Berends-Giele Recursion and a $1/N_c$ Expansion in MadGraph

23RD MCNET MEETING 6-8 DECEMBER 2021 - ANDREW LIFSON

IN COLLABORATION WITH OLIVIER MATTELAER

LUND UNIVERSITY

UCLouvain

Outline of Presentation

- Describe aim of project
- Remind colour ordering, how to calculate matrix element
- Remind about Berends-Giele recursions
- The QCD Colour bottleneck
- Remind about the $1/N_c$ expansion
- Introduce the `color_ordering` branch
  - Uses Berends-Giele recursion
  - Calculates matrix elements at given order in N_c
  - Show some results

The Project Idea: Beyond Feynman in MadGraph

- Was interested to help MadGraph go to higher multiplicities of final states
- Olivier mentioned there exists an old branch using Berends-Giele recursion
- Number of Feynman diagrams grows factorially with multiplicity

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Aim of project

Revive this old branch, get faster MEs

How to Calculate ME in MadGraph

Colour and kinematics factorise

$\sum_{\text{col}} |M|^2 = A_i^* C_{ij} A_j$

$C_{ij} \equiv$ colour matrix, in MadGraph uses trace basis

$A_i \equiv$ kinematic colour-ordered amplitude (use e.g. Feynman, Berends-Giele)

Trace basis summary:

pros:

- Nice symmetries of kinematics
- Easy to understand physically
- Planar diagrams and $1/N_c$ expansion

cons:

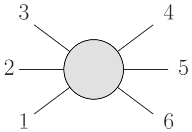
- Overcomplete basis
- Not orthogonal
- Squaring $\sim n! \times n!$ matrix

Standard MadGraph Kinematics: the Helas Routine

Basic principals of Helas

Calculate and store propagators as off-shell currents
Recycle these currents where possible

Example: 6g amplitude



$\rightarrow$

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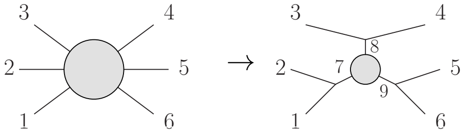

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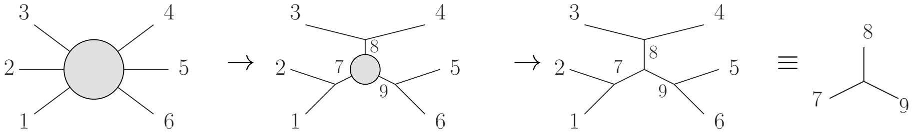

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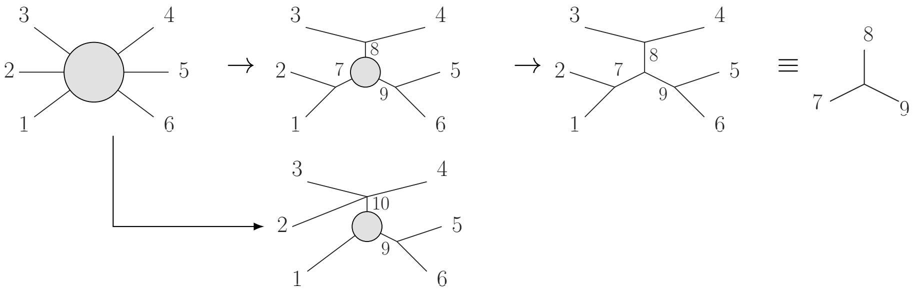

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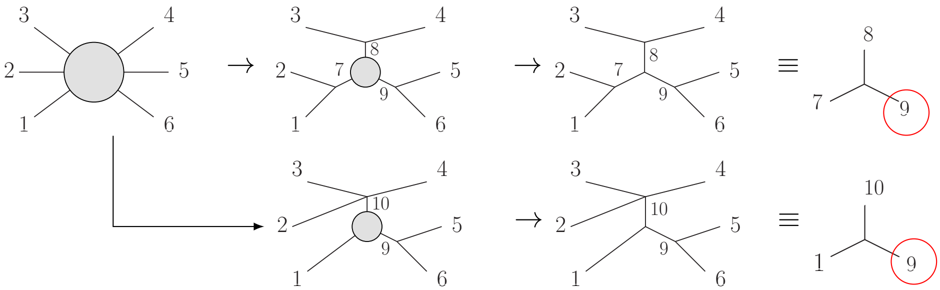
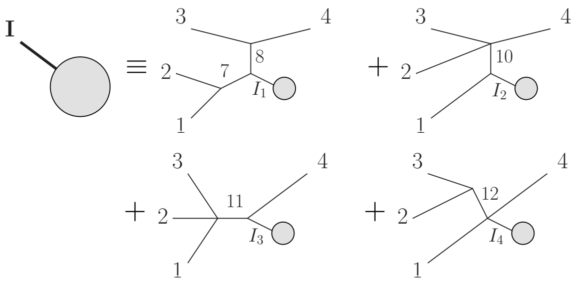


Alternative Kinematics: Berends-Giele Recursion

Basic Principal of Berends-Giele Recursion

Combine multiple off-shell currents into one

Example: 6g amplitude, 4 Feyn diags $\rightarrow$ 1 diag



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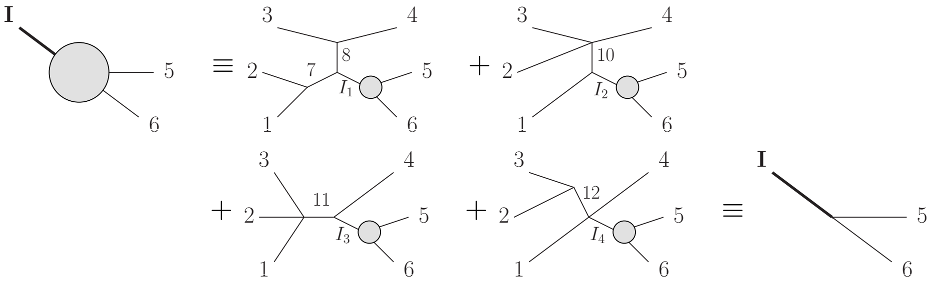

The Accidental Project: Fixed Colour Expansion

This old branch can truncate colour matrix to fixed order in N_c

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This old branch can truncate colour matrix to fixed order in N_c

Schematic colour matrix:

$$C \sim N_c^n \begin{pmatrix} \mathcal{O}(1) & \dots & \mathcal{O}(\frac{1}{N_c^2}) & \dots & \mathcal{O}(\frac{1}{N_c^4}) & \dots \\ \vdots & \ddots & & & & \vdots \\ & & & & & \\ \dots & \mathcal{O}(\frac{1}{N_c^2}) & \dots & \mathcal{O}(\frac{1}{N_c^4}) & \dots & \mathcal{O}(1) \end{pmatrix}$$

- Colour matrix elements are polynomials in N_c
- Diagonal is leading in colour
- Off-diagonal elements at least 2 powers of N_c smaller than diagonal.

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$$\text{LC: } C \sim N_c^n \begin{pmatrix} \mathcal{O}(1) & \cdots & \mathcal{O}(\frac{1}{N_c^2}) & \cdots & \mathcal{O}(\frac{1}{N_c^4}) & \cdots \\ \vdots & \ddots & & & & \vdots \\ & & & & & \\ \cdots & \mathcal{O}(\frac{1}{N_c^2}) & \cdots & \mathcal{O}(\frac{1}{N_c^4}) & \cdots & \mathcal{O}(1) \end{pmatrix}$$

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Schematic colour matrix:

$$\text{NLC: } C \sim N_c^n \begin{pmatrix} \mathcal{O}(1) & \cdots & \mathcal{O}(\frac{1}{N_c^2}) & \cdots & \mathcal{O}(\frac{1}{N_c^4}) & \cdots \\ \vdots & \ddots & & & & \vdots \\ & & & & & \\ \cdots & \mathcal{O}(\frac{1}{N_c^2}) & \cdots & \mathcal{O}(\frac{1}{N_c^4}) & \cdots & \mathcal{O}(1) \end{pmatrix}$$

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$$\text{N}^2\text{LC: } C \sim N_c^n \begin{pmatrix} \mathcal{O}(1) & \cdots & \mathcal{O}(\frac{1}{N_c^2}) & \cdots & \mathcal{O}(\frac{1}{N_c^4}) & \cdots \\ \vdots & \ddots & & & & \vdots \\ & & & & & \\ \cdots & \mathcal{O}(\frac{1}{N_c^2}) & \cdots & \mathcal{O}(\frac{1}{N_c^4}) & \cdots & \mathcal{O}(1) \end{pmatrix}$$

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Excellent! Colour is the QCD Bottleneck!

	$gg \to t\bar{t}$	$gg \to t\bar{t}gg$	$gg \to t\bar{t}ggg$
	Instructions	Instructions	Instructions
madevent	11G	180G	5T
matrix1	1G (9.3%)	160G (90%)	4.9T (98%)
↳ ext	76M (<1%)	100M (<1%)	110M (<1%)
↳ int	540M (4.8%)	16G (8.9%)	180 G (3.6%)
↳ amp	280M (2.6%)	77G (42%)	1.7T (33%)

Mattelaer and Ostralenk, 2021

- Calculating kinematics of Feynman diagrams up to $\sim 50\%$
- Factorial-squared colour matrix/sum takes up most of remaining time
- Can we improve speed without large impact on accuracy?

Current Progress and Status of Code

- One of the first branches ever in MadGraph, now revived
  - Nice to see version control working
  - Had to get it working with modern MadGraph, Python, etc.
  - Fixed some outstanding bugs
- Was a complete lack of caching
  - Was much slower than standard MadGraph to evaluate matrix elements
  - Changed the algorithm to include caching
- Validated, some final optimisations in progress
- So far mostly for standalone
  - Plans to combine with phase-space symmetry and quicker colour factor determination for MadEvent (with Rikkert Frederix and Timea Vitos)

Speed of $1/N_c$ expansion

Preliminary standalone speed tests (some optimisation remains)

$gg \to 5g$, single phase-space point

Colour	ME E-7	Gen time	Eval time
old	6.674	6m 25s	0.491s
modLC	6.591	2m 31s	0.350s
NLC	5.794	2m 44s	0.426s
N^2LC	6.612	2m 46s	0.453s
N^3LC	6.671	2m 48s	0.513s
N^4LC	6.674	2m 50s	0.544s

- N^nLC eval speeds $\sim$ std mg
- N^nLC gen speeds $\ll$ std mg

Speed of $1/N_c$ expansion

Preliminary standalone speed tests (some optimisation remains)

- N^nLC eval speeds $\sim$ std mg
- N^nLC gen speeds $\ll$ std mg

$gg \to 6g$, single phase-space point

Colour	ME E-9	Gen time	Eval time
old		$>$ 2 days	
modLC	3.157	340m 9s	9.096s
NLC			
⋮			
N^5LC	3.369	348m 59s	36.286s

Conclusion from speed

Similarly fast so far
Can often go one particle further, e.g. $2g \to 6g$ now possible!

Accuracy of $1/N_c$ expansion: all-gluon

Definitions

LC: Is modified s.t. $|M|^2_{\mathrm{LC}} = N_c^{n-2}(N_c^2 - 1)\sum_i |A_i|^2$
not LC: standard trace-basis definition

Accuracy in RAMBO flat phase space

Process	NEvents	Ave FC/LC	SD FC/LC	Ave FC/NLC	SD FC/NLC
2g >2g	1E05	1.0	0	1.0	0
2g >3g	1E05	1.0	0	1.031	2E-11
2g >4g	1E04	1.011	0.019	1.085	0.013
2g >5g	1E04	1.042	0.043	1.154	0.018

Conclusions: all-gluon amplitudes

Modified LC better than NLC
Low multiplicity, modified LC at few percent level or better

Accuracy of $1/N_c$ expansion: single quark line

Definitions

Use standard trace-basis definition

Accuracy in RAMBO flat phase space

Process	NEvents	Ave FC/LC	SD FC/LC	Ave FC/NLC	SD FC/NLC
$u\bar{u}$ >2g	1E05	0.929	0.040	1.0	0
$u\bar{u}$ >3g	1E05	0.979	0.054	1.011	0.007
$u\bar{u}$ >4g	1E04	1.072	0.085	1.005	0.009
$u\bar{u}$ >5g	1E04	1.208	0.116	1.008	0.012

Conclusions: single quark line

NLC about percent level or better in flat phase space

Accuracy of $1/N_c$ expansion: 2 quark lines

Definitions

Use standard trace-basis definition

Process	NEvents	Ave FC/LC	SD FC/LC	Ave FC/NLC	SD FC/NLC
$u\bar{u} > d\bar{d}$ 0g	1E05	0.889	0	1.0	0
$u\bar{u} > d\bar{d}$ 1g	1E05	0.974	2.399	1	0
$u\bar{u} > d\bar{d}$ 2g	1E05	1.098	1.678	1.009	0.025
$u\bar{u} > d\bar{d}$ 3g	1E04	1.225	2.064	1.017	0.039

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Conclusions: two quark lines

LC horribly unstable! NLC about percent level or better in flat phase space

Modified colour ordering for multiple quark lines

LC: only consider diagonal (off-diagonal is sub-leading, set to 0)

$$\mathrm{LC}: C_{ij}A_j \sim N_c^n \begin{pmatrix} \mathcal{O}(1) & 0 & \dots & & & 0 \\ 0 & \ddots & 0 & \dots & & 0 \\ 0 & \dots & \mathcal{O}(1) & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathcal{O}(\frac{1}{N_c^2}) & 0 & 0 \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \dots & & & 0 & \mathcal{O}(\frac{1}{N_c^2}) \end{pmatrix} \begin{pmatrix} A_1 \\ \\ \\ \vdots \\ \\ A_n \end{pmatrix}$$

Likely explanation

Strict LC completely misses some poles

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Likely explanation

Strict LC completely misses some poles

Accuracy of $1/N_c$ expansion: 2 quark lines mod col

Definitions

Use full diagonal $\equiv$ modified colour ordering

Process	NEvents	Ave FC/LC	SD FC/LC
$u\bar{u} > d\bar{d}$ 0g	1E05	0.8	0
$u\bar{u} > d\bar{d}$ 1g	1E05	0.740	0.089
$u\bar{u} > d\bar{d}$ 2g	1E05	0.743	0.120
$u\bar{u} > d\bar{d}$ 3g	1E04	0.785	0.159

Accuracy of $1/N_c$ expansion: 2 quark lines mod col

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Conclusions: two quark lines mod colour

LC now stable, systematically about 25% too small, can correct?

Conclusion and Outlook

- For faster code, higher multiplicity, need to improve both colour and kinematics
- Berends-Giele allows for more streamlined kinematics
- Time taken to create and build Helas routine is a bottleneck
- $1/N_c$ expansion allows faster code for good accuracy
- New `color_ordering` branch can do this expansion for you, with good and improving speed
- Outlook: Include phase-space symmetry $\Rightarrow$ cross-section scale like n^4, not $n!$ (Rikkert Frederix and Timea Vitos)
- Outlook: `color_ordering` branch will be completed/released soon (in few months)

The $1/N_c$ Expansion

Schematic example

The n-gluon colour matrix C in fundamental basis

$$C \sim N_c^n \begin{pmatrix} \mathcal{O}(1) & \dots & \mathcal{O}(\frac{1}{N_c^2}) & \dots & \mathcal{O}(\frac{1}{N_c^4}) & \dots \\ \vdots & \ddots & & & & \vdots \\ & & & & & \\ \dots & \mathcal{O}(\frac{1}{N_c^2}) & \dots & \mathcal{O}(\frac{1}{N_c^4}) & \dots & \mathcal{O}(1) \end{pmatrix}$$

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Feynman Diagrams → Berends-Giele Recursion

Keys results of Berends-Giele recursion

Berends-Giele recursion = supercharging power of off-shell currents
Leads to shorter and quicker kinematics/programs

E.g. all-gluon Berends-Giele current

$$J_{n,\mu}(1^{h_1},\ldots,n^{h_n}) = \frac{-i}{P_{1,n}^2}\left\{\sum_{i=1}^{n-1} V_{3,\mu,\nu\rho}(P_{1,i},P_{i+1,n})J^{\nu}(1^{h_1},\ldots,i^{h_i})J^{\rho}((i+1)^{h_{i+1}},\ldots,n^{h_n})\right.$$
$$\left. + \sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2} V_{4,\mu\nu\rho\sigma}J^{\nu}(1^{h_1},\ldots,i^{h_i})J^{\rho}((i+1)^{h_{i+1}},\ldots,j^{h_j})J^{\sigma}((j+1)^{h_{j+1}},\ldots,n^{h_n})\right\}$$

Phase-Space Symmetry and Faster Colour Factors

- Frederix, Vitos 2021 show how to quickly find which kinematic amplitudes are LC, NLC
  - Plan to put this into `color_ordering` branch

- Interchanging momenta of identical final-sate particles gives same ME
  - $\Rightarrow$ don't need to compute ME twice in PS integral
  - $\Rightarrow$ only need small subset of colour matrix for cross-section
  - Have power-like growth of colour, not factorial-like!

- Plan is to combine this with `color_ordering` branch
  - Currently in preliminary stages

The Current Limits of MadGraph

- Multi-jet processes at ever-increasing energies $\Rightarrow$ many hard partons
- MG currently limited in being able to deal with that
- Say boundaries for many processes, e.g. all-gluon = dominant
- Can we push this boundary back somewhat?
- Can we improve speed without large impact on accuracy?

Current Code Status and Future Plans

- Currently can calculate any non-decay-chain process SA ME to a given colour order
- Can also be used in MadEvent but not as well tested/optimised
- 1 major optimisation remains, then plan to publish (many smaller optimisations still available)

Basics of the Algorithm: Flow definition

- Explain the concept of a flow in this branch (go through all-g, 2q, 4q examples to illuminate)
- Say that each flow is calculated separately

Basics of the Algorithm: the 'Colour' matrix

- For a given flow, calculate all `JAMPS` once
- Calculate the colour for the first row of the colour matrix to a specified power of $1/N_c$, calculate ME for that row explicitly
- Go through rows of colour matrix by permuting `JAMP` numbers

Colour: the QCD Bottleneck

- Show results from Olivier/Kiran's paper
- Conclude summing over colour matrix starts to dominate the time taken
- Remind that adding partons is theoretically possible, but in practice already at $2 \to 6$ gluons the code is too large to compile in a reasonable amount of time
- Can we improve speed without large impact on accuracy?

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