



PennState



University of
Pittsburgh

Adventures in Perturbation Theory

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Phenomenology 2022

University of Pittsburgh



The Niels Bohr
International Academy

10 May, 2022



Roadmap



◆ *Spiritus Movens*

- ▶ the *shocking simplicity* of QFT
- ▶ what do amplitudes *look like*—functionally?
- ▶ why is perturbation theory *so hard*?
—and how can we make it *easier*?

◆ Improving Integration with Better **Integrand Bases**

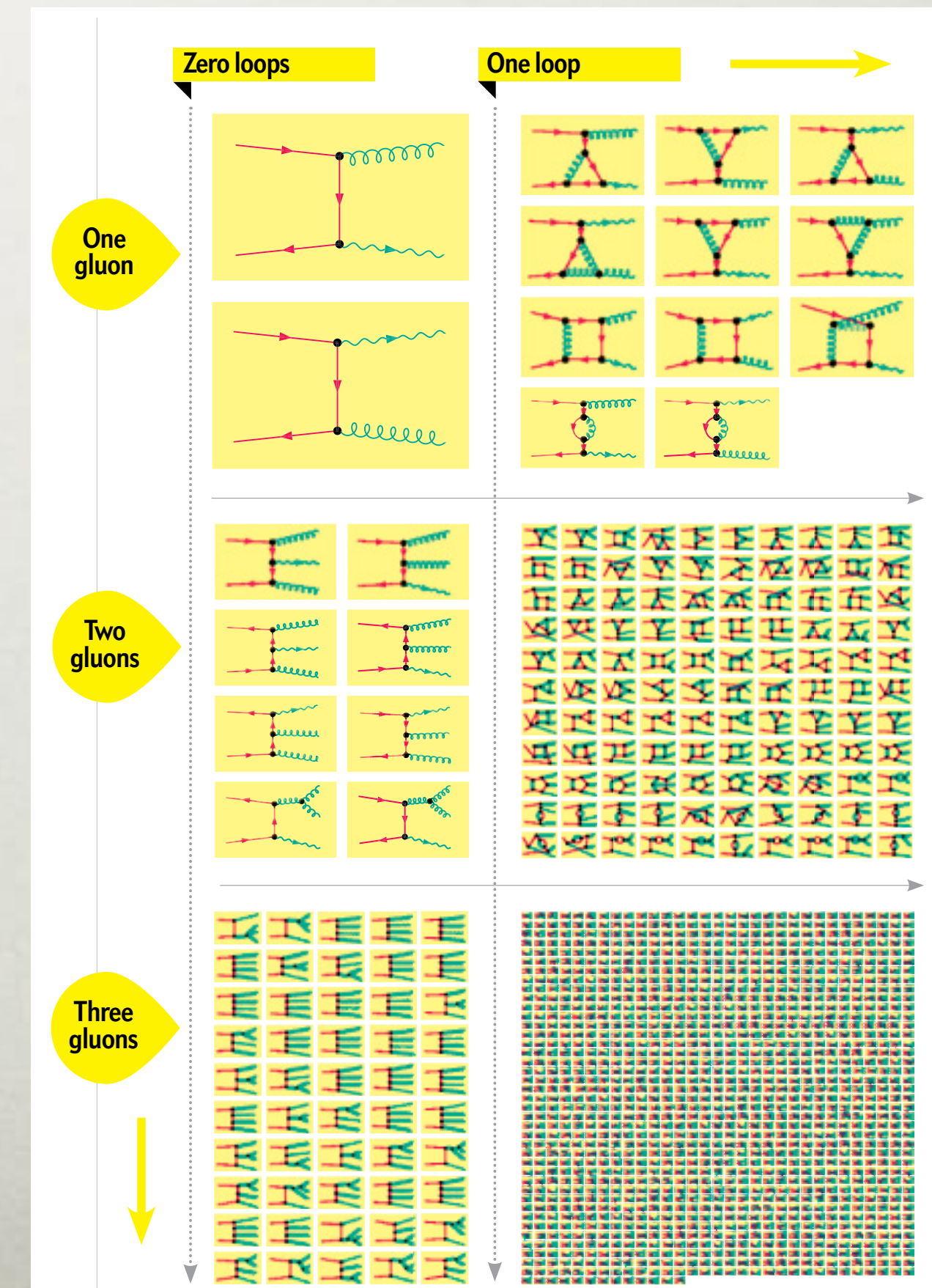
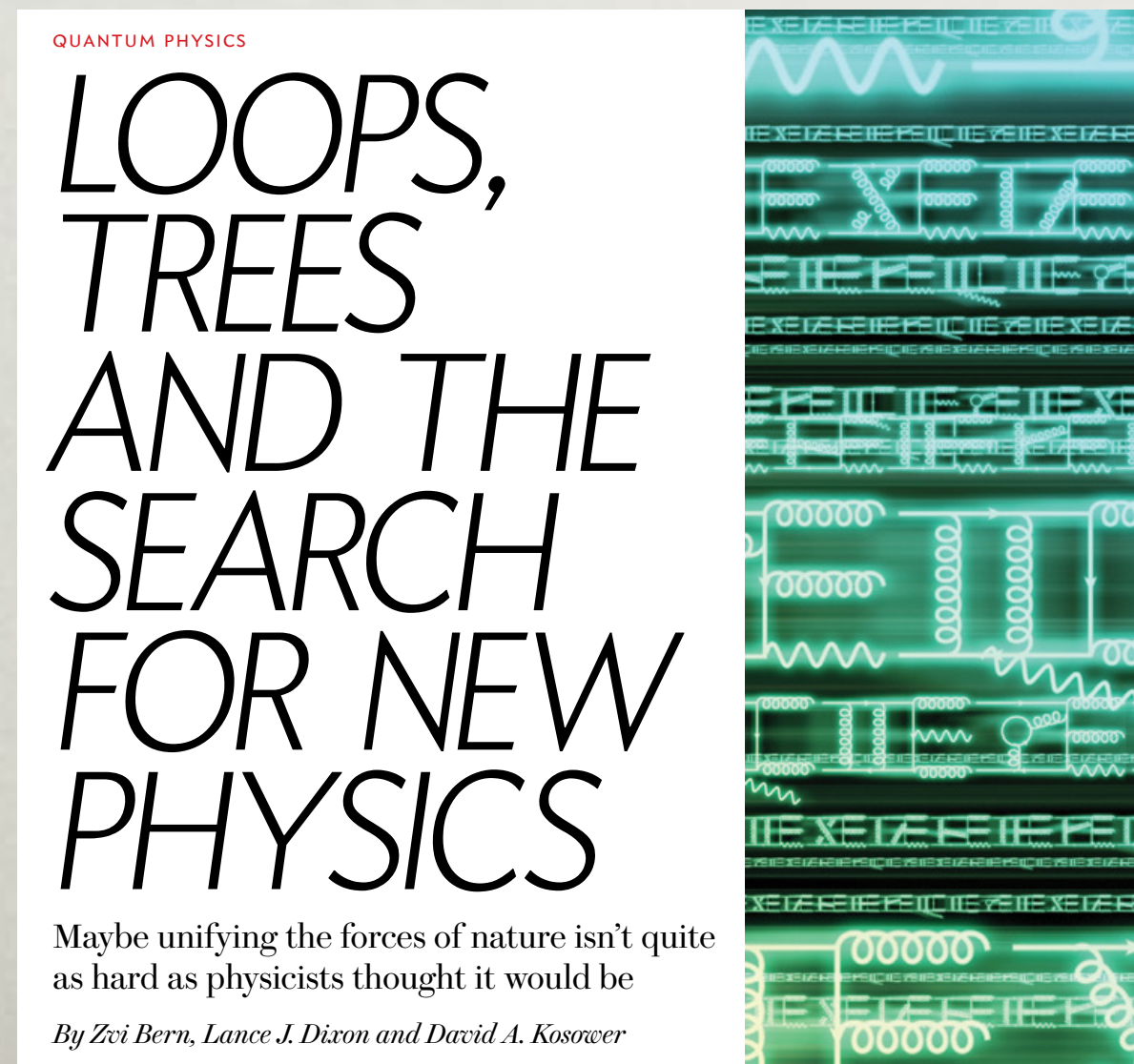
- ▶ what makes for a *good* basis of Feynman integrals?
- ▶ *stratifying* theories and stratified integrand bases
- ▶ generalized vs *prescriptive unitarity*



Explosions of Complexity

- ◆ While ultimately correct, the Feynman expansion renders *all but the most trivial* predictions—
involving the **fewest particles**, at the **lowest orders** of perturbation—

or **computationally intractable**
theoretically inscrutable

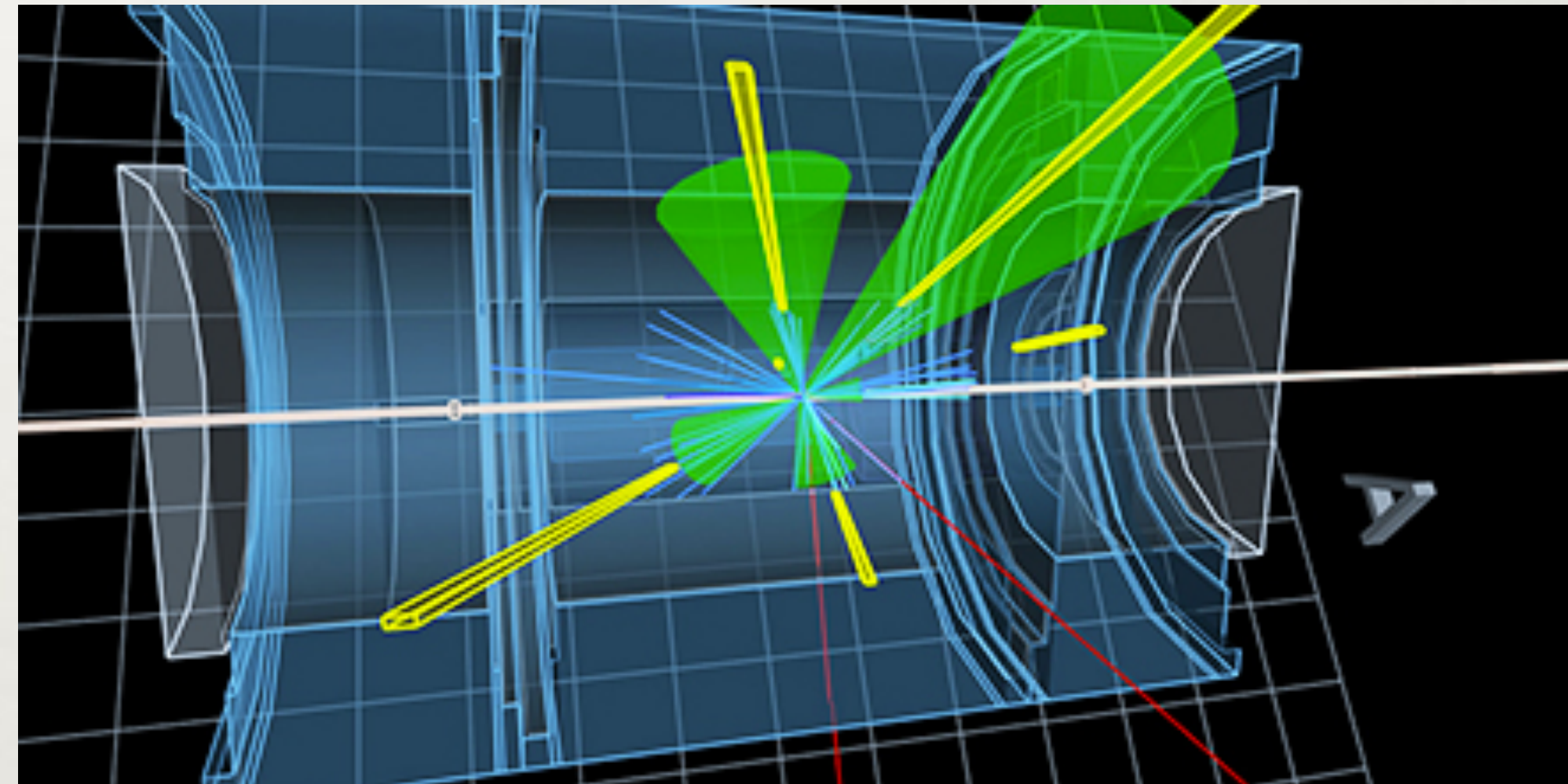


[Bern, Dixon, Kosower, *Scientific American* (2012)]



Needs (Once) Beyond Our Reach

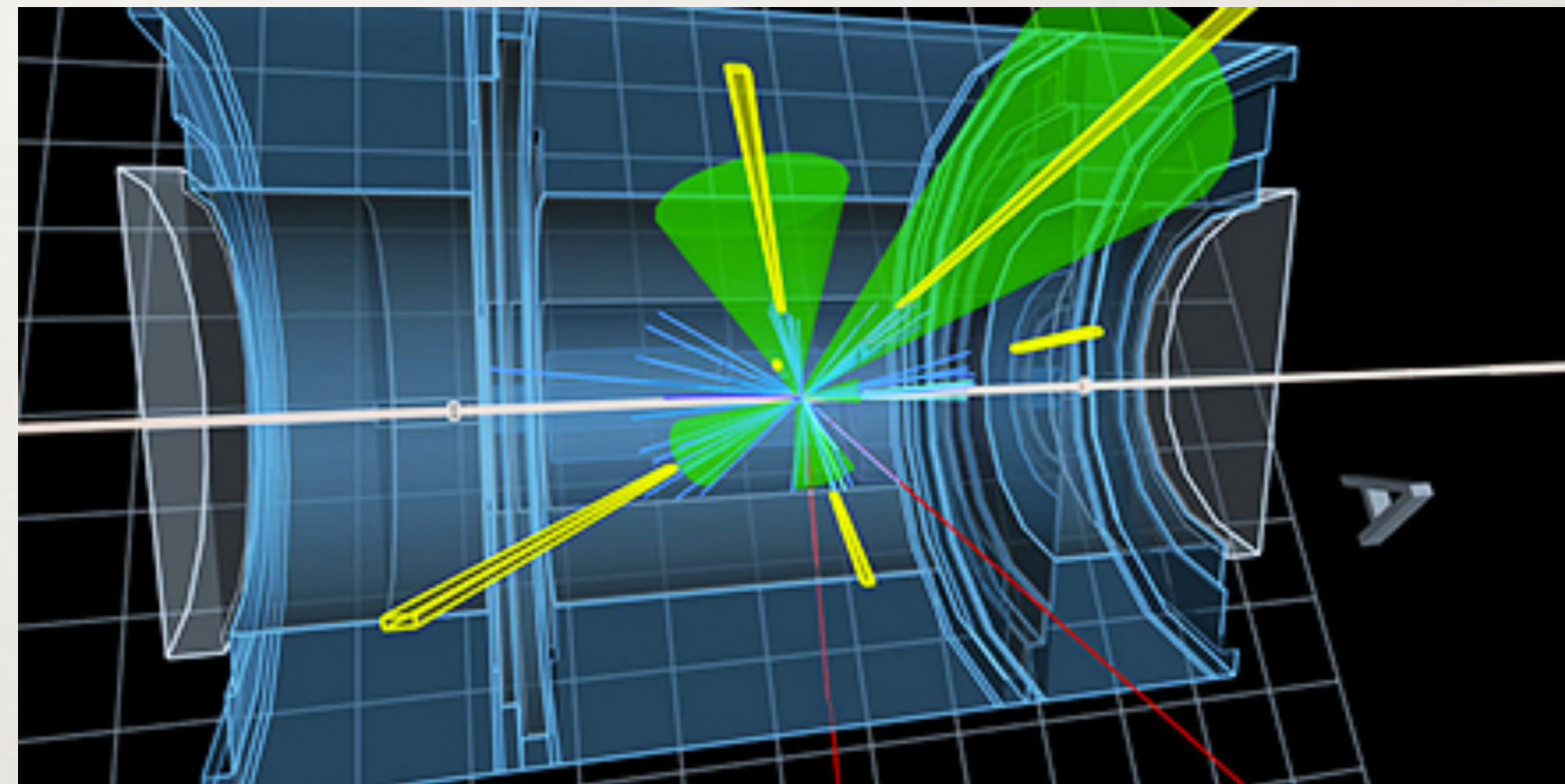
- ◆ Background amplitudes **crucial** for *e.g.* colliders





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Needs (Once) Beyond Our Reach

- ◆ Background amplitudes crucial for *e.g.* colliders

Supercollider physics [Rev.Mod.Phys. 56 (1984)]

E. Eichten

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe

Lawrence Berkeley Laboratory, Berkeley, California 94720

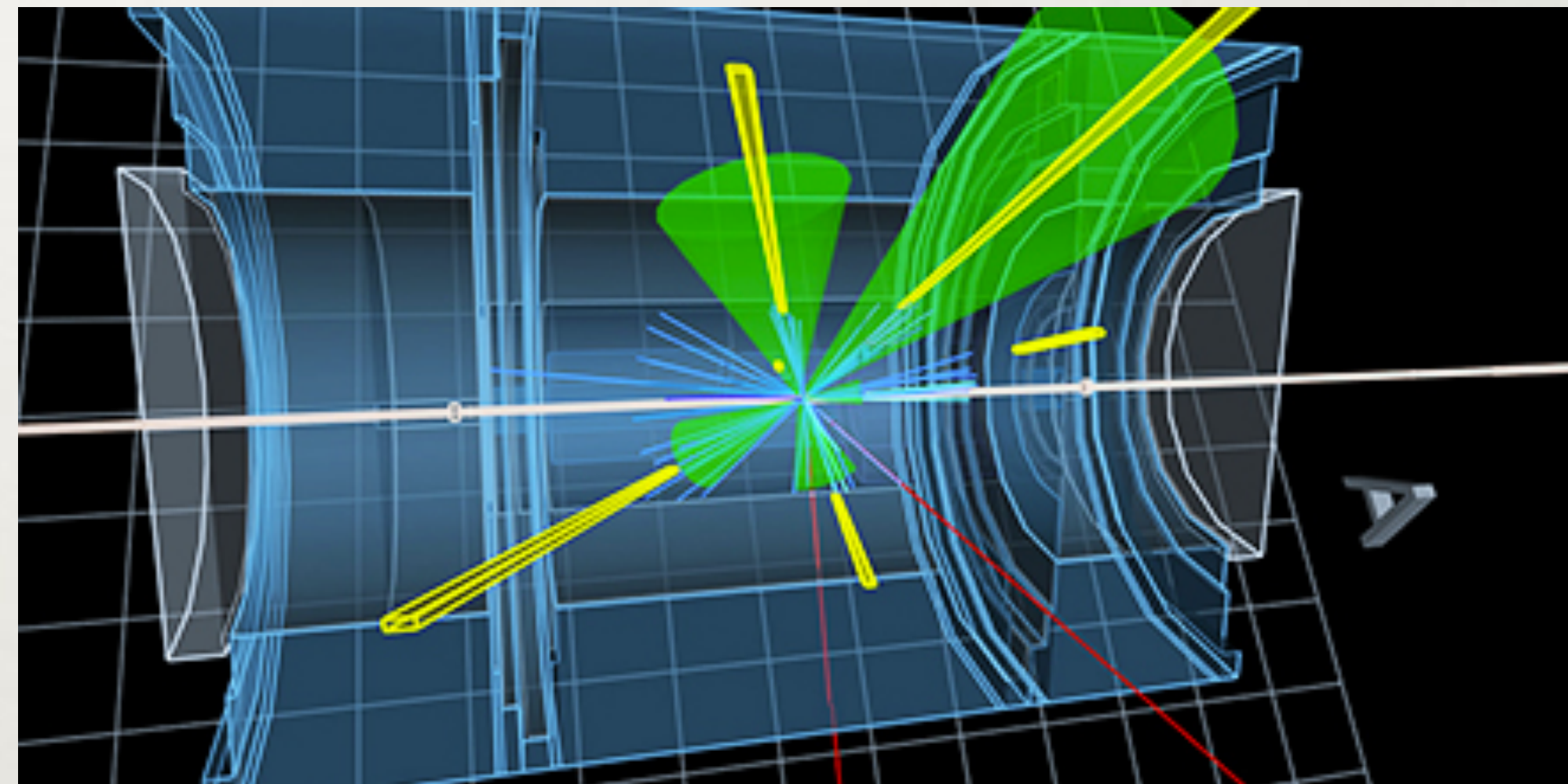
K. Lane

The Ohio State University, Columbus, Ohio 43210

C. Quigg

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Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.



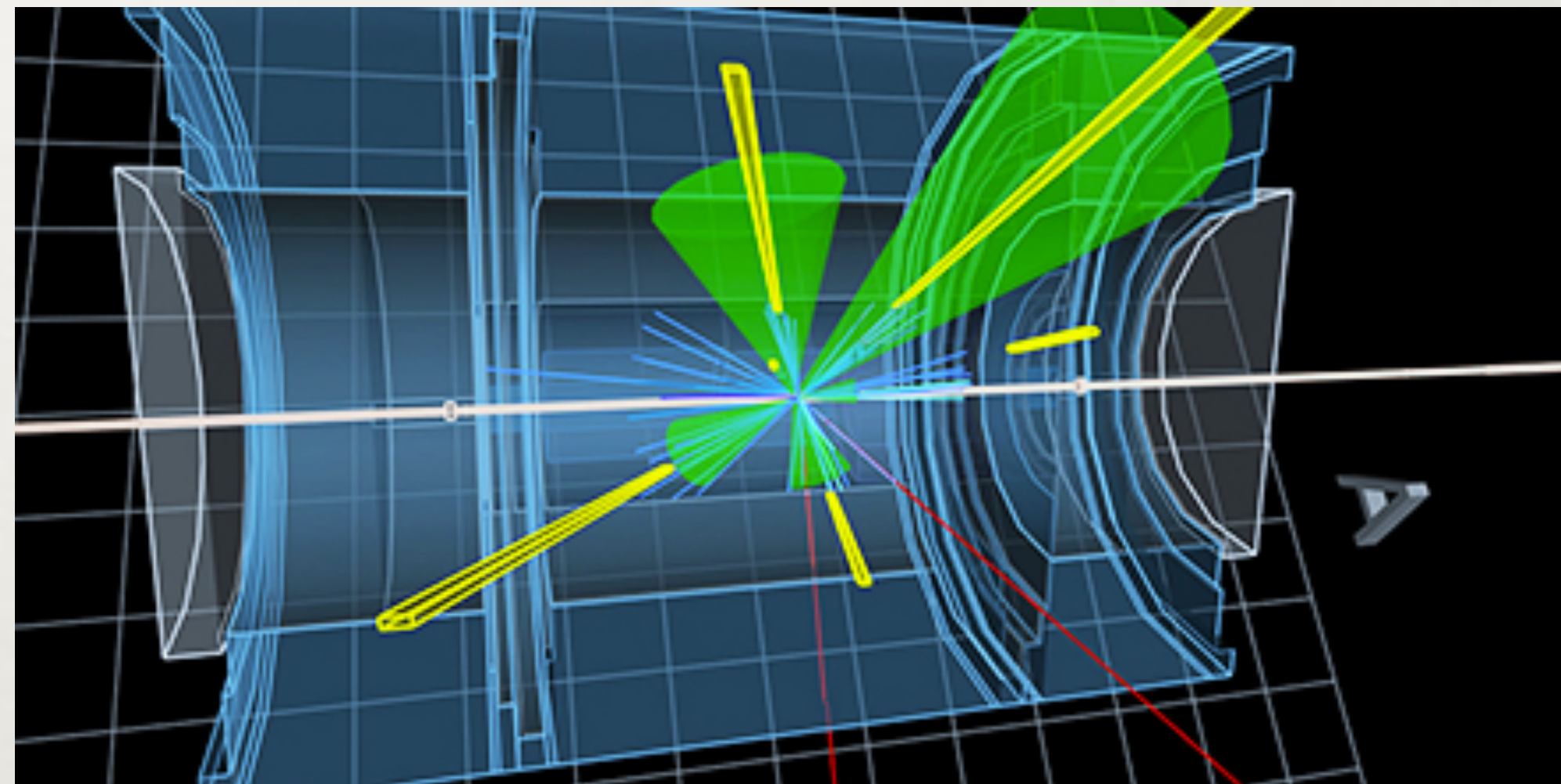
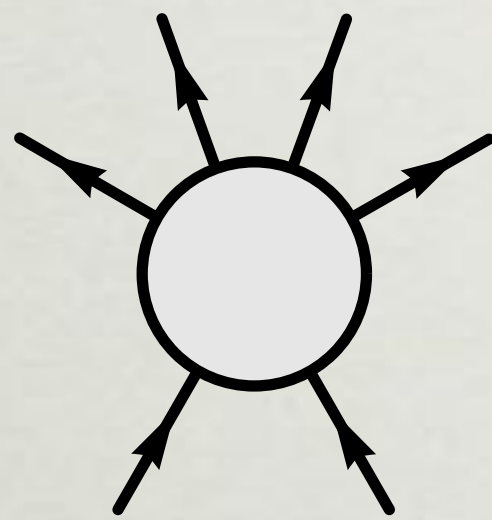
- ◆ Once considered *computationally intractable*

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.



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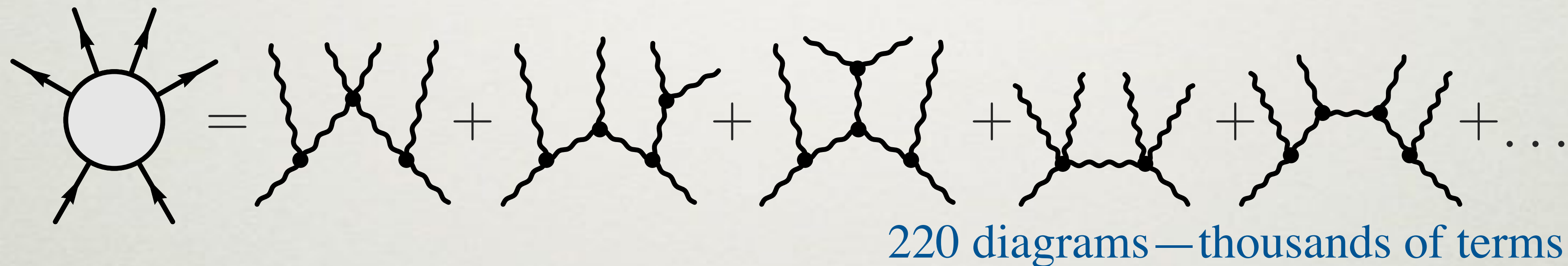
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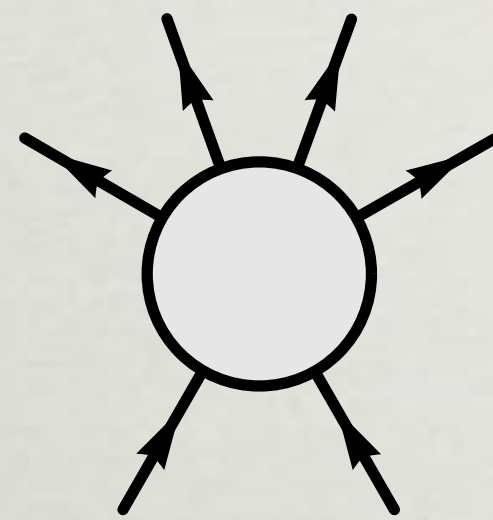
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THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON-GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

[*Nucl.Phys.* **B269** (1985)]

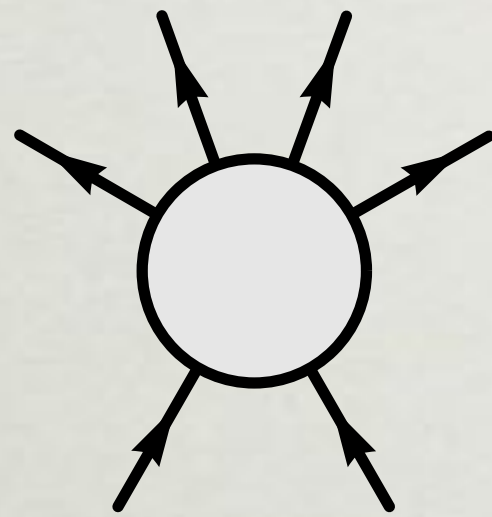
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Needs (Once) Beyond Our Reach

◆ Background amplitudes crucial for e.g. colliders



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gluons. The cross section for the scattering of two gluons with momenta p_1, p_2 into four gluons with momenta p_3, p_4, p_5, p_6 is obtained from eq. (5) by setting $I=2$ and replacing the momenta p_1, p_2, p_3, p_4 by $-p_1, -p_2, -p_3, -p_4$.

As the result of the computation of two hundred and forty Feynman diagrams, we obtain

$$A_{(2)}^2(p_1, p_2, p_3, p_4, p_5, p_6) = (\mathcal{D}^1, \mathcal{D}^2, \mathcal{D}^3, \mathcal{D}^4, \mathcal{D}^5) \cdot \begin{pmatrix} K & K_c & K_c & K_c \\ K_c & K & K & K_c \\ K_c & K & K & K_c \\ K_c & K_c & K & K \end{pmatrix} \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \\ \mathcal{D}_3 \\ \mathcal{D}_4 \end{pmatrix} \quad (6)$$

where $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and \mathcal{D}_4 are 11-component complex vector functions of the momenta p_1, p_2, p_3, p_4, p_5 and p_6 , and K, K_c, K_c and K_c are constant 11×11 symmetric matrices. The vectors $\mathcal{D}_1, \mathcal{D}_2$ and \mathcal{D}_3 are obtained from the vector \mathcal{D} by the permutations $(p_3 \leftrightarrow p_4), (p_5 \leftrightarrow p_6)$ and $(p_3 \leftrightarrow p_4, p_5 \leftrightarrow p_6)$, respectively, of the momentum variables in \mathcal{D} . The individual components of the vector \mathcal{D} represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices K , which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to $N(N^2-1)$ and $N(N^2-1)$, respectively (N is the number of colors, $N=3$ for QCD).

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TABLE I
Matrices $K(I, J, I=1-11, J=1-11)$.

Matrix $K^{(1)}$					Matrix $K^{(2)}$				
8	4	-2	-1	2	0	1	0	0	-1
4	8	-1	1	-1	0	2	1	0	-1
-2	-1	4	4	1	1	2	2	1	2
2	1	4	8	2	-1	4	1	1	1
-1	-1	4	2	8	1	2	4	-1	4
2	0	1	-1	1	8	4	-1	0	1
0	2	-1	2	4	8	-2	0	0	0
1	1	2	4	4	-1	-2	8	-1	-2
0	0	2	1	-2	0	1	4	-2	0
0	1	1	-1	1	0	-1	4	8	-1
-1	-2	1	4	0	0	2	-1	-1	8

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TABLE I (continued)

Matrix $K^{(3)}$					Matrix $K^{(4)}$				
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

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where ϵ is the totally antisymmetric tensor, $\epsilon_{333}=1$. For the future use, we define one more function,

$$F(p_1, p_2) = ((p_1, p_2)(p_3, p_4) + (p_1, p_3)(p_2, p_4) - (p_1, p_4)(p_2, p_3)) / (p_1, p_2) \quad (10)$$

Note that when evaluating A_3 and A_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions E, F and G on the momenta p_1, p_2, p_3, p_4 .

The diagrams D_i^2 are listed below:

$$D_1^2(1) = -\frac{\delta_1}{s_{14}s_{23}s_{34}} [((p_1 - p_2)(p_3 - p_4))((p_1 - p_4)(p_2 + p_3)) - ((p_1 - p_3)(p_2 + p_4)) \times ((p_1 - p_4)(p_2 - p_3)) + ((p_2 + p_3)(p_1 - p_4))((p_1 - p_4)(p_2 - p_3))]$$

$$D_2^2(2) = \frac{1}{s_{12}s_{34}} (2E(p_2 - p_3, p_1 - p_4) - 2E(p_1 - p_4, p_2 - p_3) + \delta_1((p_1 - p_2)(p_3 - p_4)))$$

$$D_3^2(3) = -\frac{4}{s_{14}s_{23}s_{123}} [((p_1 + p_2 - p_3)(p_4 + p_1 - p_4))E(p_1, p_2) - ((p_1 + p_2 - p_3)(p_4 - p_1 + p_4))E(p_1, p_2) - ((p_1 - p_2 + p_3)(p_4 + p_1 - p_4))E(p_1, p_2)]$$

[3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has successfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge the hospitality of Aspen Center for Physics, where this work was being completed

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$$D_1^2(16) = \frac{-4}{s_{12}s_{34}s_{124}} [s_{33} - s_{34} + s_{34}]E(p_1, p_2)$$

$$D_1^2(17) = \frac{4}{s_{12}s_{34}s_{124}} [s_{23} - s_{34} - s_{34}]E(p_1, p_2)$$

$$D_1^2(18) = \frac{-4}{s_{12}s_{34}s_{124}} [2(p_1 + p_2)(p_3 - p_4) - s_{34}]E(p_1, p_2)$$

$$D_1^2(19) = \frac{-2}{s_{12}s_{34}} [p_2 - p_1 - p_4]$$

$$D_1^2(20) = \frac{2}{s_{12}s_{34}} [p_2 - p_1 - p_4]$$

$$D_1^2(21) = \frac{-4}{s_{12}s_{34}s_{134}} [s_{24} - s_{34} + s_{34}]E(p_1, p_2)$$

$$D_1^2(22) = \frac{4}{s_{12}s_{34}s_{144}} [s_{23} - s_{34} - s_{34}]E(p_1, p_2)$$

$$D_1^2(23) = \frac{4}{s_{12}s_{34}s_{144}} [2(p_1 + p_2)(p_3 - p_4) + s_{23}]E(p_1, p_2)$$

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$$D_1^2(32) = \frac{4}{s_{12}s_{34}s_{123}} [((p_1 - p_2 + p_3)(p_4 + p_1 - p_4) + s_{123})E(p_1, p_2)] \quad (11)$$

where $\delta_1 = 1$.

The diagrams D_i^2 are obtained from D_1^2 by replacing δ_1 by $\delta_i = 0$ and the functions $E(p_1, p_2)$ by $G(p_1, p_2)$.

The diagrams D_i^2 are listed below:

$$D_2^2(1) = \frac{4}{s_{12}s_{34}s_{123}} [F(p_1, p_2)E(p_1, p_2) - F(p_1, p_2)E(p_1, p_2) + [F(p_1, p_2) + s_{34}]E(p_1, p_2)]$$

$$D_2^2(2) = \frac{-4}{s_{12}s_{34}s_{123}} [F(p_1, p_2) + \frac{1}{2}s_{34}]E(p_1, p_2) + [F(p_1, p_2) + \frac{1}{2}s_{34}]E(p_1, p_2) - F(p_1, p_2)E(p_1, p_2)]$$

$$D_2^2(3) = \frac{4}{s_{12}s_{34}s_{123}} [F(p_1, p_2)E(p_1, p_2) - F(p_1, p_2)E(p_1, p_2) - [F(p_1, p_2) - \frac{1}{2}s_{34} + \frac{1}{2}s_{34}]E(p_1, p_2)]$$

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$$D_2^2(11) = \frac{1}{2s_{12}s_{34}s_{123}} [s_{23} + s_{34} - s_{34}]E(p_1, p_2) - [s_{23} + s_{34} - s_{34}]E(p_1, p_2) - [s_{23} + s_{34} - s_{34}]E(p_1, p_2) \quad (12)$$

The diagrams D_i^2 are listed below:

$$D_2^2(1) = \frac{1}{s_{12}s_{34}s_{123}} [s_{24} - s_{34} + s_{34}]E(p_1, p_2) - [s_{24} - s_{34} + s_{34}]E(p_1, p_2)$$

$$D_2^2(2) = \frac{1}{s_{12}s_{34}s_{123}} [s_{23} - s_{34} + s_{34}]E(p_1, p_2) - [s_{23} - s_{34} + s_{34}]E(p_1, p_2)$$

$$D_2^2(3) = \frac{1}{s_{12}s_{34}s_{123}} [s_{23} + s_{34} + s_{34}]E(p_1, p_2) - [s_{23} + s_{34} + s_{34}]E(p_1, p_2)$$

$$D_2^2(4) = \frac{1}{s_{12}s_{34}s_{123}} [s_{23} + s_{34} - s_{34}]E(p_1, p_2) - [s_{23} + s_{34} - s_{34}]E(p_1, p_2)$$

$$D_2^2(5) = \frac{1}{s_{12}s_{34}s_{123}} [s_{24} - s_{34} - s_{34}]E(p_1, p_2) - [s_{24} - s_{34} - s_{34}]E(p_1, p_2)$$

$$D_2^2(6) = \frac{1}{s_{12}s_{34}s_{123}} [s_{24} - s_{34} - s_{34}]E(p_1, p_2) - [s_{24} - s_{34} - s_{34}]E(p_1, p_2)$$

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$$D_2^2(15) = \frac{1}{s_{14}s_{23}s_{34}} [((p_1 + p_2)(p_3 - p_4))((p_1 - p_4)(p_2 - p_3)) + ((p_2 - p_3)(p_1 - p_4))((p_1 - p_4)(p_2 + p_3)) + ((p_1 + p_2)(p_3 - p_4))((p_1 - p_4)(p_2 - p_3))]$$

$$D_2^2(16) = \frac{2}{s_{14}s_{23}s_{34}} [((p_1 - p_2)(p_3 + p_4))((p_1 - p_4)(p_2 - p_3)) + ((p_1 + p_2)(p_3 - p_4))((p_1 - p_4)(p_2 - p_3)) + ((p_1 - p_2)(p_3 + p_4))((p_1 - p_4)(p_2 - p_3))] \quad (13)$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams \mathcal{D} are calculated by using eqs. (11)-(13). The result is substituted to eq. (8) to obtain the vectors \mathcal{D}_1 and \mathcal{D}_2 . After generating the vectors $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ and \mathcal{D}_5 by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_3 and A_2 . Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi-gluon amplitudes are tested by checking the gauge invariance. Due to the specific

Discovery of Shocking Simplicity

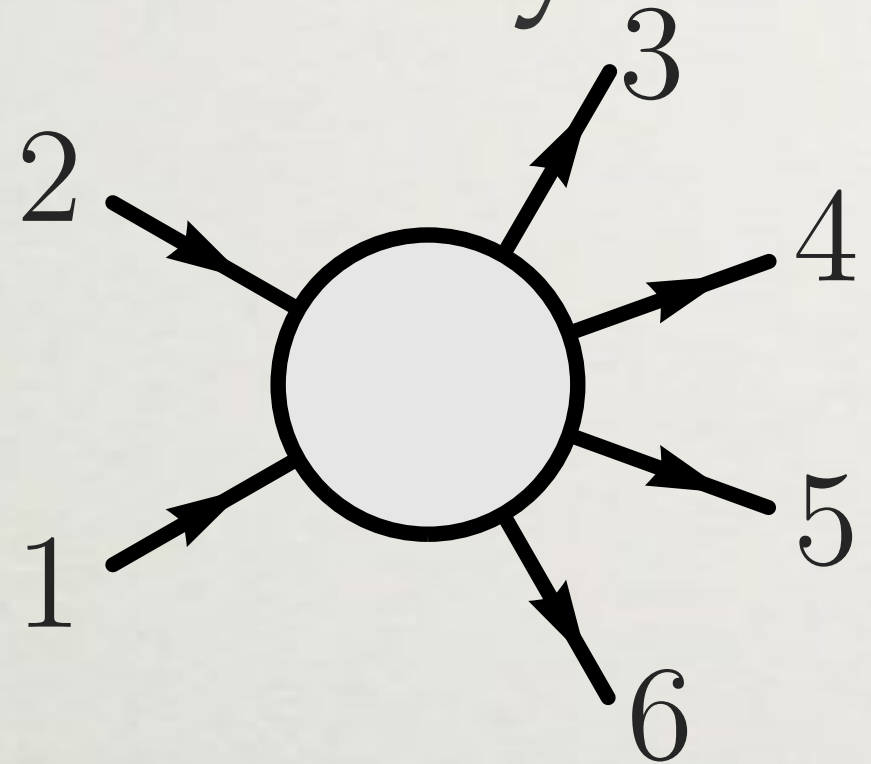


- ◆ Within six months, Parke-Taylor stumbled on a simple guess —unquestionably a *theorist's delight*:



Discovery of Shocking Simplicity

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$$= \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \langle 5 6 \rangle \langle 6 1 \rangle}$$

$$p_a^\mu \equiv \sigma_{\alpha\dot{\alpha}}^\mu \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}}$$

$$\langle a b \rangle \equiv \det(\lambda_a, \lambda_b)$$

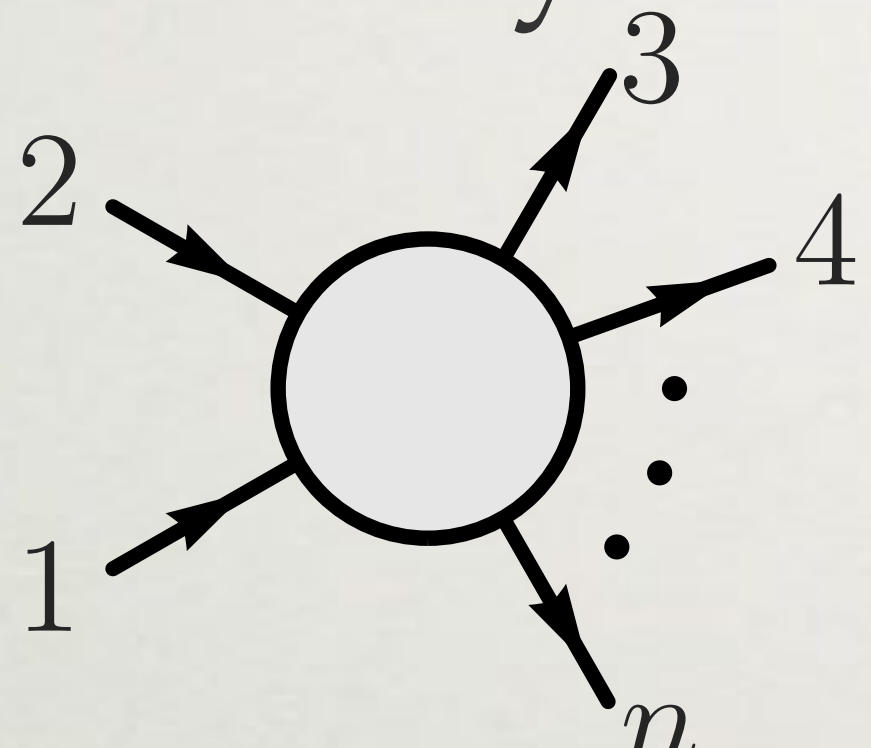
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[van der Waerden (1929)]



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Amplitude for n -Gluon Scattering [PRL 56 (1986)]

Stephen J. Parke and T. R. Taylor
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
 (Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

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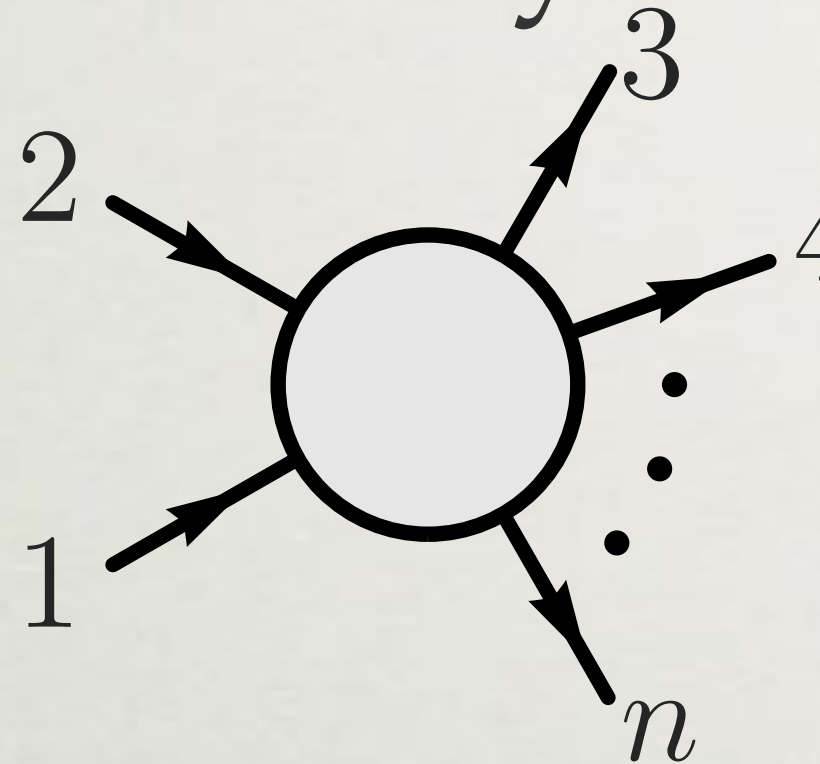
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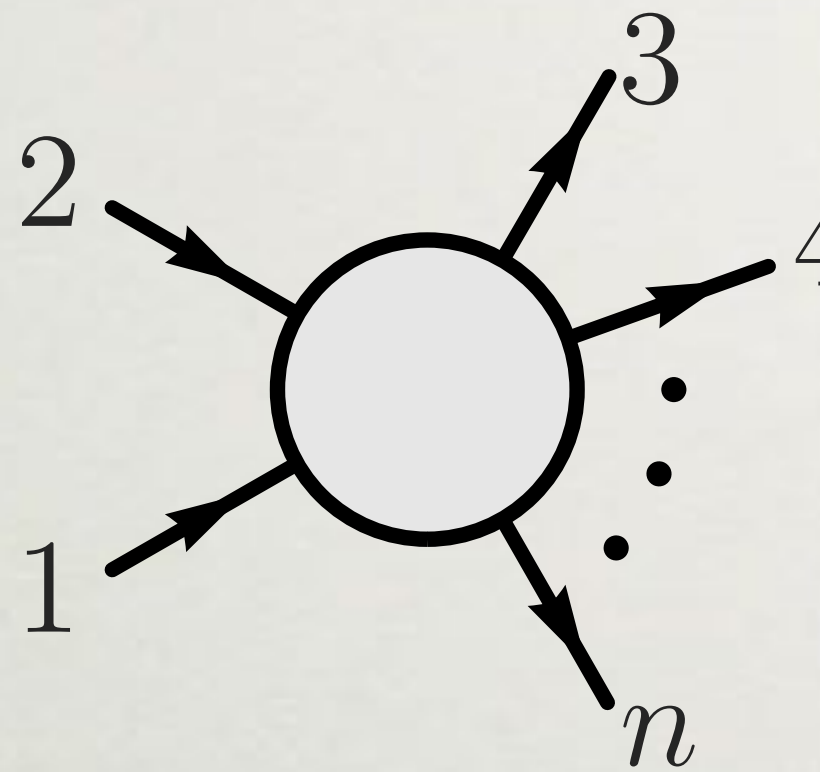

$$= \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle}$$

Goal: make the **simplicity** of amplitudes **manifest** in the way we compute them, **dramatically** extending the reach of the predictions we can make for experiment

Perturbations of Parke/Taylor's Guess



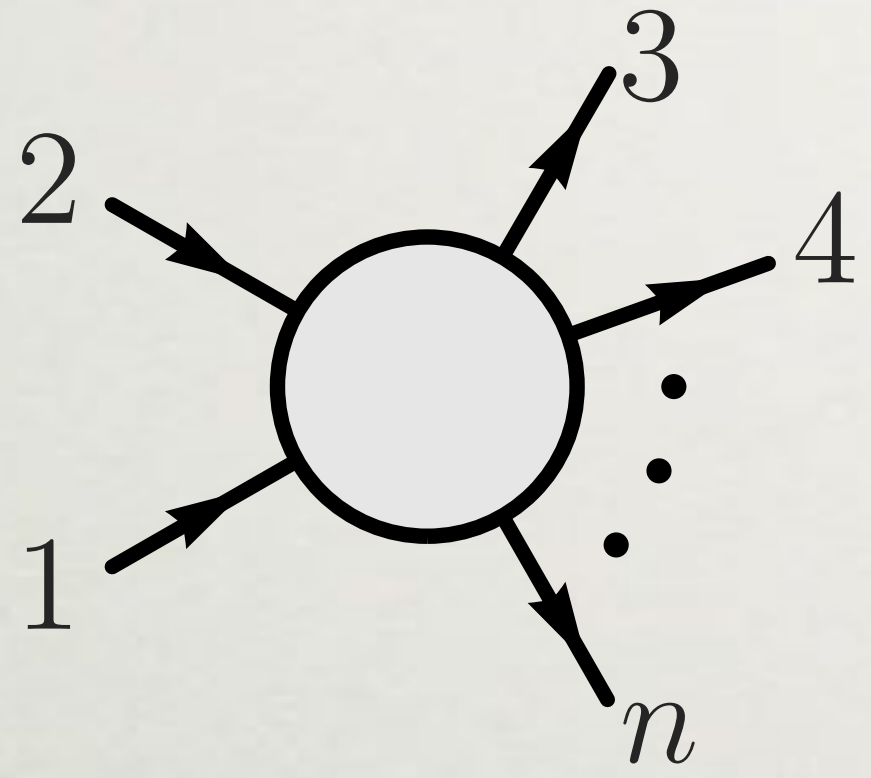
- ◆ What about beyond the leading order of approximation?


$$= \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle}$$

Perturbations of Parke/Taylor's Guess



- ◆ What about beyond the leading order of approximation?


$$= \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \times$$
$$\left\{ 1 + \dots \right\}$$

Perturbations of Parke/Taylor's Guess



- ◆ What about beyond the leading order of approximation?

[Bern, Dixon, Dunbar, Kosower (1994)]

$$\begin{aligned}
 & \left(\begin{array}{c} \text{Diagram: A circle with } n \text{ external legs labeled } 1, 2, 3, 4, \dots, n \end{array} \right) = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \times \\
 & \left\{ 1 + \sum_{a < b} \left(\begin{array}{c} \text{Diagram: A square with vertices } a \text{ and } b \text{ and a wavy line connecting them} \end{array} \right) + \dots \right\}
 \end{aligned}$$

Perturbations of Parke/Taylor's Guess



◆ What about beyond the leading order of approximation?

[Vergu (2009)]

3

complexity of the computations. It has also been useful to use the results for the cuts already computed when computing the coefficients of integrals detected by new cuts. In this way, one can linearize the consistency of results from different cuts and reduce the number of unknowns at the same time.

Let us make a further comment about our computation procedure. The conformal integrals with pentagon loops have numerators containing the loop momenta in combinations like $(k+l)^2$, where l is the loop momentum and k is an external on-shell momentum. If the propagator with momentum l is cut then, on that cut, one cannot distinguish between $(k+l)^2$ and $2k \cdot l$. However, it is easy to see that one can choose to cut another propagator and in that case this ambiguity does not arise and the numerator factor is uniquely defined.

IV. RESULTS

We use dual variable notation (see Ref. [48]) for the integrals. The external dual variables are listed in clockwise direction. To the left loop we associate the dual variable x_p and to the right loop we associate the dual variable x_q . We use the notation $x_{ij} = x_i - x_j$.

We introduce the following notation which will be useful in the following

$$\left[\begin{matrix} a & b & c & \dots \\ d & e & f & \dots \end{matrix} \right] = x_{da} x_{eb} x_{fc} \dots \pm (\text{permutations of } \{d, e, f, \dots\}). \quad (6)$$

The sign \pm above takes into account the signature of the permutation of $\{d, e, f, \dots\}$. It is easy to show that

$$\left[\begin{matrix} a & b & c & \dots \\ d & e & f & \dots \end{matrix} \right] = \frac{\det(x_{ij})}{x_{ij} x_{jk} x_{kl} \dots} x_{ij}^2. \quad (7)$$

For some topologies, the expansion of the $\left[\begin{matrix} \dots \\ \dots \end{matrix} \right]$ symbol yields terms that would cancel propagators. For those cases we make the convention that all the terms that would cancel propagators are absent. In fact, as we will see, terms that would cancel propagators of the double pentagon topologies naturally yield coefficients for some of the topologies with a smaller number of propagators.

A. Double box topologies

In the case of the double box topologies the massive legs attached to the vertices incident with the common edge have to be a sum of at least three massless momenta. The cases where these massive legs are the sum of two massless momenta are treated separately in the subsection. IV A 7. This distinction only arises for the double box topologies.

1. No legs attached

$$\frac{1}{2} (x_{2a+1}^2)^2 x_{2-1a+1}^2 \quad (8)$$

$$\frac{1}{4} (x_{2a}^2)^2 x_{2-1a+1}^2 \quad (9)$$

$$-\frac{1}{4} x_{2a}^2 (x_{2a-1}^2 x_{2-1a}^2 - x_{2a-1}^2 x_{2-1a+1}^2) \quad (10)$$

2. One massless leg attached

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 - x_{2a-1}^2 x_{2a+1}^2) x_{2-1a+1}^2 \quad (11)$$

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 + x_{2a-1}^2 x_{2a+1}^2 - x_{2a-1}^2 x_{2a+1}^2) \quad (12)$$

$$-\frac{1}{4} x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 \quad (13)$$

$$-\frac{1}{4} x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2 \quad (14)$$

3. Two massless legs attached

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 - x_{2a-1}^2 x_{2a+1}^2) x_{2-1a+1}^2 \quad (15)$$

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 - x_{2a-1}^2 x_{2a+1}^2 + x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 + x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2) \quad (16)$$

$$\frac{1}{4} (-x_{2a+1}^2 x_{2a-1}^2 x_{2a-1}^2 + x_{2a-1}^2 x_{2a+1}^2 x_{2a-1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2a-1}^2) \quad (17)$$

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 x_{2a-1}^2 - 2x_{2a-1}^2 x_{2a+1}^2 x_{2a-1}^2 + x_{2a-1}^2 x_{2a+1}^2 x_{2a-1}^2 + x_{2a+1}^2 x_{2a-1}^2 x_{2a-1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2a-1}^2) \quad (18)$$

4. One massive leg attached

$$\frac{1}{4} x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 \quad (19)$$

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2) \quad (20)$$

$$0 \quad (21)$$

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2) \quad (22)$$

5. One massless leg and one massive leg attached

$$0 \quad (23)$$

$$0 \quad (24)$$

$$-\frac{1}{4} x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 \quad (25)$$

$$0 \quad (26)$$

$$\frac{1}{4} x_{2a+1}^2 (x_{2a-1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2a+1}^2) \quad (27)$$

$$\frac{1}{4} (-x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 + x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2 + x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2) \quad (28)$$

$$0 \quad (29)$$

$$0 \quad (30)$$

$$0 \quad (31)$$

6. Two massive legs attached

$$0 \quad (32)$$

$$0 \quad (33)$$

$$0 \quad (34)$$

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & a+2 \\ a+3 & a+4 & a-2 \end{matrix} \right] \quad (42)$$

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2) x_{2-1a+1}^2 \quad (43)$$

$$-\frac{1}{4} \left[\begin{matrix} a-1 & a & a+1 \\ a+3 & a+4 & a-3 \end{matrix} \right] \quad (44)$$

$$0 \quad (45)$$

$$0 \quad (46)$$

$$-\frac{1}{2} \left[\begin{matrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{matrix} \right] \quad (47)$$

$$0 \quad (48)$$

$$-\frac{1}{4} \left[\begin{matrix} a-2 & a-1 & a \\ a+2 & b-1 & b \end{matrix} \right] \quad (49)$$

$$-\frac{1}{4} \left[\begin{matrix} a-3 & a-2 & a-1 \\ a+1 & a+2 & a+3 \end{matrix} \right] \quad (50)$$

$$0 \quad (51)$$

$$0 \quad (52)$$

7. Extra double boxes

$$\frac{1}{4} (-x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 + x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2 + x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2) \quad (53)$$

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 \\ b & b+1 & a-1 \end{matrix} \right] \quad (54)$$

$$0 \quad (55)$$

$$-\frac{1}{4} \left[\begin{matrix} a+1 & a+2 & b-1 & b \\ b & b+1 & a-1 & a \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a & a+1 \\ b & b+1 \end{matrix} \right] \left[\begin{matrix} b & b+1 \\ a-1 & a \end{matrix} \right] \quad (56)$$

$$-\frac{1}{4} \left[\begin{matrix} a+1 & a+2 & b-1 & b \\ b+1 & b+2 & a-1 & a \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a+1 & a+2 \\ b-1 & b \end{matrix} \right] \left[\begin{matrix} b+1 & b+2 \\ a-1 & a \end{matrix} \right] \quad (57)$$

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 & b \\ b+1 & b+2 & c-1 & c \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a & a+1 \\ b-1 & b \end{matrix} \right] \left[\begin{matrix} b+1 & b+2 \\ c-1 & c \end{matrix} \right] \quad (58)$$

$$-\frac{1}{4} \left[\begin{matrix} a+1 & b-1 & b \\ c & c+1 & d-1 & d \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a & a+1 \\ b-1 & b \end{matrix} \right] \left[\begin{matrix} c & c+1 \\ d-1 & d \end{matrix} \right] \quad (59)$$

B. Kissing double-box topologies

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 \\ b+1 & c-1 & c \end{matrix} \right] \quad (53)$$

$$-\frac{1}{4} \left[\begin{matrix} a+1 & b-1 & b \\ b & b+1 & a-1 \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a & a+1 \\ b & b+1 \end{matrix} \right] \left[\begin{matrix} b & b+1 \\ a-1 & a \end{matrix} \right] \quad (54)$$

$$-\frac{1}{4} \left[\begin{matrix} a+1 & a+2 & b-1 & b \\ b & b+1 & a-1 & a \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a-1 & a \\ b & b+1 \end{matrix} \right] \left[\begin{matrix} a+1 & a+2 \\ b-1 & b \end{matrix} \right] \quad (55)$$

$$-\frac{1}{4} \left[\begin{matrix} a+1 & a+2 & b-1 & b \\ b+1 & b+2 & a-1 & a \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a+1 & a+2 \\ b-1 & b \end{matrix} \right] \left[\begin{matrix} b+1 & b+2 \\ a-1 & a \end{matrix} \right] \quad (56)$$

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 & b \\ b & b+1 & c-1 & c \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a & a+1 \\ b & b+1 \end{matrix} \right] \left[\begin{matrix} b & b+1 \\ c-1 & c \end{matrix} \right] \quad (57)$$

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 & b \\ b+1 & b+2 & c-1 & c \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a & a+1 \\ b & b+1 \end{matrix} \right] \left[\begin{matrix} b+1 & b+2 \\ c-1 & c \end{matrix} \right] \quad (58)$$

$$-\frac{1}{4} \left[\begin{matrix} a+1 & b-1 & b \\ c & c+1 & d-1 & d \end{matrix} \right] + \frac{1}{4} \left[\begin{matrix} a & a+1 \\ b-1 & b \end{matrix} \right] \left[\begin{matrix} c & c+1 \\ d-1 & d \end{matrix} \right] \quad (59)$$

C. Hex-pentagon topologies

1. No legs attached

$$\frac{1}{2} x_{2a+1}^2 x_{2a-1}^2 (x_{2a+1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2-1a+1}^2) \quad (60)$$

$$\frac{1}{2} x_{2a+1}^2 x_{2a-1}^2 (x_{2a+1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2-1a+1}^2) \quad (61)$$

2. One massless leg attached

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 - x_{2a-1}^2 x_{2a+1}^2) (x_{2a+1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2-1a+1}^2) \quad (62)$$

$$\frac{1}{4} x_{2a+1}^2 x_{2a-1}^2 (x_{2a+1}^2 x_{2-1a+1}^2 + x_{2a-1}^2 x_{2-1a+1}^2 - x_{2a+1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2-1a+1}^2) \quad (63)$$

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2 + 2x_{2a+1}^2 x_{2a-1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2a+1}^2 x_{2-1a+1}^2) \quad (64)$$

3. One massive leg attached

$$0 \quad (65)$$

$$\frac{1}{4} (x_{2a+1}^2 x_{2a-1}^2 - x_{2a-1}^2 x_{2a+1}^2) (x_{2a+1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2-1a+1}^2) \quad (66)$$

$$\frac{1}{4} x_{2a+1}^2 x_{2a-1}^2 (x_{2a+1}^2 x_{2-1a+1}^2 - x_{2a-1}^2 x_{2-1a+1}^2) \quad (67)$$

4. One massless, one massive leg attached

$$0 \quad (68)$$

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b & b+1 \\ b+2 & c-1 & c & q \end{matrix} \right] \quad (69)$$

Note that in the previous formula we suppress the terms containing x_{2a+1}^2 which would otherwise cancel a propagator of the underlying topology. When expanded out, the expression above has 12 terms.

$$-\frac{1}{4} \left[\begin{matrix} a-2 & a-1 & a+1 \\ a+2 & b-1 & b & q \end{matrix} \right] \quad (70)$$

In the previous formula we suppress the terms containing x_{2a+1}^2 which would otherwise cancel a propagator of the underlying topology.

5. Two massless legs attached

$$\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 & b \\ b+1 & b+2 & c-1 & q \end{matrix} \right] \quad (71)$$

In the previous formula we suppress the terms containing x_{2a+1}^2 which would otherwise cancel a propagator of the underlying topology.

D. Double pentagon topologies

1. No legs attached

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 & b \\ b+1 & b+2 & c-1 & q \end{matrix} \right] \quad (73)$$

In the expansion of the above formula we drop terms that would cancel propagators (in this case, the terms containing x_{2a}^2 , x_{2b}^2 , x_{2c}^2 , or x_{2q}^2). This expression has 6 terms when expanded.

2. One massless leg attached

$$-\frac{1}{4} \left[\begin{matrix} a+1 & a+2 & b-1 & b & p \\ b & b+1 & a-1 & a & q \end{matrix} \right] \quad (74)$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing x_{2a}^2 , x_{2b}^2 , and x_{2q}^2). This expression has 15 terms when expanded.

3. One massive leg attached

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 & b & p \\ b & b+1 & c-1 & c & q \end{matrix} \right] \quad (75)$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing x_{2a}^2 , x_{2b}^2 , or x_{2q}^2). This expression has 16 terms when expanded.

4. Two massless legs attached

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 & b & p \\ b+1 & b+2 & c-1 & a & q \end{matrix} \right] \quad (76)$$

In the formula we drop terms that would cancel propagators (in this case, the terms containing x_{2a}^2). This expression has 64 terms when expanded.

5. One massless, one massive leg attached

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 & b & p \\ b+1 & b+2 & c-1 & c & q \end{matrix} \right] \quad (77)$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing x_{2a}^2). This expression has 78 terms when expanded.

6. Two massive legs attached

$$-\frac{1}{4} \left[\begin{matrix} a & a+1 & b-1 & b & p \\ c & c+1 & d-1 & d & q \end{matrix} \right] \quad (78)$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing x_{2a}^2). When expanded, the above expression contains 96 terms. The number of conformal drawings is 160 (the number of coefficients unrelated by symmetries is lower).

E. Assembly of the result

As explained in Sec. II, for the MHV amplitudes the ratio between the l -loop amplitude and the tree-level amplitude can be written as a sum between parity even and parity odd contributions

$$M_n^{(l)} = M_n^{(l, \text{even})} + M_n^{(l, \text{odd})}. \quad (79)$$

Then, the even part can be written

$$M_n^{(l, \text{even})} = -\pi^{2-2n} \int d^4 x \mu^2 \sum_{a \in \text{even}} \sum_{c_i \in I_i} c_i I_i. \quad (80)$$

where the first sum runs over cyclic and anti-cyclic permutations of the external legs, the second sum runs over all the topologies, a_i is a symmetry factor associated to topology i , c_i is the numerator of the topology i , as listed in Sec. IV and I_i is the denominator or the product of propagators in the topology i .

Apart from the parity odd part which we have not computed, there is also a contribution which is not detectable from four-dimensional cuts, denoted by $M_n^{(l, \text{odd})}$. This part of the result is such that its integral vanishes in four dimensions, but the integral itself can give contributions to the divergent and finite parts. In Ref. [52], for $n=6$ case, this part of the result was found to be closely related to $\text{Cr}(C)$ contributions at one loop, $M_n^{(l, \text{odd})}$.

Based on previous computations we expect that the odd part and the μ integrals will not be needed in order to compare with the Wilson loop results. The odd parts could be

Perturbations of Parke/Taylor's Guess



- ◆ What about beyond the leading order of approximation?

[Arkani-Hamed, **JB**, Cachazo, Trnka (2010)]

$$\begin{aligned}
 & \text{Diagram: A circle with four external legs labeled 1, 2, 3, 4 and a vertical ellipsis of dots. A label 'n' is placed near the bottom of the circle, with a line connecting it to the first diagram in the sum below.} \\
 & = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \times \\
 & \left\{ 1 + \sum_{a < b} \text{Diagram 1} + \sum_{a < b < c < d} \text{Diagram 2} + \dots \right\}
 \end{aligned}$$

Diagram 1: A square with vertices \$a\$ (top-left), \$b\$ (top-right), and two black dots at the bottom. A wavy line connects the two bottom dots.

Diagram 2: A hexagon with vertices \$a\$ (top-left), \$b\$ (top-right), \$c\$ (bottom-right), and \$d\$ (bottom-left). Two black dots are on the left and right vertical edges. Wavy lines connect the top and bottom dots.

Perturbations of Parke/Taylor's Guess



- ◆ What about beyond the leading order of approximation?

[Arkani-Hamed, **JB**, Cachazo, Trnka (2011)]

$$\begin{aligned}
 & \text{Diagram: A circle with } n \text{ external legs labeled } 1, 2, 3, 4, \dots, n. \\
 & = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \dots \langle n 1 \rangle} \times \\
 & \left\{ 1 + \sum_{a < b} \text{Diagram 1} + \sum_{a < b < c < d} \text{Diagram 2} \right. \\
 & \quad + \sum_{\substack{a < b \leq c < \\ < d \leq e < f}} \text{Diagram 3} + \left. \sum_{\substack{a \leq b < c < \\ < d \leq e < f}} \text{Diagram 4} + \dots \right\}
 \end{aligned}$$

The diagrams are Feynman-like diagrams with external legs labeled a, b, c, d, e, f . Diagram 1 shows a box with a wavy line. Diagram 2 shows a box with two wavy lines. Diagram 3 shows a box with four wavy lines. Diagram 4 shows a box with three wavy lines.

What Form do Observables Take?

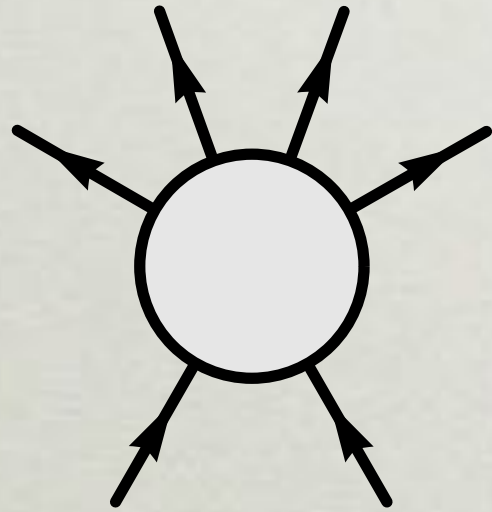


- ◆ But what about after **regularization** and *loop integration*?
What is the *mathematical form* of the predictions made by QFT?

What Form do Observables Take?



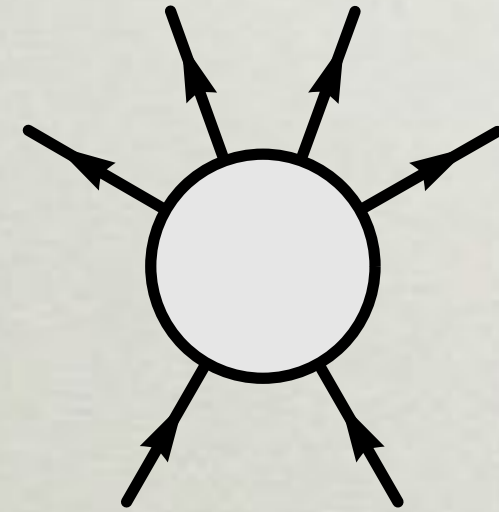
- ◆ But what about after **regularization** and *loop integration*?
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What Form do Observables Take?



- ◆ But what about after **regularization** and *loop integration*?
What is the *mathematical form* of the predictions made by QFT?



The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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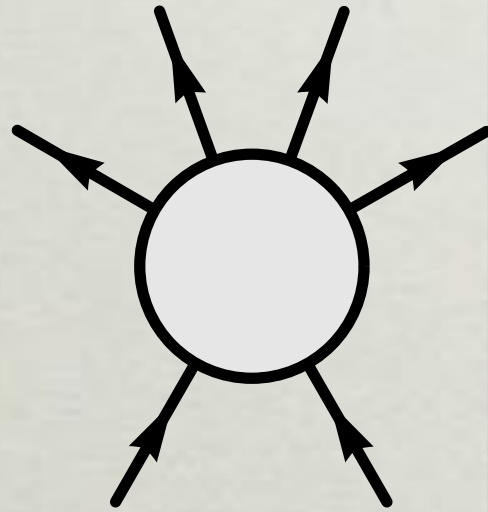
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[Del Duca, Duhr, Smirnov (2010)]

What Form do Observables Take?



◆ But what about after **regularization** and **loop integration**?
What is the **mathematical form** of the predictions made by QFT?



Claude Duhr

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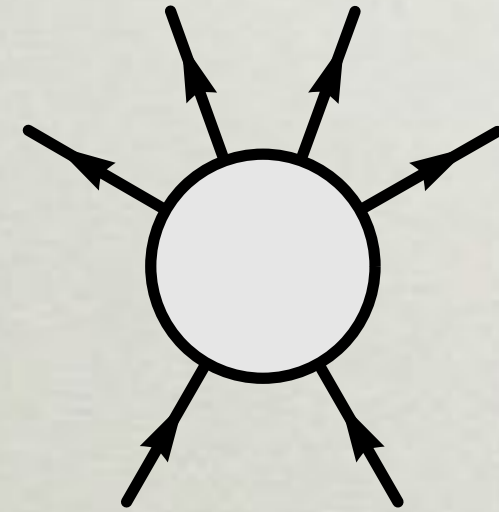
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What Form do Observables Take?



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What is the *mathematical form* of the predictions made by QFT?



Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,¹ M. Spradlin,² C. Vergu,² and A. Volovich²

¹*Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*

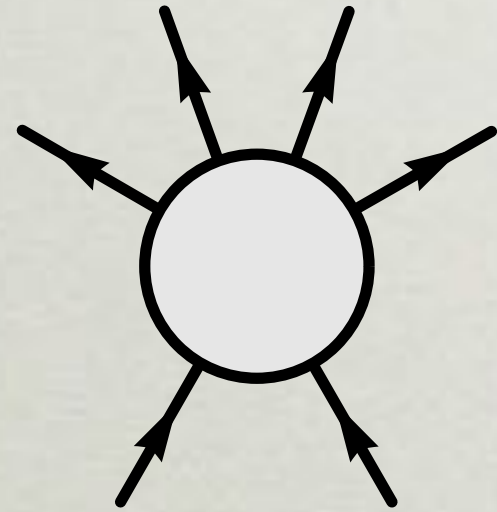
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We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

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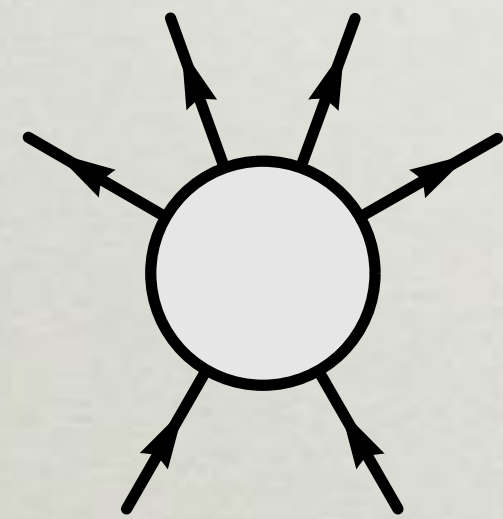
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$$R(u_1, u_2, u_3) = \sum_{i=1}^3 \left(L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) - \frac{1}{8} \left(\sum_{i=1}^3 \text{Li}_2(1 - 1/u_i) \right)^2 + \frac{J^4}{24} + \frac{1}{2} \zeta_2 (J^2 + \zeta_2)$$

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State of the art:

6-point (N)MHV @ (6) 7 loops(!!!)

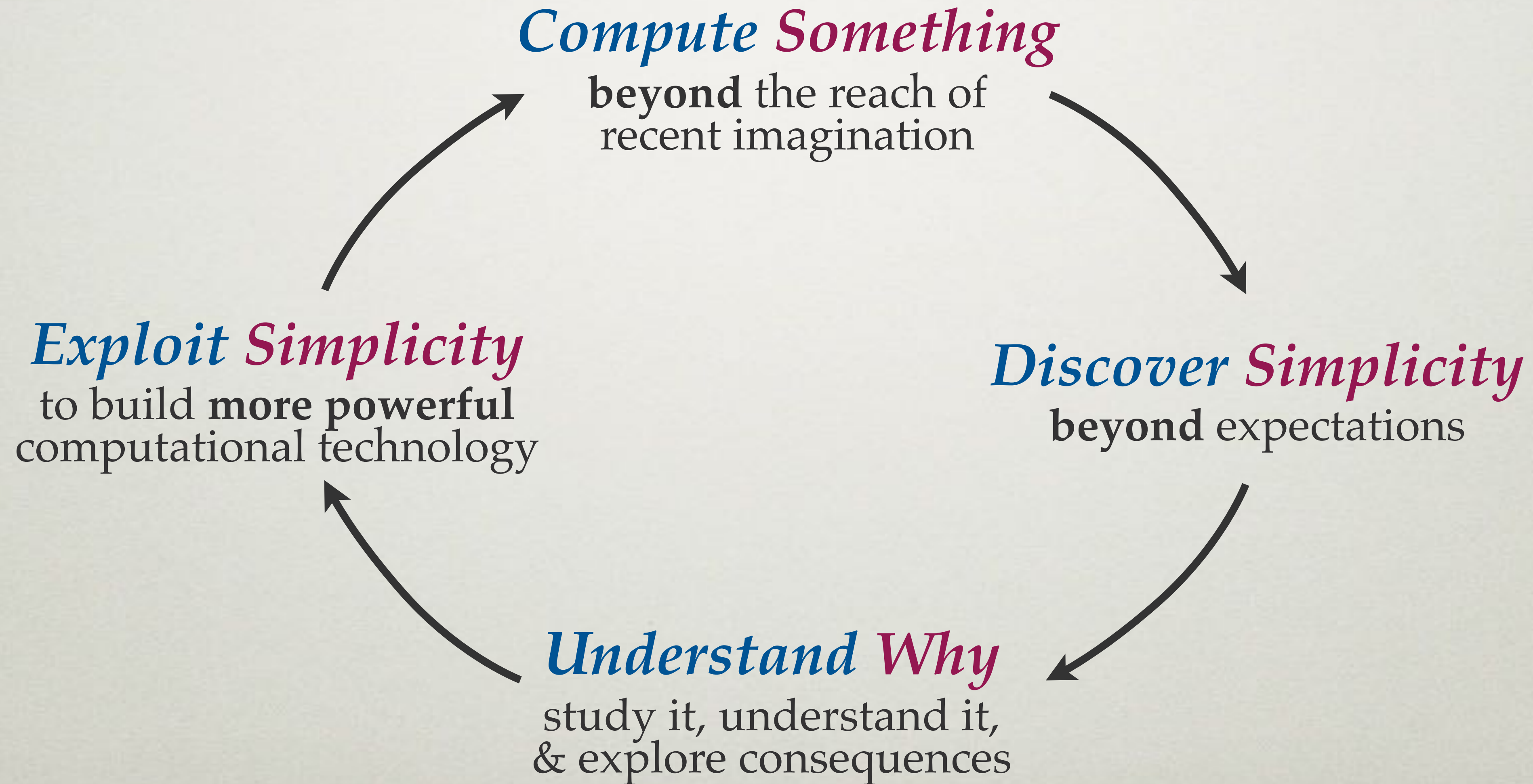
7-point (N)MHV @ 4 loops (symbol-level)

[Dixon, *et al* (2019);...]

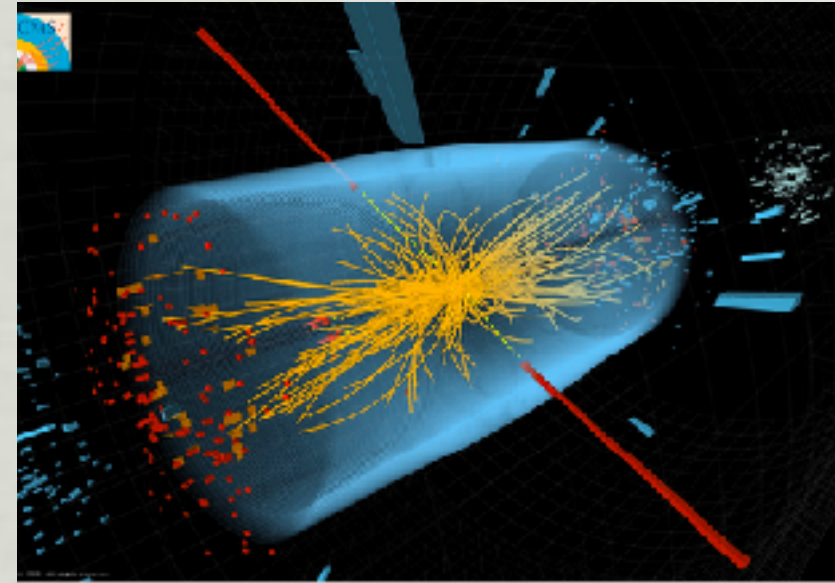
Amplitudes: a Virtuous Cycle



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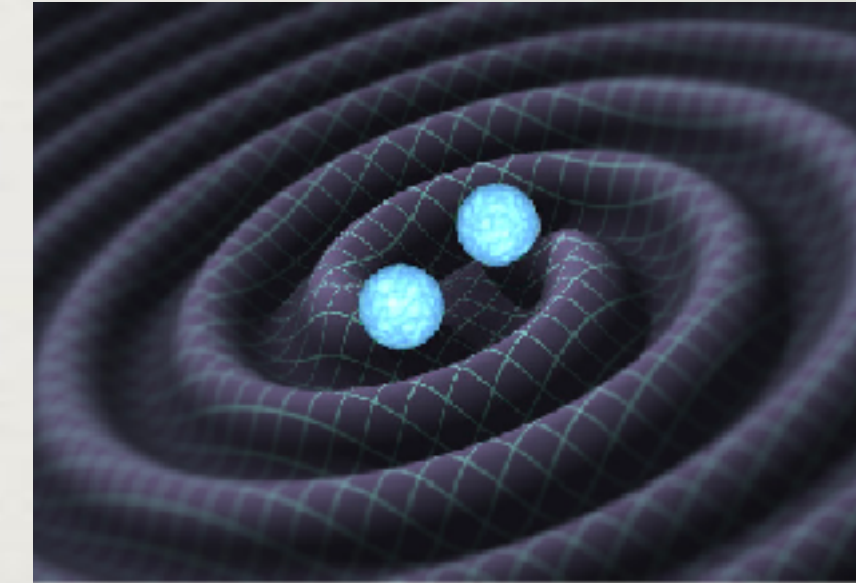


Amplitudes: a Virtuous Cycle



Compute Something

beyond the reach of
recent imagination

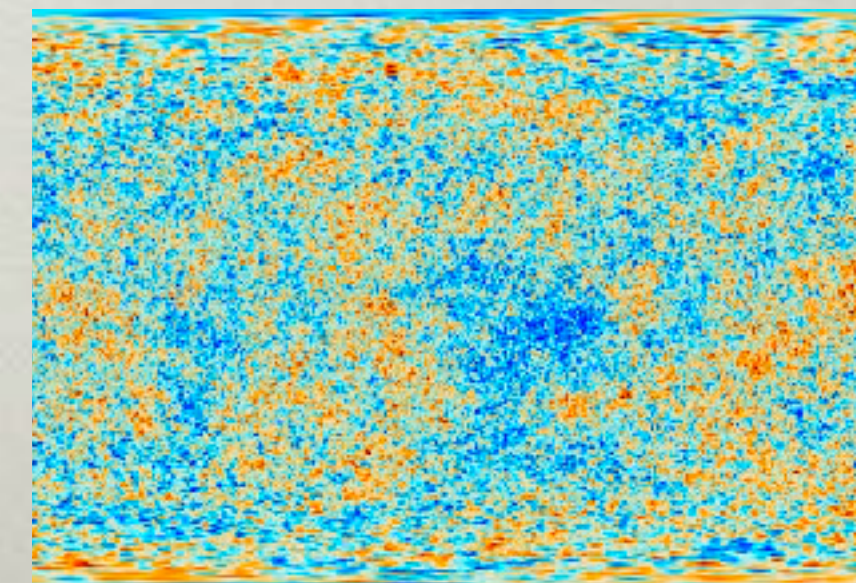


Exploit Simplicity
to build more powerful
computational technology

Discover Simplicity
beyond expectations



Understand Why
study it, understand it,
& explore consequences



Why is Perturbation Theory so Hard?



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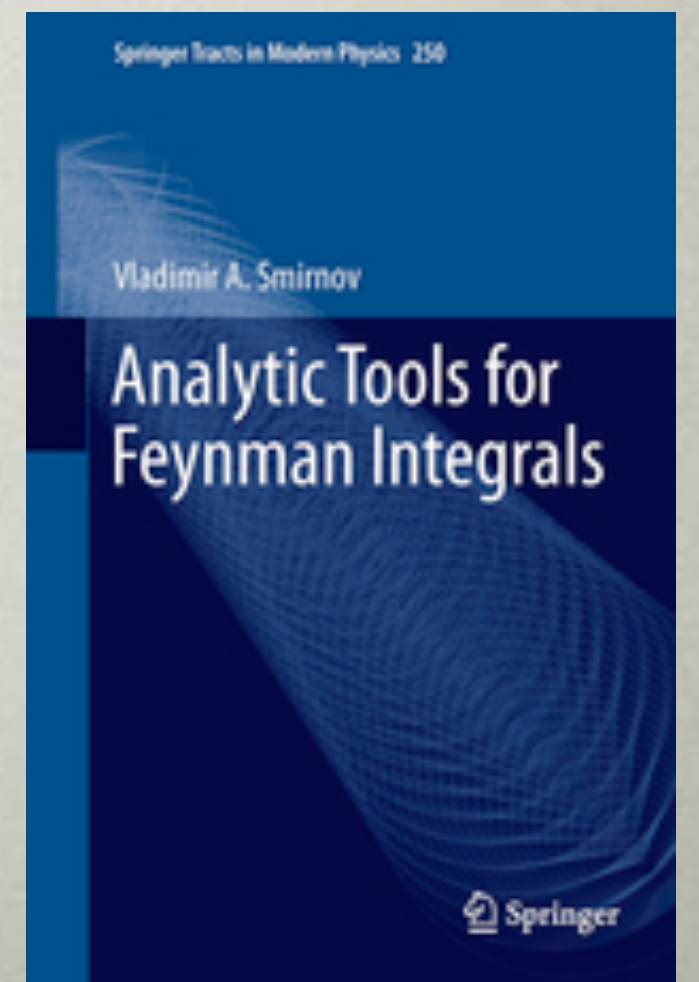


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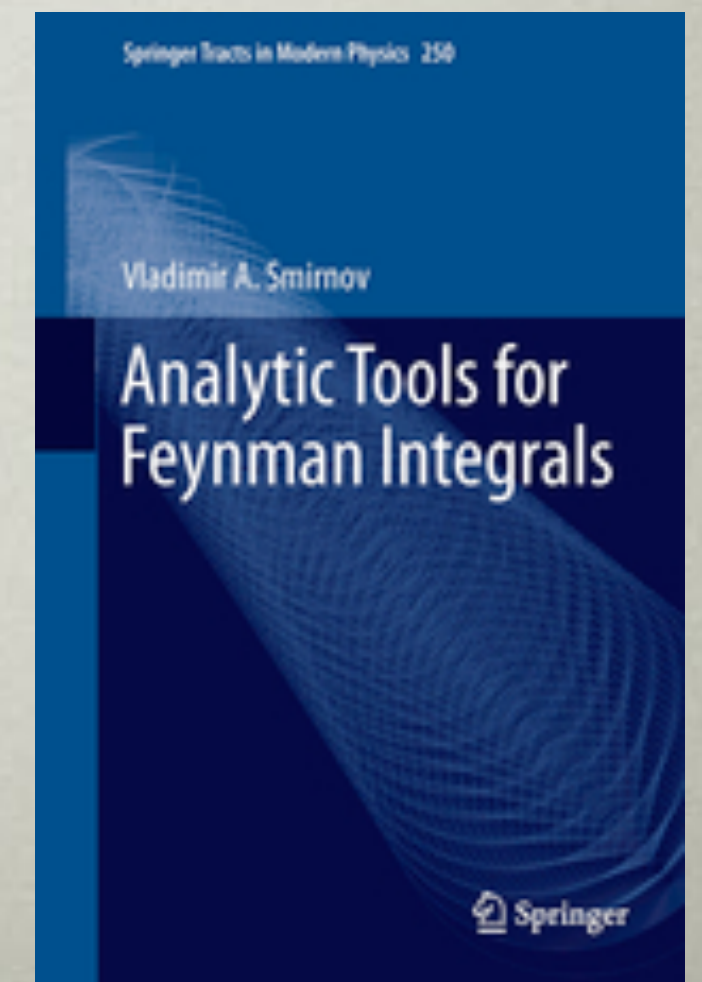
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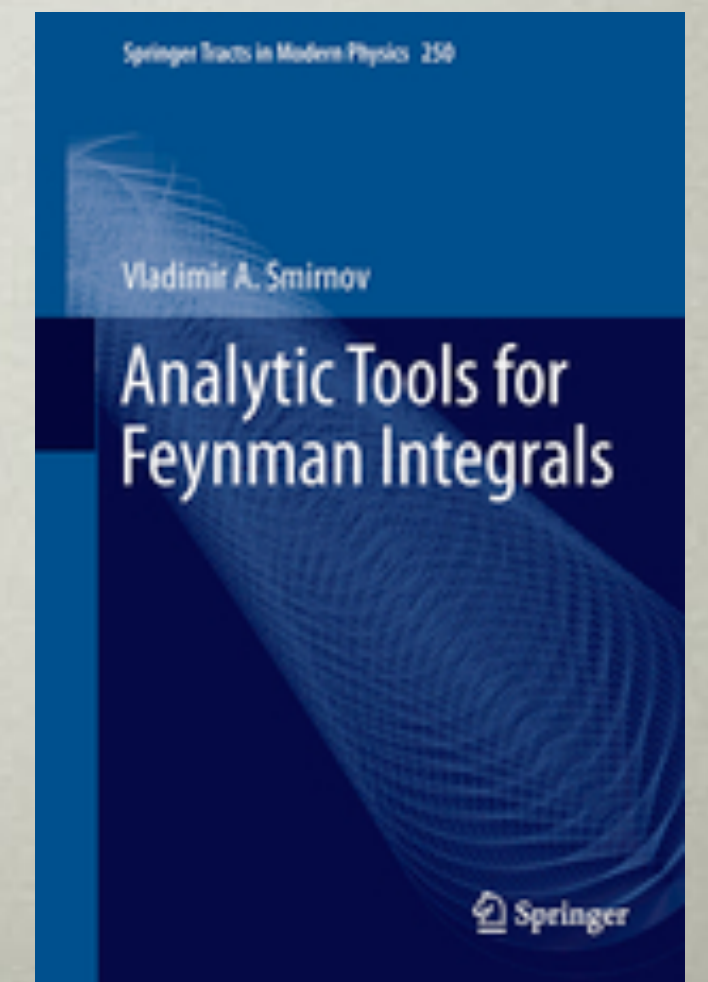
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$$\frac{\partial}{\partial s} [g(s) \log(f(s))] = g'(s) \log(f(s)) + g(s) f'(s) / f(s)$$

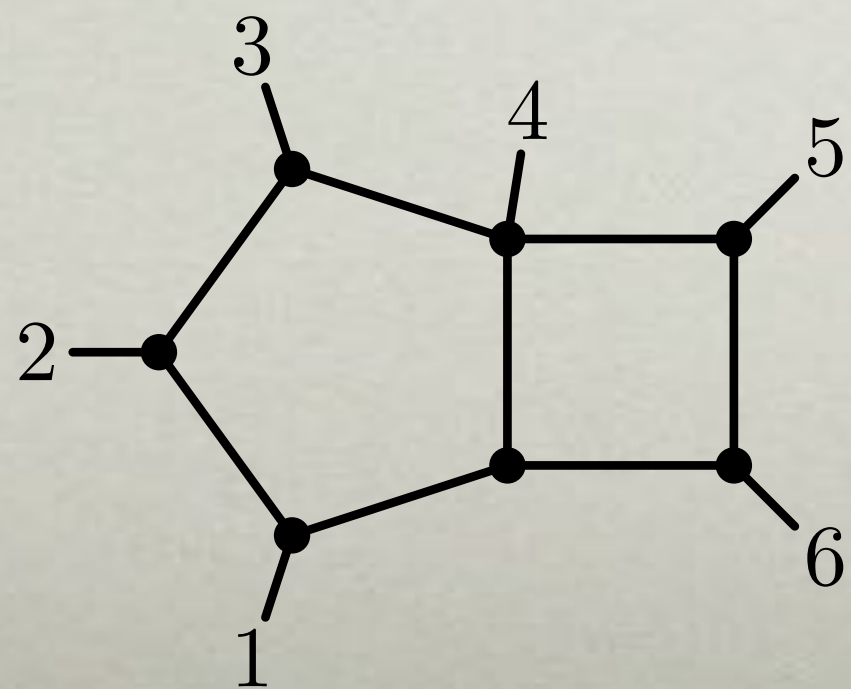


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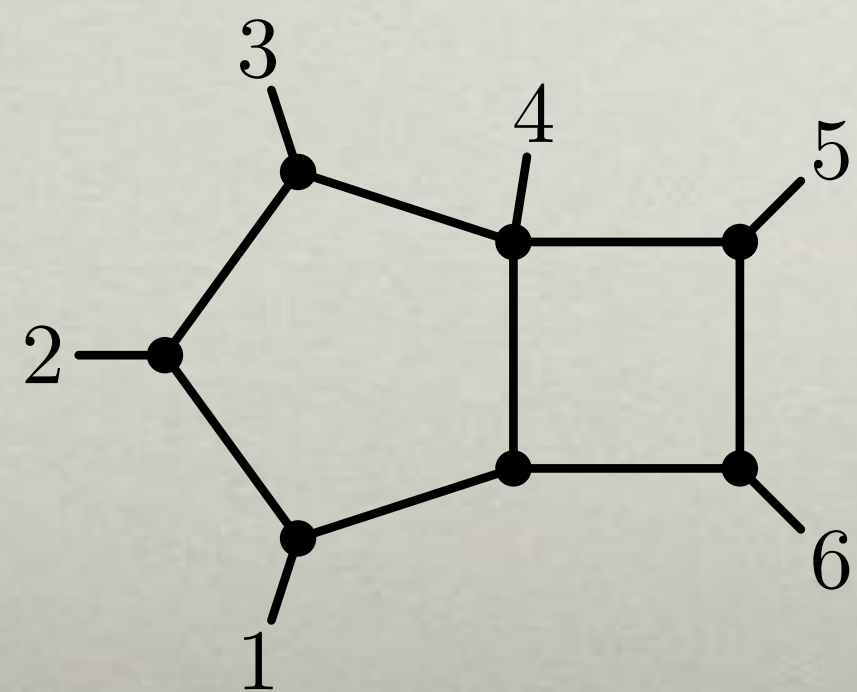


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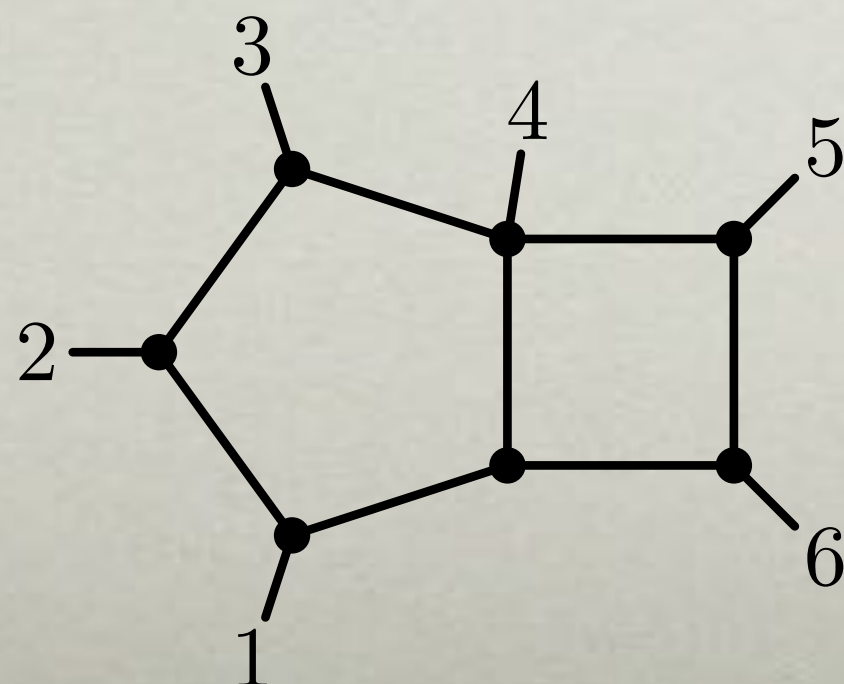


$$\Rightarrow \frac{s_{13}}{s_{12} s_{23} s_{56}} \left(\begin{aligned} &\text{tr}_+ [p_3, p_{12}, p_6, p_1] \left(\text{Li}_4(\dots)'s + \dots \right) \\ &+ \text{tr}_+ [p_{12}, p_6, p_1, p_3] \left(\text{Li}_4(\dots)'s + \dots \right) \end{aligned} \right)$$



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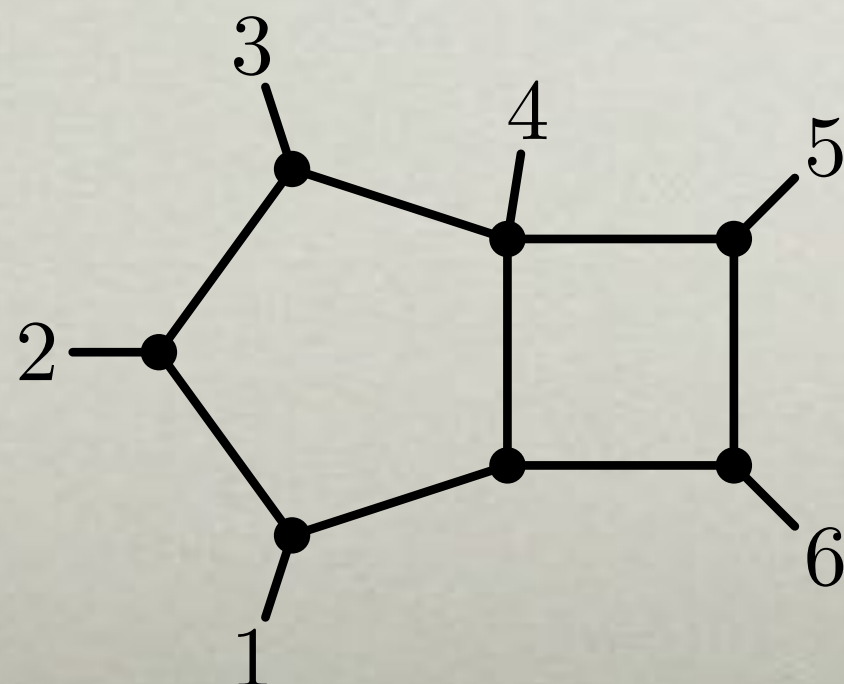
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- ◆ **Avoid regularization** whenever possible:

- ▶ can all(?) *finite* quantities be computed *without regularization*?
 - without expanding them in terms of divergent integrals?

(Answer: sometimes)

[JB, Langer, Patatoukos (2021); ...]



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*Improving Loop Integration by
Building Better Bases*

Generalized Unitarity: a modern take



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- ◆ **Key observation:** viewed as a potential element of *some* basis, *every* Feynman integrand can be interesting!
 - ▶ Why not try to find the *best/easiest* integrands—and use these?

Stratifying Quantum Field Theories



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$$\mathfrak{B}^{\text{SM}} \supset \mathfrak{B}^{\mathcal{N}=2} \supset \mathfrak{B}^{\mathcal{N}=4}$$



Stratifying Integrand Bases

- ◆ Suppose that a basis could be carved up into **subspaces** (by any arbitrary means):

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 - ▶ recently, we gave an **intrinsically graph-theoretic** definition of power-counting for ***non-planar*** integrand bases

[JB, Herrmann, Langer, Trnka (2020)]



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- $\{\text{finite}\} \oplus \{\text{divergent}\}$

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$$\left\{ \text{finite} \right\} \oplus \left\{ \left(\text{UV-divergent} \right) \right\} \oplus \left\{ \left(\text{IR-divergent} \right) \right\}$$

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- ◆ ¿Is it possible to stratify integrand bases by *physical structure*?

$$\left\{ \text{finite} \right\} \oplus \left\{ \left(\mathcal{O}(1/\epsilon^{2L})\text{-divergent} \right) \oplus \left(\mathcal{O}(1/\epsilon^{2L-1})\text{-divergent} \right) \oplus \cdots \oplus \left(\mathcal{O}(1/\epsilon)\text{-divergent} \right) \right\}$$
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- Can we further stratify each part by *transcendental structure*?

{ finite }

{ max-weight } ⊕ { next-to-max-weight } ⊕ ⋯ ⊕ { rational }

{ polylogs } ⊕ { elliptic-polylogs } ⊕ { K3-polylogs } ⊕ ⋯

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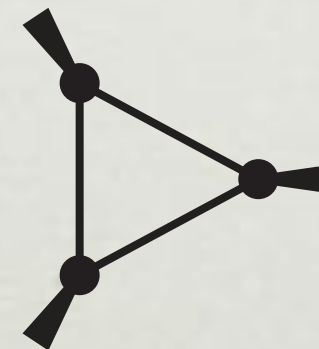
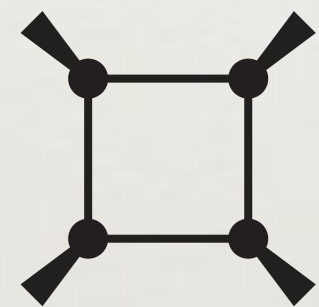


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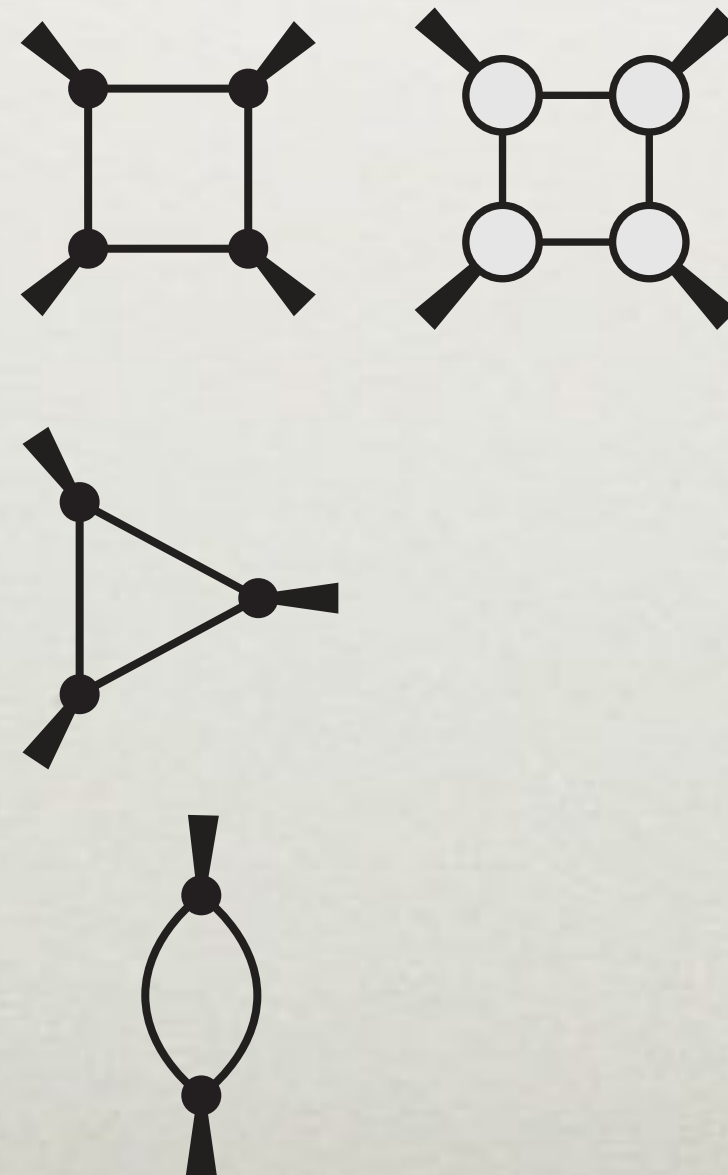


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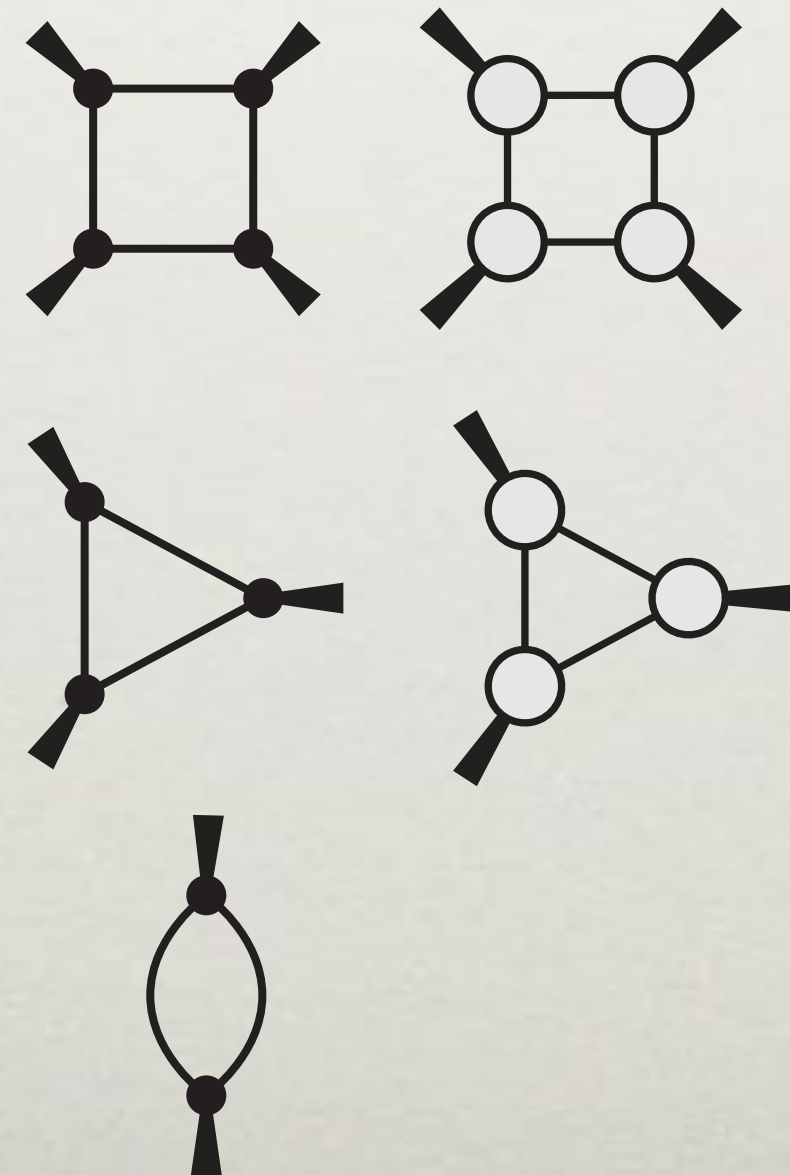


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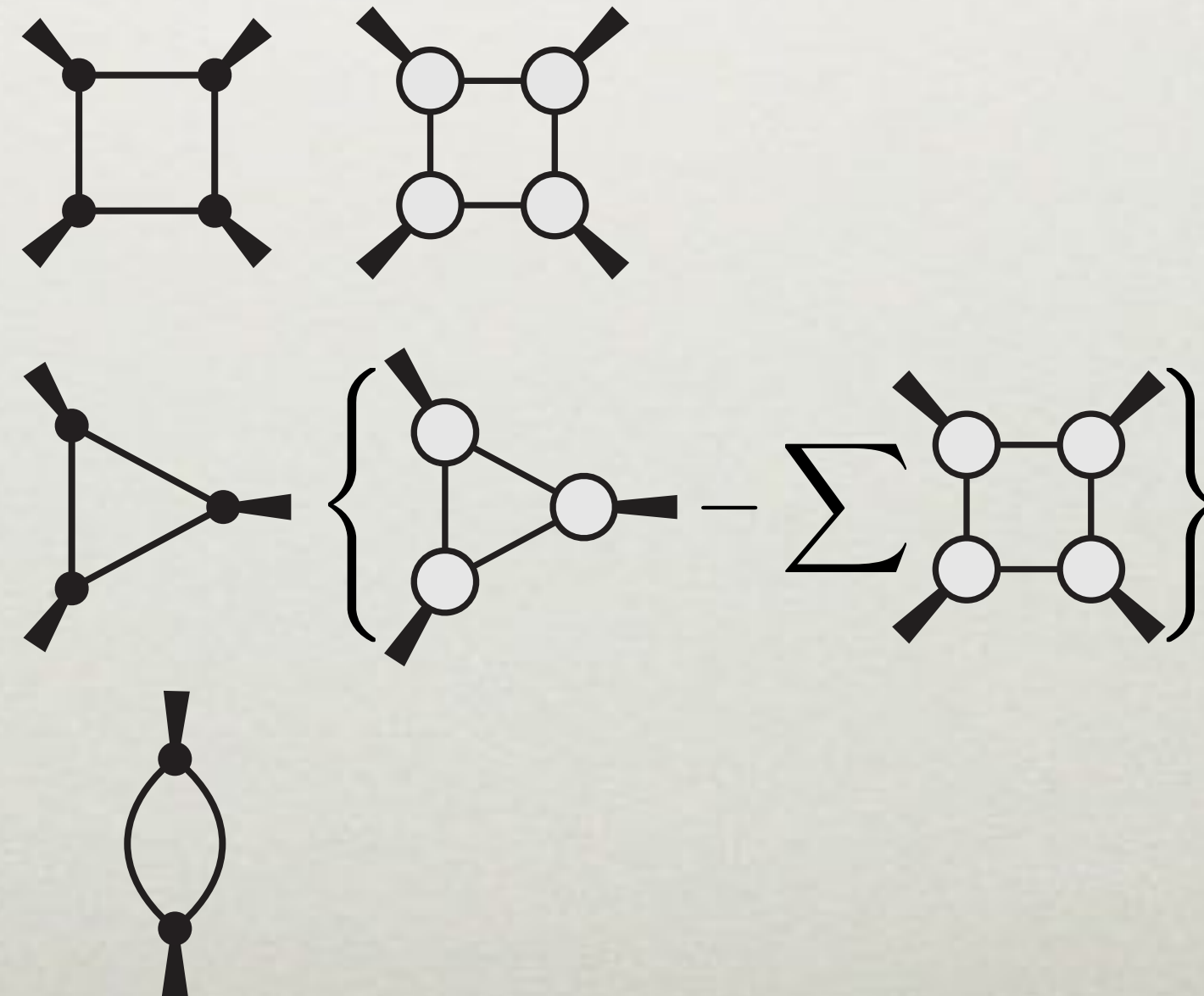


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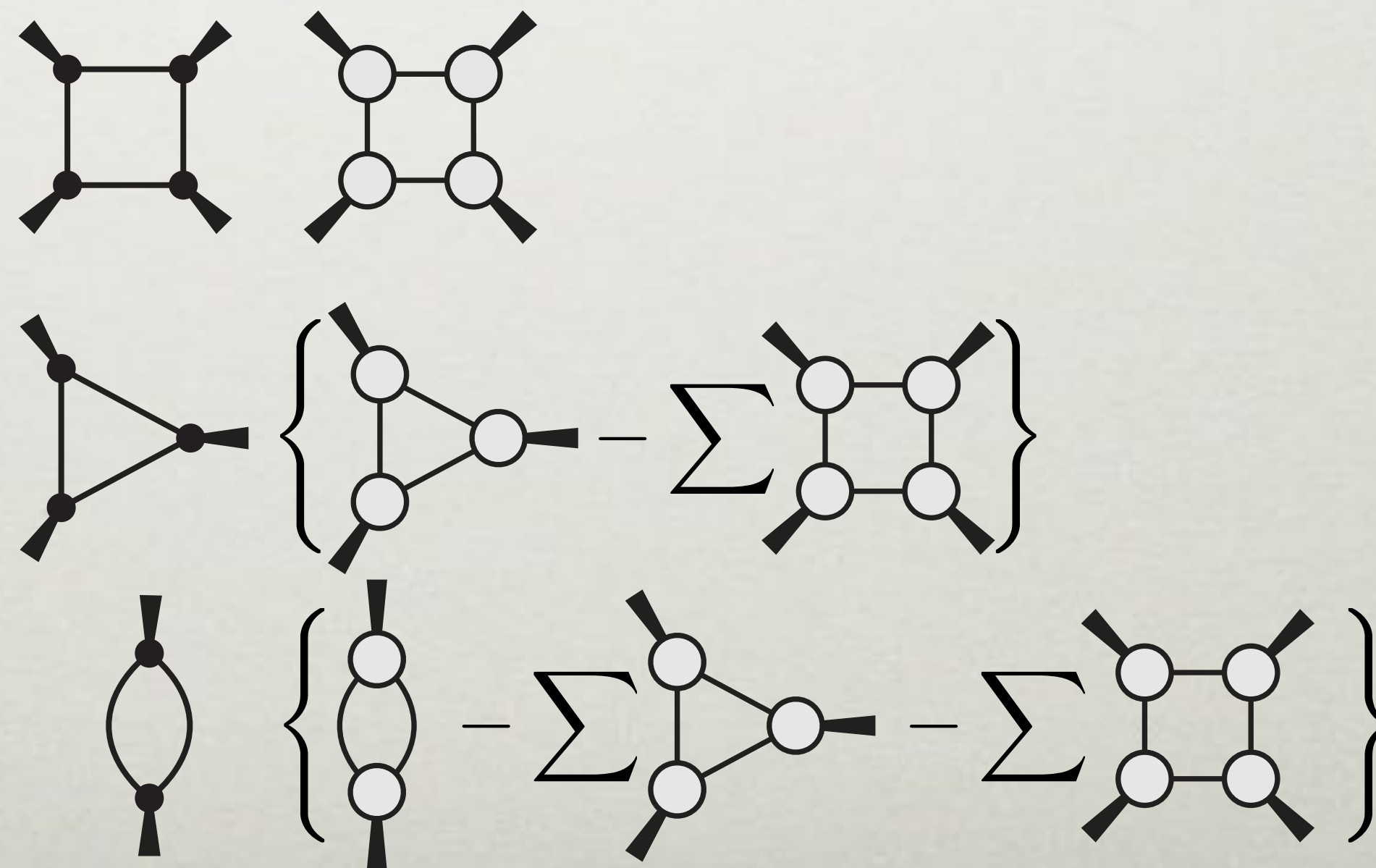


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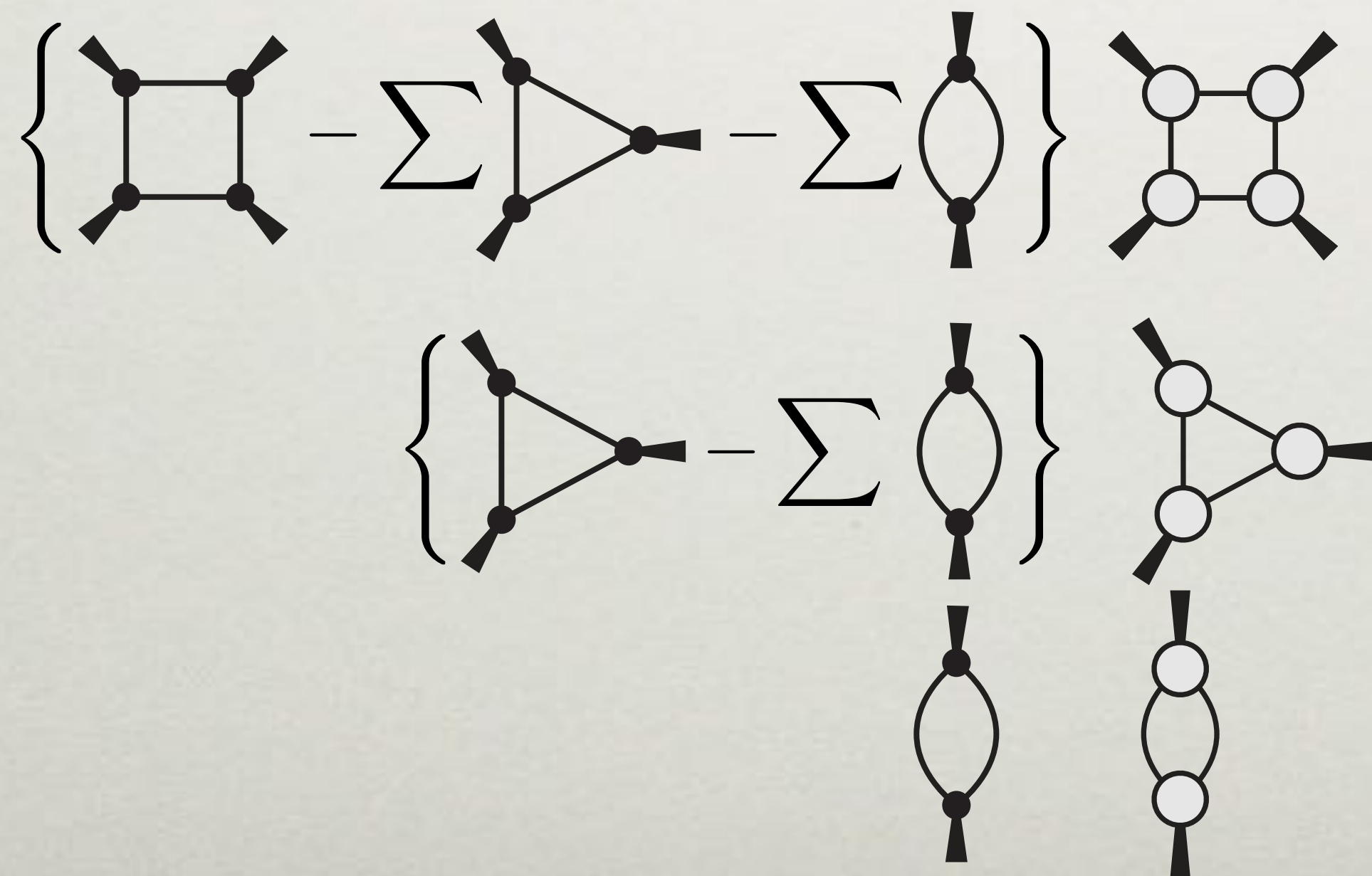


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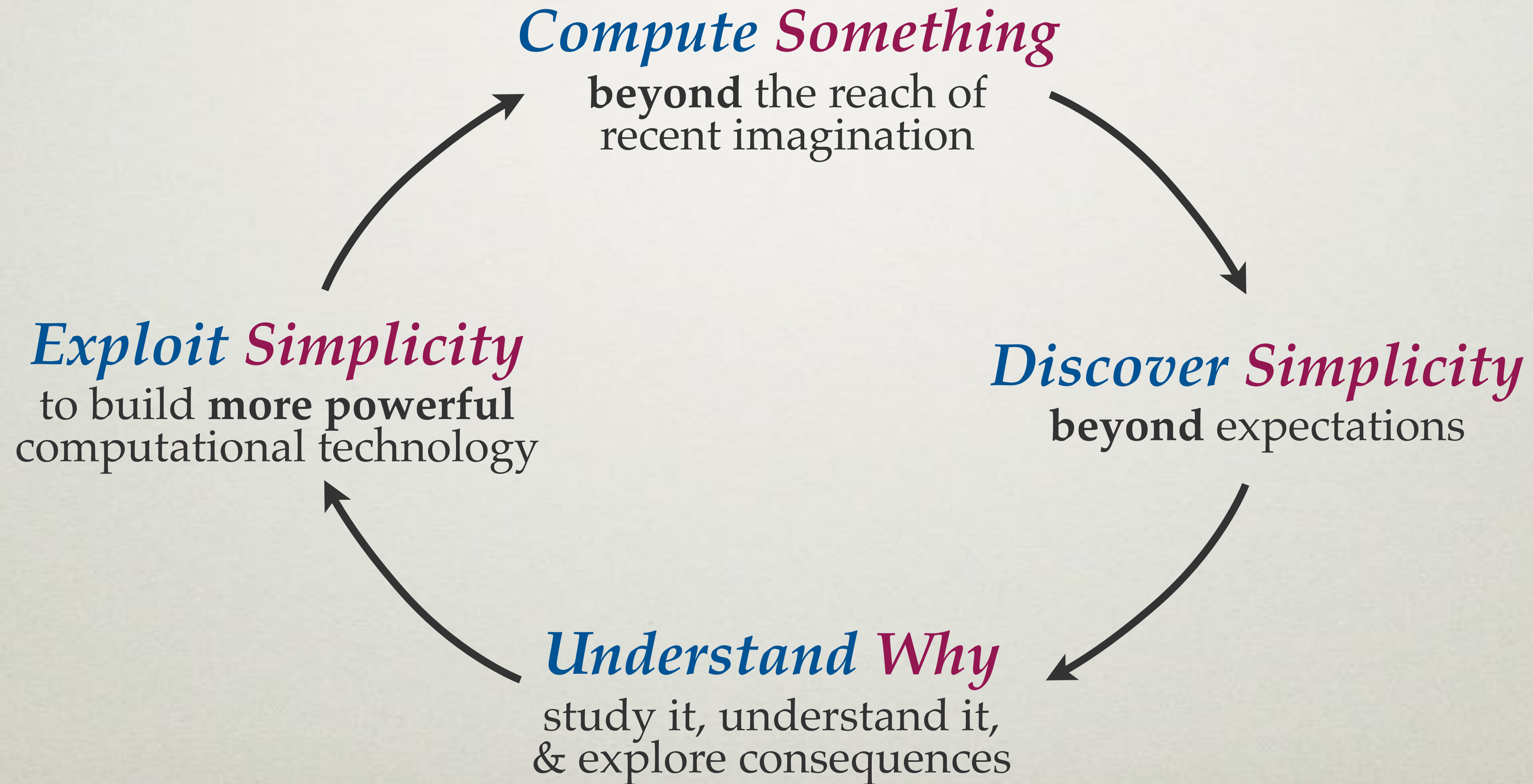
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 - ▶ the **coefficient** of any amplitude in this basis will simply be the *on-shell function* evaluated on the contour (a **leading singularity**)
- ◆ Choosing a **maximal** set of IR / UV-**divergence-probing contours** ensures(?) that the basis is split into finite / divergent subspaces

Amplitudes: a Virtuous Cycle



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Thank you!