Lectures Notes

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- Zero-Form and One-Form Symmetries
- Examples of Theories with One-Form Symmetries
- Line Operators in Different Phases of QFT
- Line Operators in Conformal Theories

There are certainly many typos, errors etc.

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Chapter 1
Symmetries and Extended Symmetries

In classical field theory, if an action S admits continuous transformations leaving it invariant then we have a conserved current $\partial j = 0$ and this famously [1] allows to define a conserved charge

$$Q = \int_{\Sigma_{d-1}} \star j$$

where Σ_{d-1} is a closed $d-1$ dimensional manifold. In the quantum theory a continuous transformation leaving the action invariant may or may not be a symmetry,[1] but assuming that it is, $\partial j = 0$ is promoted to an operator equation valid at separated points in all correlation functions. Then from Q we can construct a unitary acting on the Hilbert space of the theory

$$U_\alpha = e^{i\alpha \int_{\Sigma_{d-1}} \star j} \ ,$$

The operators U_α obey a group multiplication law (the group may not be commutative) and the operators transform under the symmetry via $\mathscr{O} \to U_\alpha \mathscr{O} U_\alpha^\dagger$, furnishing linear representations on which the U_α act faithfully. It is important that the Hilbert space (with or without various defects) does not have to be in linear representations of U_α – only the local operators have to be. For instance, in QM, the operators U_α can be realized projectively. Another important fine print is that requiring that symmetries act faithfully on local operators is not essential. For instance, in TFT, there are no local operators but we still discuss the action of symmetries on extended operators.

A hallmark of the unitary U_α is that it depends (as an operator acting on the Hilbert space) on Σ_{d-1} only topologically – that is, it is independent of Σ_{d-1} for small deformations. This allows us to define the more general notion of a zero-form symmetry, as any operator that depends topologically on Σ_{d-1}. This clearly allows to discuss the U_α that correspond to discrete symmetries, even though we do not have conserved currents for discrete symmetries. Such generalized symmetries do

[1] The ABJ anomaly is the quantum effect which violates a classically valid symmetry.

not have to be unitary, or even invertible, matrices, in which case we refer to them as non-invertible zero-form symmetries.

Another possible generalization of the concept of symmetry is to consider operators that depend topologically on a $d-2$ dimensional surface, Σ_{d-2}. Such operators are called one-form symmetry. We will denote then by $U_\alpha^{(1)}$. We will assume that they are unitary. Then it is easy to see that they must form an Abelian group since we can pass $d-2$ dimensional surfaces past each other without intersection:

$$U_{\alpha'}^{(1)}(\Sigma'_{d-2})U_\alpha^{(1)}(\Sigma_{d-2}) = U_\alpha^{(1)}(\Sigma_{d-2})U_{\alpha'}^{(1)}(\Sigma'_{d-2}) \; , \tag{1}$$

above, $\Sigma_{d-2}, \Sigma'_{d-2}$ are parallel space-like surfaces at different times, acting on the same Hilbert space. In the same way as zero-form symmetry acts on local operators, the one-form symmetry acts on line operators by some linear action which is obtained by wrapping the line operator L with Σ_{d-2} or, equivalently, by writing $L \to U_\alpha^{(1)}(\Sigma_{d-2})\, L\, U_\alpha^{\dagger(1)}(\Sigma'_{d-2})$ for Σ'_{d-2} in the future of L and Σ_{d-2} in the past of L such that they are transverse to L in the space directions.

The equation (1) should again not be interpreted as an action on the Hilbert space, but rather as an action on the line operators L. This action provides Ward identities for correlation functions of L. There could be various central extensions of (1) when studied as an action on the Hilbert space but it is exact as the action on line operators inside correlation functions. We can again insist that the one form symmetry $U_\alpha^{(1)}$ acts faithfully on the line operators of the theory.

In the same way that zero-form symmetry has been a useful tool in the study of local operators, one form symmetry is an important concept in the study of line operators.

Chapter 2
Examples

In this chapter we cover simple examples of QFTs with one form symmetry and line operators charged under those one form symmetries. A very important concept we will keep coming back to is that once a line operator is charged under one-form symmetry, it cannot be a trivial line operator and in particular must have some effect at long distances. This is easily proven by observing that we can wrap the line with the one-form symmetry $d-2$ dimensional surface which is arbitrarily far away and yet this must yield a nontrivial result according to the representation of the line operator under the one-form symmetry. (Since the one-form symmetry group is Abelian, we can take all the line operators in one-dimensional representations and hence they simply pick up phases according to their charge.) How the line defect affects the physics at long distances varies between different phases of QFT.

2d $U(1)$ Theory

$$S=\int d^2x\left(\frac{-1}{2e^2}F_{01}^2+\frac{\theta}{2\pi}F_{01}\right)\ . \tag{2}$$

We will quantize the theory on $S^1\times\mathbb{R}$ where the circle has radius R. Pick a gauge $A_0=0$ and then $\partial_1 F_{01}$ follows as a constraint – which is nothing but the statement that the electric field is constant. We can parameterize the remaining variable as $A_1(x,t)=G(t)$, with $G\simeq G+\frac{1}{R}\mathbb{Z}$. The latter identification follows from a residual large gauge transformations with parameter $e^{i\Omega}=e^{ikx/R}$ for an arbitrary integer k. This large gauge transformation is allowed since x and $x+2\pi R\mathbb{Z}$ are identified.

The model therefore reduces to quantum mechanics of $G(t)$ which is not terribly surprising given that the photon has no actual propagating polarizations in 2d. We find the action and Hamiltonian

$$S=2\pi R\int dt\left(\frac{1}{2e^2}\dot{G}^2+\frac{\theta}{2\pi}\dot{G}\right)\ .$$

$$H=\frac{1}{2}\frac{e^2}{2\pi R}\left(\Pi_G-\theta R\right)^2\ .$$

The eigenfunctions are $\Psi_n(G) = e^{2\pi inGR}$ with energies $E_n = \frac{1}{2}\frac{e^2R}{2\pi}(2\pi n - \theta)^2$. Note that the spectrum is invariant under $\theta \to \theta + 2\pi$. The expectation value of the electric field in this state is

$$\langle n|F_{01}|n\rangle = e^2\left(n - \frac{\theta}{2\pi}\right) .$$

These states describe configurations with spatially constant electric field and spatially constant energy-density.[2]

Since $\partial_1 F_{01} = 0$ holds as a constraint and $\partial_0 F_{01} = 0$ follows as an equation of motion, we see that the operator F_{01} is topological in the sense that its correlation functions are independent of space or time. We can therefore interpret

$$U_\alpha = e^{i\alpha\left(\frac{F_{01}}{e^2} + \frac{\theta}{2\pi}\right)}$$

as a unitary corresponding to a $U(1)$ one-form symmetry charge.

It describes the conservation of electric field which results from the absence of dynamical charged particles. More generally, the electric one-form symmetry group is the subgroup of the center of the gauge group which is not acting on any of the dynamical matter fields. The line operators charged under the one-form symmetry are the electric Wilson lines

$$L_q = Pe^{iq\int A_0 dt} .$$

Repeating the quantization above in the presence of the Wilson line at $x = 0$ we find $\frac{1}{e^2}\partial_1 F_{01} = q\delta(x)$ which means that $F_{01}(x > 0) - F_{01}(x < 0) = e^2 q$. This induces therefore a jump by q units in the electric field. Of course the spectrum is empty on S^1 since the two ends cannot be identified any longer. But in infinite volume the ground state of the theory with the Wilson line is well defined – the Wilson line induces a jump by q units among the lowest lying possible values of the electric field.[3]

The jump in the electric field due to the Wilson line can be written algebraically as

$$e^{i\alpha\left(\frac{F_{01}}{e^2} + \frac{\theta}{2\pi}\right)}(x < 0)L_q(x = 0) = e^{-i\alpha q}\, L_q(x = 0)e^{i\alpha\left(\frac{F_{01}}{e^2} + \frac{\theta}{2\pi}\right)}(x > 0) ,$$

and the phase $e^{-i\alpha q}$ on the right hand side is interpreted as the action of the $U(1)$ one-form symmetry, endowing the Wilson line with charge q under the one-form $U(1)$ symmetry. The Wilson line has a notable effect arbitrarily far from it – it changes the value of the electric field in one side of space, consistently with it being charged under one-form symmetry.[4]

3d $U(1)$ Theory

[2] Sometimes these states are called universes since they are Poincaré invariant.

[3] Note that $L_{\pm 1}$ does not increase the energy compared to the usual homogenous ground state for $\theta = \pi$ – this is sometimes called de-confinement since it means that there is no cost in placing an external particle of charge ± 1 at $\theta = \pi$.

[4] This is true also for $L_{\pm 1}$ at $\theta = \pi$, where the energy density does not change, but the electric field does change.

$$S = \int d^3x \, \frac{-1}{4e^2} F^2 \ . \qquad (3)$$

Similarly to the 2d theory, there is a $U(1)$ one form symmetry in this theory since there are no dynamical charges. The current is a two-index operator $J_{\mu\nu} = F_{\mu\nu}$, which is conserved by virtue of the equations of motion. This current can be integrated on one-dimensional curves, leading to a topological operator:

$$\int_{\Sigma_1} \varepsilon_{\mu\nu\rho} F_{\nu\rho} d\Sigma_1^{\mu} \ ,$$

where $d\Sigma_1^{\mu}$ is a vector tangent to the curve. Since the one-form $\star F$ is closed, this integral depends on Σ_1 topologically. As usual, exponentiating it leads to a unitary operator. We can properly normalize this unitary such that if Σ_1 wraps the wordline of a charged particle $L(\gamma)$ with charge q, $L(\gamma) = Pe^{iq\int_\gamma A}$ then we find

$$U_\alpha(\Sigma_1) L(\gamma) = e^{iq\alpha} L(\gamma) \ , \qquad (4)$$

i.e. the Wilson line of charge q carries charge q under the $U(1)$ one-form symmetry, as in 2d. Note that in (4) we do not study an equal time commutator as in some of the formulas before but a Euclidean configuration where Σ_1 wraps γ. This becomes a statement about a commutator in Lorentzian signature.

There are several things to remark about this theory.

- Placing a Wilson line of charge q at $\vec{x} = 0$ leads to electric field $F_{0i} \sim qx_i/x^2$ and electric potential $A_0 \sim q \log a^{-1}|\vec{x}|$, with a some UV cutoff scale. This famously leads to a divergent energy for a single charged particle both in the UV and the infrared. If we study the dipole configuration with the charges separated distance D apart then the infrared divergence is removed and we find a logarithmic binding energy signaling that the particles are very difficult to separate. However, strictly speaking, the theory is still deconfined since such a logarithmic potential does not lead to an area law, instead, the rectangular Wilson loop has expectation value $e^{-T \log D}$ where T is the extent in time and D the separation between the charges. (This is valid for $T \gg D$.)
- The theory (3) has interesting local operators $\mathscr{M}_n$ charged under the $U(1)$ zero-form symmetry $J_\mu = \frac{1}{2\pi}\varepsilon_{\mu\nu\rho}F^{\nu\rho}$. These are the monopole local operators (disorder operators), defined by removing a point from space-time and forcing the flux through a small S^2 surrounding it to be $\frac{1}{2\pi}\int_{S^2} F = 2\pi n$.
- We can deform the action (3) by adding monopole operators as follows:

$$S = \int d^3x \, \left(\frac{-1}{4e^2} F^2 + \sum_n a_n \mathscr{M}_n(x) + c.c. \right) \ . \qquad (5)$$

  The one-form symmetry is preserved and remains $U(1)$ since there are no dynamical charged particles but the zero-form symmetry is broken by the monopole operators. We can again study the effect of placing a single Wilson loop L_q at

$\vec{x}=0$ and then a dipole with the Wilson loops distance D apart. The monopole operators lead to dramatic effects on both questions.

To understand how the monopole terms change the dynamics of the theory it is convenient to dualize the gauge field to a compact scalar φ of radius 2π via $\frac{1}{2\pi}F_{\mu\nu}=\varepsilon_{\mu\nu\rho}\partial^\rho\varphi$ with a standard kinetic term and potential $V(\varphi)=\sum_n a_n e^{in\varphi}+c.c.$. This potential generically gaps the φ excitation (i.e., in the dual language, the photon becomes massive due to monopole proliferation!) and it leads to a confining, approximately one-dimensional, string emanating from a probe charge and a simlar string now connects a dipole. To see that, note that an electric probe is a source for a monodromy of the φ field (i.e. a vortex) and due to the potential $V(\varphi)$, away from the sources the field would prefer to be in the ground state almost everywhere, as much as possible. The vortex therefore has to be the end point of a string where φ rapidly jumps. This leads to a linear energy in the length of space for an isolated probe charge, and the string, which continues to infinity, ensures that the one form symmetry charges are reproduced while for a dipole the energy is linear in D and hence the rectangular Wilson loop behaves as e^{-cDT}, with a coefficient c that is identified with the tension of the string.

We have seen in this example two different realizations of the infrared behavior of line operators charged under one-form symmetry: in one case there was a linearly decaying, isotropic electric field and the theory is gapless, and in the second case, the theory is gapped and we have electric field confined to a line and

continuing forever without decaying in magnitude, until it ends on an opposite charge.

<u>4d $U(1)$ gauge field</u>

As in our prevous examples, the free $U(1)$ gauge theory has an electric one-form symmetry. This one-form symmery charge is obtained by integrating $\star F$ over two-dimensional surfaces (Gauss' law). The Wilson line of a probe of charge q carries charge q under the electric one-form symmetry. Famously one find that this leads to the electric potential

$$A_0=\frac{e^2q}{4\pi r}\ ,$$

which is a scale invariant decay since both the left hand side and the ride hand side have dimension 1. This would be one of our simplest examples of a conformal line operator (in the bulk CFT which is the free photon theory). A noteworthy point about the 4d $U(1)$ guage field is that it has, in fact, $U(1)\times U(1)$ one-form symmetry, where the second $U(1)$ factor originates from the magnetic one-form symmetry, whose charges are obtained by integrating F over two-dimensional surfaces. This is conserved due to the absence of dynamical magnetic charges. The charged line operators are the 't Hooft lines, defined similarly to the disorder operators in our 2+1 dimensional $U(1)$ gauge theory. These are again conformal line operators leading to a scale invariant magnetic field $B\sim 1/r^2$.

<u>$SU(N)$ gauge theory in 4d</u>

The action is given by

$$S = \int d^4x \left(\frac{-1}{4g_{YM}^2} F^2 + \frac{\theta}{8\pi^2} F \wedge F \right) . \tag{6}$$

As in our 2d example, θ is a 2π periodic parameter. Since the gluons are in the adjoint representation of the gauge group one cannot hope for a Gauss law measuring the precise representation of a source, however, the N-ality of the representation (i.e. the number of boxes mod N) is conserved, hence, the theory has a $\mathbb{Z}_N$ one-form symmetry. The Wilson line in any representation which has a nonzero number of boxes mod N is charged under $\mathbb{Z}_N$ one-form symmetry. Therefore, it must have some long distance effects. Famously, it is conjectured that the theory (6) is confined and gapped.[5] A Wilson line charged under one-form symmetry must be the end point of a two dimensional sheet of the confining string. More precisely, a Wilson line with charge k mod N must be the end point of the two-dimensional sheet of the confining k-string. The sheet can end either on another Wilson line with the opposite charge, or it has to continue to infinity.

[5] Though at $\theta = \pi$ (6) is most likely not gapped at finite volume due to the spontaneous breaking of time reversal symmetry. Each of the corresponding vacua at infinite volume is most likely gapped.

Chapter 3
Line Operators in Conformal Theories

An interesting phase encountered above is that of a conformal line operator in the free 4d $U(1)$ gauge theory. Our aim in this chapter is to study conformal line operators in more detail. Our setup is a conformal field theory in d space-time dimensions with a line operator extended in time and localized at $\vec{x}=0$. While our discussion thus far emphasized one-form symmetry, there are many

conformal line operators that are uncharged under one- form symmetry and there are also many conformal line

operators which are nontrivial in theories that have no one-form symmetry. This is analogous to the situation with local operators, which can be non-trivial even if the are not charged under any symmetry. Starting from the bulk $so(d+1,1)$ conformal symmetry, a conformal wilson line would preserve the maximal allowed subgroup that leaves $\vec{x}=0$ invariant. The allowed transformations therefore consist of dilations, translations in time, and out of the special conformal transofrmaitons,

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2} ,$$

we must only allow those with $\vec{b}=0$ and hence, denoting $b^0 = \beta$ we have the transformation (evaluated at $\vec{x}=0$)

$$\vec{x}=0: \quad t' = \frac{t}{1-\beta t} .$$

The three transformations we have found comprise an $sl(2,\mathbb{R})$ subgroup which at $\vec{x}=0$ acts as

$$t' = \frac{at+b}{ct+d} , \qquad ad-bc=1 .$$

A conformal line operator in a CFT is any line operator that preserves this subgroup. The theory in the presence of such a conformal line operator is called Defect Conformal Field Theory (DCFT).[6]

The main actors in DCFT are the bulk local operators $\mathscr{O}_i(\vec{x},t)$ and the defect local operators $\mathscr{U}_I(t)$. The defect operators $\mathscr{U}_i(t)$ have scaling dimensions corresponding to the Cartan of $sl(2,\mathbb{R})$, denoted by Δ^L, to distinguish them from the bulk scaling dimensions Δ.

For the trivial line defect, i.e. the unit line operator, the defect local operators are just the restriction of bulk local operators to $\vec{x}=0$ but more generally, there could be defect local operators which do not have anything to do with bulk local operators (such defect local operators could be thought of as acting purely in the Hilbert space of the impurity, in the Hamiltonian language).[7]

A very important gadget is the bulk-defect OPE, which allows to expand the bulk operators at small $\vec{x}$ in terms of defect operators, schematically as follows (suppressing all quantum numbers other than the scaling dimensions)

$$\mathscr{O}_i(\vec{x},t) = \sum_I \frac{a_{iI}}{|\vec{x}|^{\Delta_{\mathscr{O}_i}-\Delta^L_{\mathscr{U}_I}}} \mathscr{U}_I(t) \ . \tag{7}$$

This expansion is useful at short distances, where high- dimension defect operators make a smaller contribution.

In the vacuum, the expectation values of the $\mathscr{U}_I$ all vanish save for the unit operator. The corresponding coefficients for the unit operator on the right hand side of (7) are often denoted simply by a_i. Those describe the one-point functions of bulk operators in the vacuum with the defect L:

$$\langle \mathscr{O}_i(\vec{x},t)\rangle = \frac{a_i}{|\vec{x}|^{\mathscr{O}_i}} \ . \tag{8}$$

This is one of the key signatures of a nontrivial conformal defect: it leads to nonzero one-point functions with power laws exactly compatible with the bulk scaling dimensions.

Line operators in CFTs can be defined in many different ways (we will see some later) and it is by no means guaranteed that they are conformal. It is reasonable to expect, though that they generally flow to (possibly nontrivial) infrared DCFTs. In condensed matter

language, as we get further away from the impurity, it may not be completely screened but rather flow to a conformal impurity which affects order parameters

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[6] In addition, the transverse rotations in $so(d-1)$ leave $\vec{x}=0$ invariant, but there are $sl(2,\mathbb{R})$ invariant line operators

which break this symmetry. So we will not require $so(d-1)$ invariance as part of our definition of DCFT.

[7] More generally, it is often helpful to think about the physics of line operators in a Hamiltonian language, where the line operator is an impurity in space,

while the bulk is tuned to a second-order zero- temperature phase transition. This language makes it

very clear that RG flows on impurities are very important.

as in (8). It is an important question in condensed matter to understand the long distance effects of some impurities and, likewise, in particle physics, it is a question of obvious interest to understand the long distance limits of line operators.

RG flows on line operators can be triggered by relevant defect operators $\mathscr{U}$ with $\Delta^L < 1$ simply by integrating such operators on the defect. Therefore one canonical construction of nontrivial line operators is to begin with the trivial line defect, for which the spectrum of defect operators is given by the bulk operators restricted to $\vec{x} = 0$, and integrate on a line any such bulk operator with $\Delta < 1$. Many bulk CFTs have operators with $\Delta < 1$ and thus this gives a large family of potentially interesting conformal line operators. The physical meaning of such line defects is that we apply an external field (think magnetic field) localized in space. This is why in the condensed matter literature such line defects are often referred to as "pinning field defects."

Of course, in general, for such constructions, it is hard to derive analytically anything concrete about the infrared but we will now discuss two cases where one can say a lot.

<u>Mean Field Theory</u>

Consider the free scalar field ϕ in $2 < d \leq 4$ with a trivial line defect and deform it by the operator ϕ, which has dimension $0 < \Delta(\phi) \leq 1$ for $2 < d \leq 4$. We are therefore now calculating in free field theory with the line operator

$$Pe^{i\gamma \int dt \phi(t)} \,, \tag{9}$$

with γ a relevant coupling constant around the $\gamma = 0$ trivial line defect (it is marginal for $d = 4$ and relevant for $2 < d \leq 4$).

While this is certainly the simplest possible bulk CFT and the simplest possible line defect in this theory, unfortunately, it leads to a rather exotic infrared behavior as we will see. The infrared will not be a DCFT, except for the case of $d = 4$.

We can include the line defect in the action in order to streamline the analysis as follows:

$$S = \int d^d x \left(\frac{1}{2} (\partial \phi)^2 - \delta^{d-1}(\vec{x}) \gamma \phi \right) \,.$$

The equation of motion leads to the solution

$$\phi \sim \frac{\gamma}{|\vec{x}|^{d-3}} \,. \tag{10}$$

- For $d = 4$ this is precisely consistent with the rules for one-point functions in DCFT (8) and γ must be interpreted as an exactly marginal parameter, i.e. it does not flow and different values of γ define different DCFTs.
- For $d < 4$ the decay (10) is too slow. In fact, for $d \leq 3$, (10) is not even decaying. This pathological behavior means that in the infrared, the defect (9) affects the bulk more strongly than any conformal defect is allowed to. One can think about it as a never-ending flow, which never reaches an infrared fixed point.

Presumably, this pathological behavior does not occur in models without degenerate bulk vacua.[8]

Wilson-Fisher Fixed Points

To test the idea that the infrared is a healthy DCFT in models without bulk vacuum degeneracy, we now

consider the pinning field defect in the $O(N)$ Wilson- Fisher models. We will indeed see that we arrive at a

healthy nontrivial infrared DCFT. In other words, an external localized magnetic field is not screened in the WF critical points. Our bulk action is

$$S = \int d^d x \left(\frac{1}{2} (\partial \vec{\phi})^2 + \lambda_* (\vec{\phi}^2)^2 \right) .$$

To define the line defect, we pick an arbitrary direction $\vec{h}$ in $\mathbb{R}^N$ and define the line operator as

$$P e^{i\vec{h}\cdot\int dt \vec{\phi}(t)} , \tag{11}$$

and again, h flows to strong coupling in the infrared for $2 < d < 4$ since $\vec{\phi}$ is a relevant perturbation around the trivial line defect. The problem can be attacked analytically in three domains: in an epsilon expansion in $d = 4 - \varepsilon$ dimensions, in the large N expansion for any d, and presumably also around $d = 2$ for $N = 1$ and $N = 2$.[9] We can always rotate $\vec{h}$ so that only the component h^1 is nonzero. In the ε expansion, one finds the following fixed point:

$$h_*^2 = N + 8 + \varepsilon \frac{4N^2 + 45N + 170}{2N + 16} + O(\varepsilon^2) .$$

And scaling dimension

$$\Delta^L(\phi_1) = 1 + \varepsilon - \varepsilon^2 \frac{3N^2 + 49N + 194}{2(N+8)^2} + O(\varepsilon^3) .$$

while the operator $\phi_{a\neq 1}$ has $\Delta^L(\phi_{a\neq 1}) = 1$ from symmetry. Also the one-point function of the order parameter can be determined as

$$a_\phi^2 = \frac{N+8}{4} + \varepsilon \frac{(N+8)^2 \log 4 + N^2 - 3N - 22}{8(N+8)} + O(\varepsilon^2) .$$

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[8] Another interesting special case is $d = 3$ where the one-point function is a logarithm — this can be connected with a certain anomaly in coupling space.

[9] If the properties of the bulk CFT are continued in d, then at $d = 2$ for $N = 1$ one finds the Ising model in which (11) flows to a fusion of two Dirichlet boundary conditions and hence everything is known about it exactly. For $N = 2$ one finds a line defect at some special point (that can be determined) on the $c = 1$ conformal manifold. Either way, while the $d \to 2$ limit is known, it is less clear if it can serve as the starting point of an expansion.

In the large N limit and $d = 3$ one finds again $\Delta^L(\phi_{a\neq 1}) = 1$ and $\Delta^L(\phi_1) = 1.542...$ and $a^2_\phi = 0.55813N$. These are all roughly consistent and lead one to suspect that for any N in $d = 3$ there is an infrared stable DCFT with $\Delta^L(\phi_1) \sim 1.5 \pm 0.1$. Some more complicated line defects exist in the $O(N)$ models as well. They involve some quantum mechanical degrees of freedom coupling to the order parameter in the bulk. Of course, needless to say, it is not known what is the space of conformal line defects in these models.

Wilson Lines in Gauge Theories

We have seen that in the free $U(1)$ gauge theory in 4d the Wilson and 't Hooft lines provide some of the simplest examples of conformal line defects. In preparation for analyzing Wilson and 't Hooft lines in more complicated, interacting gauge theories, we now study Wilson lines in QED with propagating matter fields. Of course, QED with propagating matter fields in 4d is not a conformal theory (it is always infrared free) so the formalism of DCFT is not entirely appropriate. However, we will take a certain weak coupling limit where the beta function of QED will become negligible and the formalism of line defects in CFTs is valid.

We will discuss QED in 4d with a massless boson of charge 1 with action

$$S = \int d^4x \left(\frac{-1}{4e^2} F^2 + |D\phi|^2 + \lambda |\phi|^4 \right) + q \int dt A_0(\vec{x} = 0, t) \ , \tag{12}$$

The Wilson line $Pe^{iq\int dt A_0}$ can be viewed as describing a heavy nucleus of charge q. To reach a situation where the beta function of the bulk couplings e^2, λ is negligible and we can think about this setup as a line defect in a conformal theory, we take a scaling limit

$$e^2 \to 0 \ , \quad \lambda \to 0 \ , \quad q \to \infty \ , \quad \lambda/e^2 = fixed \ , \quad e^2 q = fixed \ . \tag{13}$$

Physically this means that we can think about nuclei with large charge with a sufficiently small bulk coupling in the framework of line defects in a bulk CFT.

To see that the scaling limit (13) is useful we can rewrite the action as

$$S = \frac{1}{e^2} \left[\int d^4x \left(\frac{-1}{4} F^2 + |D\phi|^2 + \frac{\lambda}{e^2} |\phi|^4 \right) + e^2 q \int dt A_0(\vec{x} = 0, t) \right] \ , \tag{14}$$

where we have rescaled $\phi \to \phi/e$ compared to (12). We do not change the notation for ϕ not to clutter the formulas. We can identify $\hbar \sim e$ and solve the model exactly in $\lambda/e^2 = fixed$ and $e^2 q = fixed$, as promised. To leading order in our scaling limit there is no beta function for the bulk coupling λ/e^2 and thus we can think of the bulk as a CFT.

To initiate an expansion in the scaling limit we must pick a saddle point and expand about it in fluctuations. The obvious saddle point is

$$A_0 = \frac{e^2 q}{|\vec{x}|} \ , \quad \phi = 0 \ . \tag{15}$$

This describes the anticipated response to a charge q probe: a standard Coulomb electric field. This could be the end of the story and we would have ended up with a conformal defect for every q. We could measure the scaling dimensions of defect operators from the fluctuations of ϕ and A. Since A appears only linearly and quadratically in the action, the fluctuations of A are insensitive to the defect at leading order and hence the operator dimensions in the bulk and defect coincide. The fluctuations of ϕ on the other hand are sensitive to the defect. The fluctuations of ϕ with angular momentum ℓ are allowed to have the following power-law behavior near the defect:

$$\phi \sim \alpha_\ell(t) r^{-\nu_\ell - 1/2} + \beta_\ell(t) r^{\nu_\ell - 1/2} ,$$

with $\nu_\ell = \sqrt{\frac{1}{4} + \ell(\ell+1) - g^4 q^2}$.

The modes α, β, by our bulk-defect OPE, correspond to operators of dimension $\Delta^L(\alpha_\ell) = 1/2 - \nu_\ell$ and $\Delta(\beta_\ell) = 1/2 + \nu_\ell$. It makes sense to think about α_ℓ as a defect operator only for $0 < \nu_\ell < 1/2$. From the s-wave modes, we see that

$$g^2 q < \frac{1}{2}$$

must be fulfilled, otherwise, all s-wave mode operators have complex scaling dimensions. Therefore we can already say that, unlike the pure $U(1)$ gauge theory, the question of which Wilson lines correspond to conformal line defects is not trivial.

It is useful to imagine that the Wilson line (i.e. our probe nucleus of charge q) has some UV cutoff, i.e. a radius r_0. The ϕ fluctuations have to be consistent with the choice of certain boundary conditions at r_0. This, in general, forces us to remove one linear combination of α_ℓ, β_ℓ. The most general boundary condition is

$$\alpha_\ell(t) = c_\ell r_0^{2\nu} \beta_\ell(t) ,$$

with some coefficient c_ℓ.

The two natural choices $c = 0$ (corresponding to imposing $\alpha = 0$) and $c = \infty$ (corresponding to imposing $\beta = 0$) are conformal boundary conditions since no explicit powers of the cutoff r_0 appear. These two correspond to two distinct conformal line defects.

- Since $g^2 q < \frac{1}{2}$, for $\ell > 0$, only the choice $\alpha = 0$ makes sense. Then we have a defect operator of dimension $\Delta(\beta_\ell) = 1/2 + \nu_\ell > 1/2 + \sqrt{2}$ for $\ell > 0$ which is an ordinary irrelevant operator (to make gauge invariant operators, bilinear operators must be considered). To be more precise, if the theory is deformed by this irrelevant operator, then the boundary condition will be with some finite nonzero c, but the statement is that one never reaches a sensible UV fixed point with $c_{\ell>0} = \infty$. In other words, it makes sense to do this only with a cutoff.
- For $\ell = 0$ the choices $c = 0$ and $c = \infty$ both make sense for any $g^2 q < \frac{1}{2}$, for $\ell > 0$, and they both correspond to a sensible UV complete DCFT. The choice $c = 0$ describes an infrared stable fixed point with an operator of dimension $\Delta^L(\beta_\ell) = 1/2 + \nu_\ell$ (again, we need to consider bilinears to make gauge invariant

combinations) while the choice $c = \infty$ corresponds to an infrared unstable DCFT with $\Delta^L(\beta_\ell) = 1/2 - \nu_\ell$ from which we can make a gauge invariant relevant bilinear $\beta^\dagger\beta$ which describes this RG flow between the UV DCFT with $\beta = 0$ and the IR DCFT with $\alpha = 0$.

In some sense it is fair to say that there is something misleading about the usual formula for the Wilson line $Pe^{iq\int dtA_0}$. In this presentation of the Wilson line, there are no free parameters that can flow (since $q \in \mathbb{Z}$ is quantized). However, there is a parameter which must be tuned to a fix point, which is the coefficient of the operator $\mu \int dt \beta^\dagger\beta = \mu r_0^{1-2\nu} \int dt \phi^\dagger\phi$. We can therefore write a more precise version of the Wilson loop as

$$Pe^{iq\int dtA_0 + ig\int dt\phi^\dagger\phi} \, . \tag{16}$$

Unlike q, the coupling g can flow and our analysis above shows that it has the following properties:

- For $g^2q > 1/2$ no fixed points exist at finite g and instead $\beta_g > 0$ and g flows to $-\infty$.
- For $g^2q = 1/2$ one fixed point with a marginal operator exists.
- For $g^2q < 1/2$ two fixed-points – one infrared stable and one unstable exist.

The fate of the flow $g \to -\infty$ can be analyzed quite explicitly and the main conclusion that a new saddle point appears, replacing (15). That saddle point leads to a trivial, completely screened line defect at long distances. Therefore for $g^2q > 1/2$ there are no nontrivial conformal Wilson lines.

This analysis does not complete the understanding of the phases of Wilson lines in QED$_4$. The infrared unstable fixed point obeys, as we saw, $\Delta^L(\phi) = 1/2 - \nu$ for the s-wave mode. Therefore, as we decrease g^2q, more and more operators become relevant at the unstable fixed point. While we know that the operator $\phi^\dagger\phi$ should trigger a flow to the stable fixed point, when $|\phi|^4$ becomes relevant the end-point is not obvious and it may or may not be the infrared stable Wilson line. Another remark is that the discussion above of Wilson lines in QED$_4$ carries over to non-Abelian conformal gauge theories such as $\mathcal{N} = 4$ SYM theory. $SO(6)_R$ invariant Wilson lines in that theory will only lead to healthy DCFTs for small enough representations and the number of such representations decreases as the coupling is increased. Presumably there are no Wilson lines at strong coupling, other than the ones protected by one-form symmetry, which we argued cannot disappear.

References

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