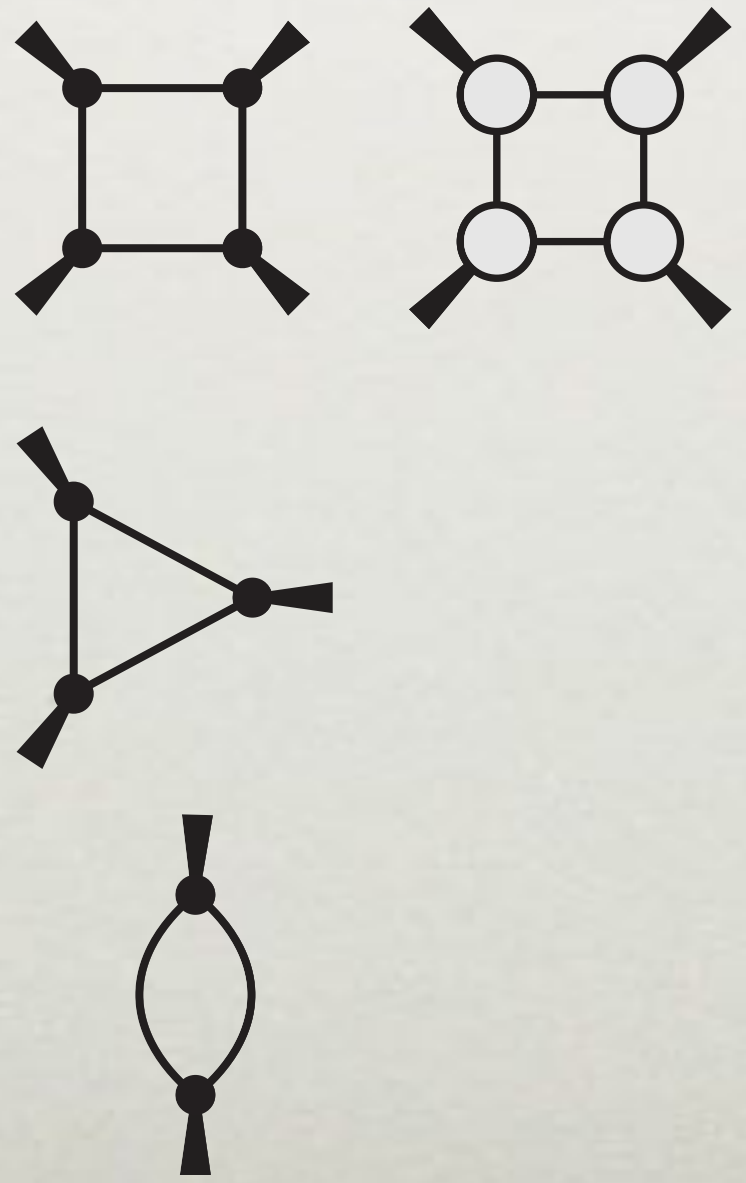

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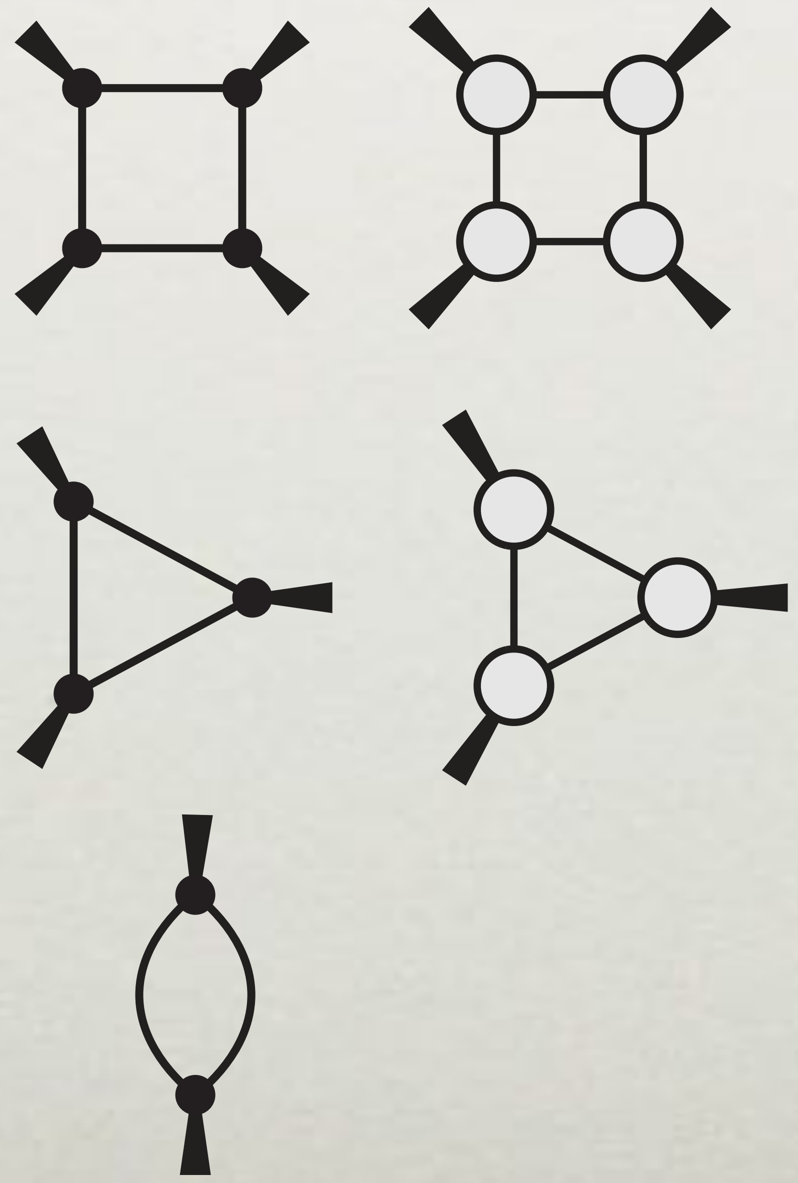

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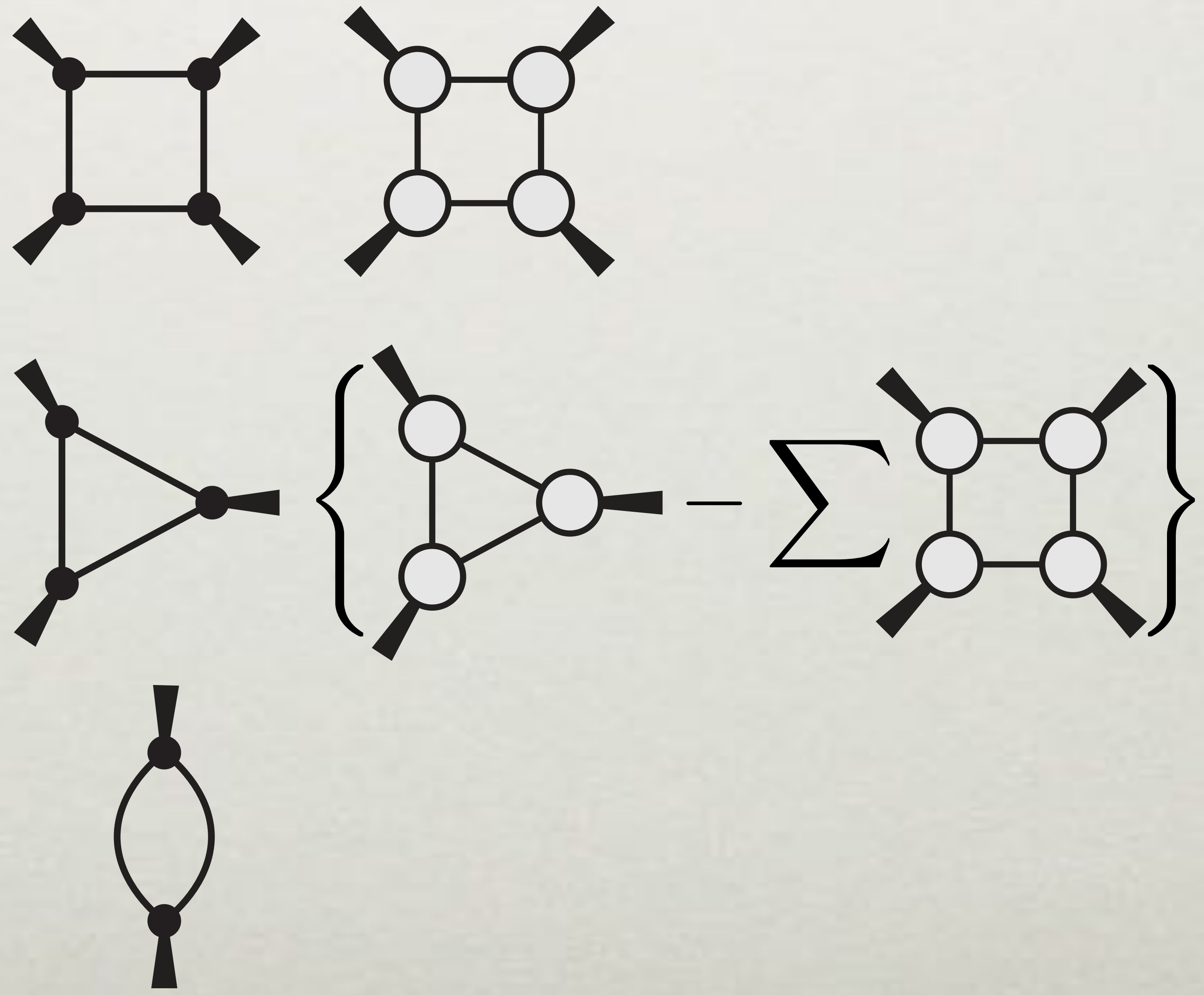

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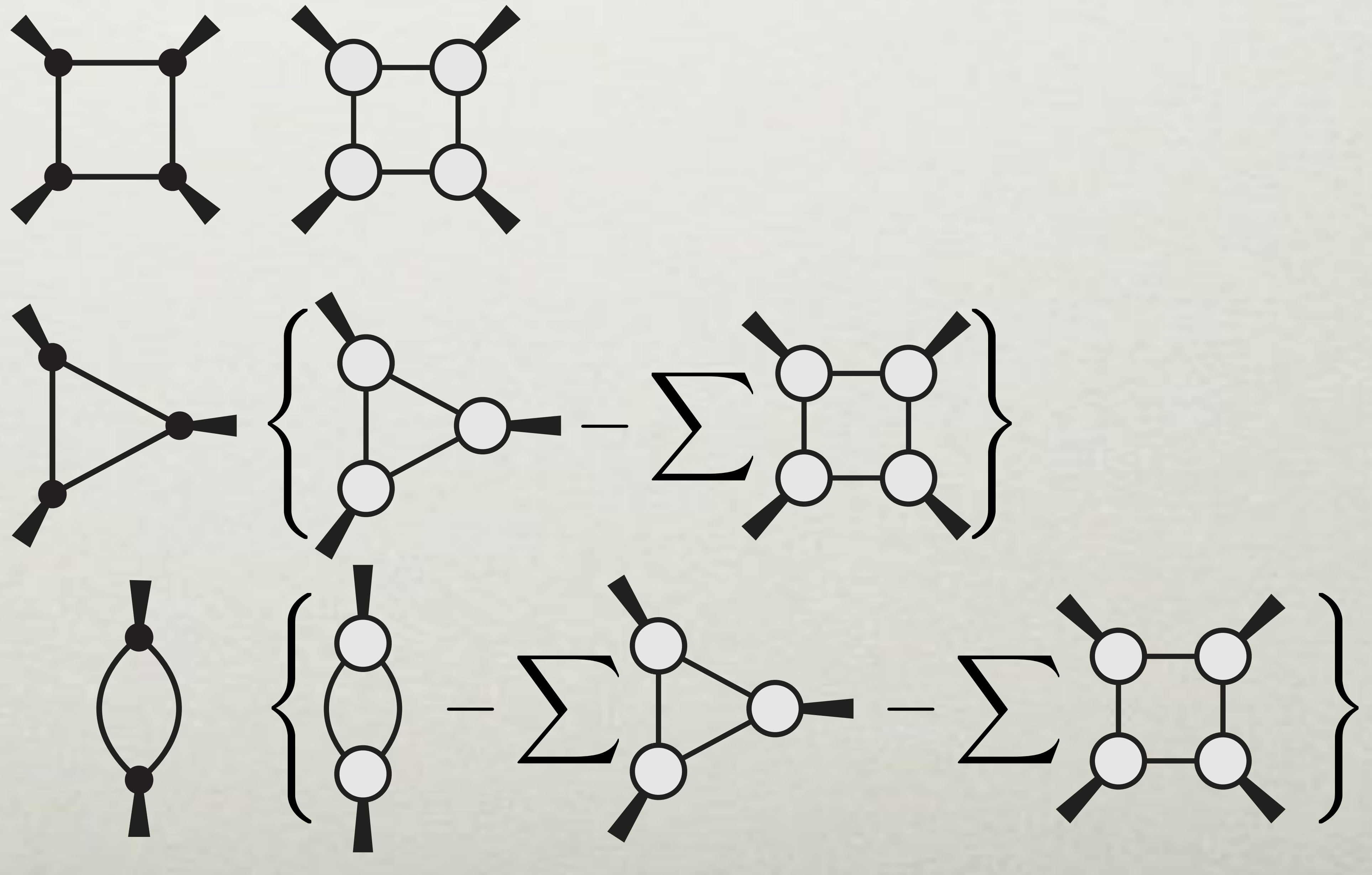

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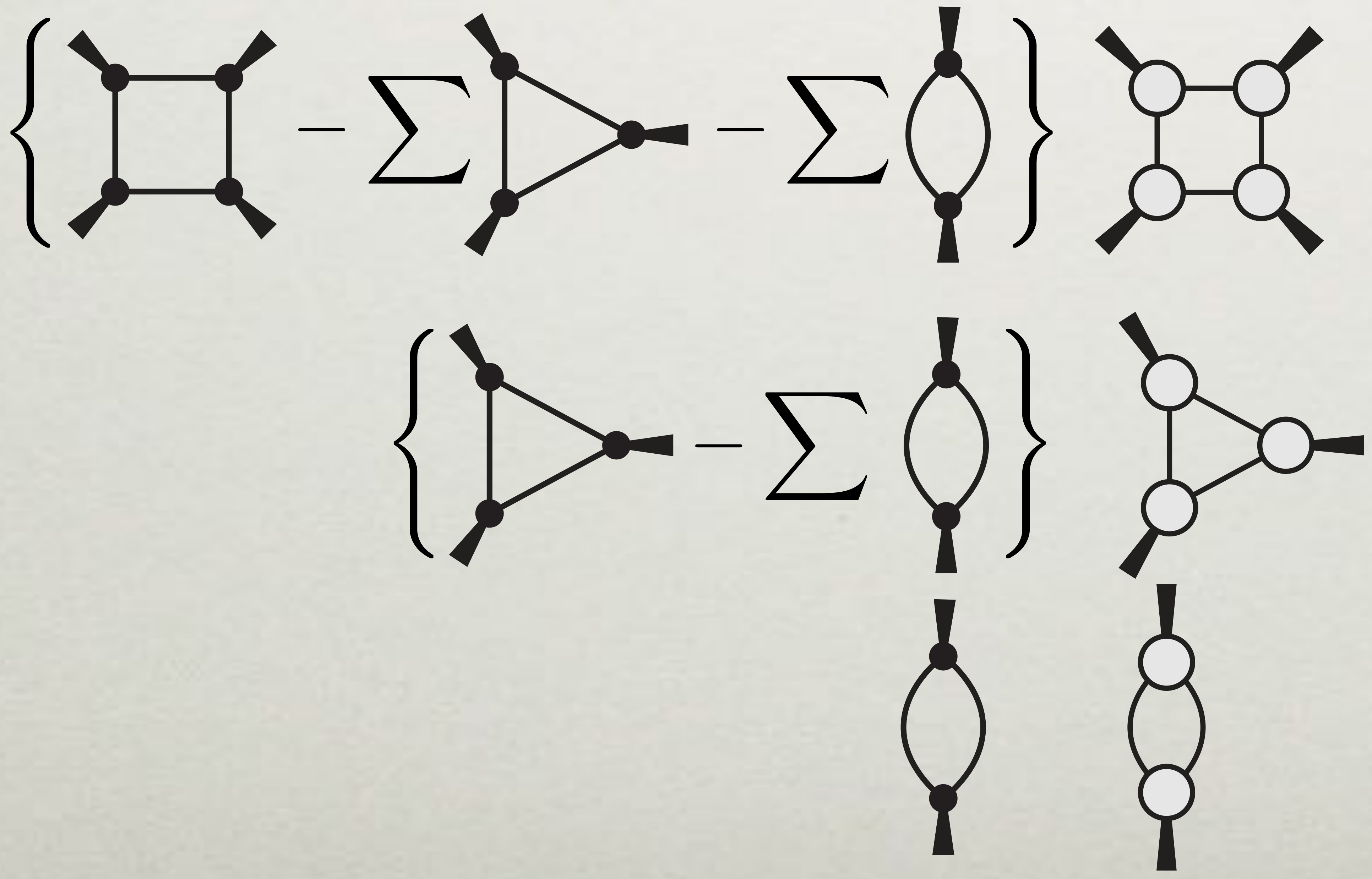

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- Choosing a maximal set of IR/UV-divergence-probing contours ensures(?) that the basis is split into finite/divergent subspaces

Amplitudes: a Virtuous Cycle

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Compute Something

beyond the reach of
recent imagination

Discover Simplicity

beyond expectations

Understand Why

study it, understand it,
& explore consequences

Exploit Simplicity

to build **more powerful**
computational technology

Thank you!