Fast Feynman integration with tropical geometry

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Based on arXiv:2008.12310

Motivation

Quantum field theory

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$\downarrow$

perturbative expansions

$$\mathcal{O}(\hbar) = \sum_{n\geq 0} A_n \hbar^n = \sum_{\text{graphs } G} \frac{I_G}{|\operatorname{Aut} G|}\hbar^{L_G}$$

where $I_G = \prod_{\ell} \int \mathrm{d}^D \boldsymbol{k}_\ell \prod_{e\in E} \frac{1}{D_e(\{\boldsymbol{k}\},\{\boldsymbol{p}\},m_e)}$.

Questions

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Lower orders $A_0, A_1, A_2, \ldots$ needed to interpret experimental data.

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How can we calculate them effectively?

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Practical question

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How can we calculate them effectively?

Associated abstract question: Computability

Is there an algorithm to compute A_n?

What is *the fastest* algorithm to compute A_n?

What is its runtime?

Is there an algorithm?

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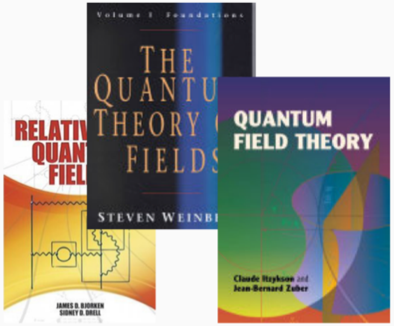
YES $\longrightarrow$



What is its runtime?

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Runtime to compute A_n for n large:

$$\mathcal{O}\Big(\alpha^n \Gamma(n+\beta) \quad \times \quad F(n) \Big)$$

→ number of Feynman graphs with n loops for $n \to \infty$
(α and β depend on theory and observable)

→ time it takes to evaluate a single Feynman integral of order n

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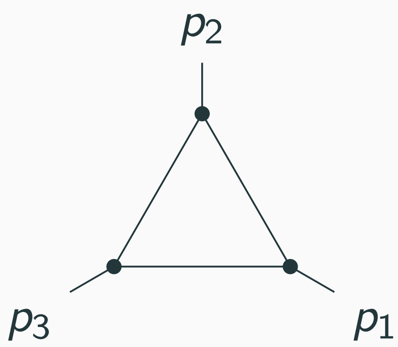
$\Rightarrow$ Tough to phrase runtime question for analytic computations.

Analytic vs numerical evaluation



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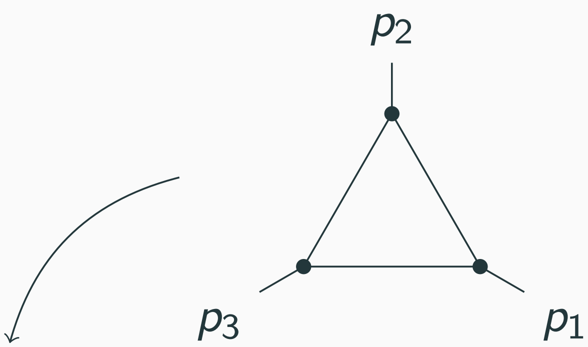

$$\int \frac{d^4k}{\pi^2} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2}$$

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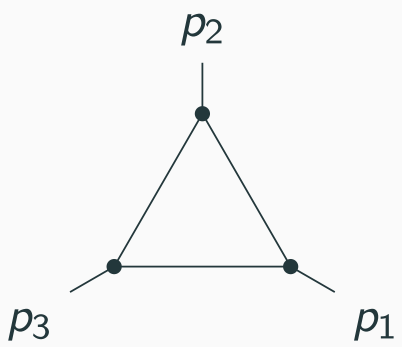

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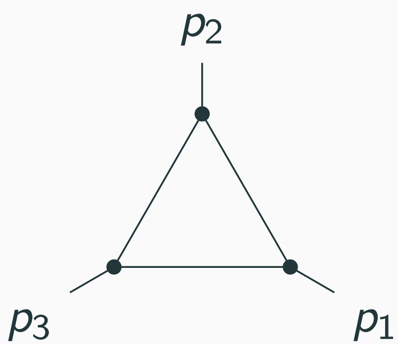

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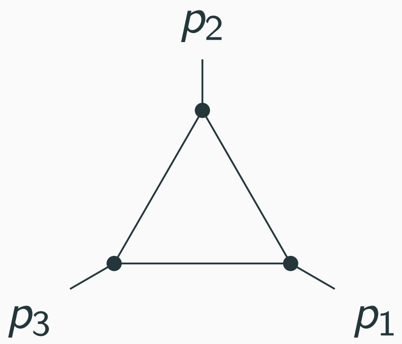

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$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

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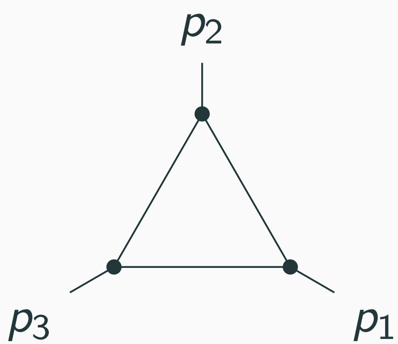

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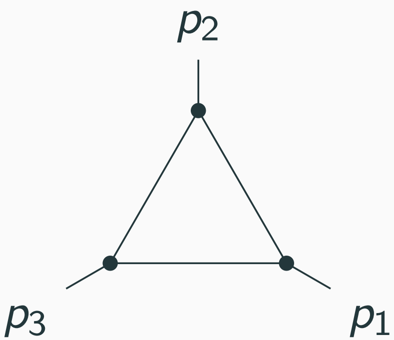
slow 'brute-force' evaluation

fast evaluation

Approach here: slow but general



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Numerical answer

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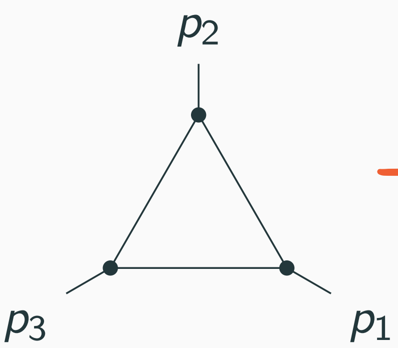

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Numerical answer

Strategy here:

Start with slow-but-general approach, then improve on it.

Direct evaluation

Algebraic geometric perspective

$$\mathcal{O}(\hbar) = \sum_{n \geq 0} A_n \hbar^n = \sum_{\text{graphs } G} \frac{I_G}{|\operatorname{Aut} G|} \hbar^{L_G}$$

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rewrite via Schwinger trick/Feynman parameters:

$$I_G = \Gamma(\omega_G) \int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{\Psi_G(\boldsymbol{x})^{D/2}} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G}$$

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where

- Ω is the standard volume form on $\mathbb{P}^{E-1}$:
  $\Omega = \sum_{k=1}^{E}(-1)^k dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_E$.
- $\Psi_G = \sum_T \prod_{e \notin T} x_e$ (sum over spanning trees)
- $\Phi_G = \sum_F \|\boldsymbol{p}(F)\|^2 \prod_{e \notin F} x_e + \Psi_G \sum_e m_e^2 x_e$(sum over 2-forests)
- Ψ_G and Φ_G are homogeneous polynomials in $x_1, \ldots, x_E$.
- We assume that the integral exists.

$$\int_{\mathbb{P}_{>0}^{E-1}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G}$$

$\Psi_G(\boldsymbol{x})$ and $\Phi_G(\boldsymbol{x})$ exhibit complicated geometric structures.

$\Rightarrow$ These integrals are hard to evaluate

$\Rightarrow$ These integrals are very interesting

$$\int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G}$$

Obstruction for direct numerical evaluation

Integrand has singularities on the boundary of $\mathbb{P}^{E-1}_{>0}$.

I.e. vanishing locus of Ψ_G and Φ_G meets the boundary of $\mathbb{P}^{E-1}_{>0}$.

$\Rightarrow$ Singularities have to be blown up first

Obstructions to numerical evaluation: Stereotypical example

$$\int_0^1 \int_0^1 \frac{\mathrm{d}x \mathrm{d}y}{x+y}$$

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$$\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{x+y} \quad \approx \quad \frac{1}{N}\sum_{n=1}^{N} \frac{1}{X_n + Y_n}$$

naive Monte Carlo evaluation

where $X_n, Y_n \in [0,1]$ are i.i.d. uniform random variables.

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$$\textbf{Var} \approx \int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{(x+y)^2} = \infty$$

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→ naive Monte Carlo fails

where $X_n, Y_n \in [0, 1]$ are i.i.d. uniform random variables.

$\Rightarrow$ even direct 'brute-force' numerical Feynman integration
is a non-trivial problem

Traditional solution

$$\int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G} \quad \rightarrow \quad \int_{\mathbb{P}^{n-1}_{>0}} \frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}} \Omega$$

Feynman integral — generalization

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Feynman integral   generalization

Sector decomposition approach

- Algorithms to perform blowups in the general case:
  Binoth, Heinrich '03; Bogner, Weinzierl '07; (Hironaka 1964)
- Simple geometric interpretation:
  Kaneko, Ueda '09

(No use of rich structure of Feynman integrals)

Projective algebraic integrals

$$\int_{\mathbb{P}_{>0}^{E-1}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G} \quad \rightarrow \quad \int_{\mathbb{P}_{>0}^{n-1}} \frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}} \Omega$$

Feynman integral | generalization

The general right hand side

- is essentially a 'Stringy integral' **Arkani-Hamed, He, Lam 2019**.
- is also a generalized Mellin transform **Nilsson, Passare 2010; Berkesch, Forsgård, Passare 2011**.
- can be interpreted in terms of toric geometry/varieties **Schultka 2018**.

Approximate runtime of the sector decomposition approach

$$\int_{\mathbb{P}^{n-1}_{>0}} \frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}} \Omega$$

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Numerical evaluation using sector decomposition for blowups:

- Runtime to evaluate the integral up to δ-accuracy

$$\approx \mathcal{O}(V^2 \cdot \delta^{-2})$$

where V is the number of monomials in $\frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}}$.

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  where V is the number of monomials in $\frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}}$.
- For Feynman integrals V grows $\approx$ exponentially with n.

  ‖
  # of edges

New results

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$\Rightarrow$ achievable accuracy 'decouples' from integral complexity.

3. Euclidean Feynman integration can be accelerated extremely:

$$\mathcal{O}(V^2 \cdot \delta^{-2}) \approx \mathcal{O}(2^{cn} \cdot \delta^{-2}) \quad \to \quad \mathcal{O}(n2^n + n^4\delta^{-2})$$

with $c \gg 1$ where n is the number of edges of the graph.

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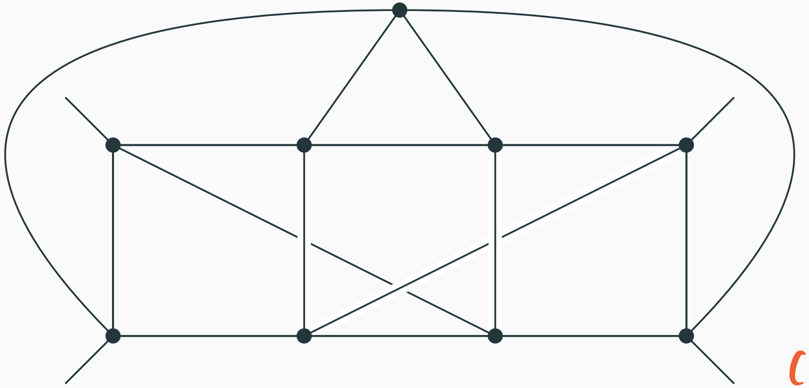
- 3 loops is already a tough challenge for existing programs.
- New: ≈ 17 loops possible (with basic implementation).
- Caveat: Only Euclidean - no Minkowski regime (so far).

Interesting example



(Brown-Schnetz)

Figure 1: A non-generalized polylog/non-MZV 8-loop φ^4-graph.

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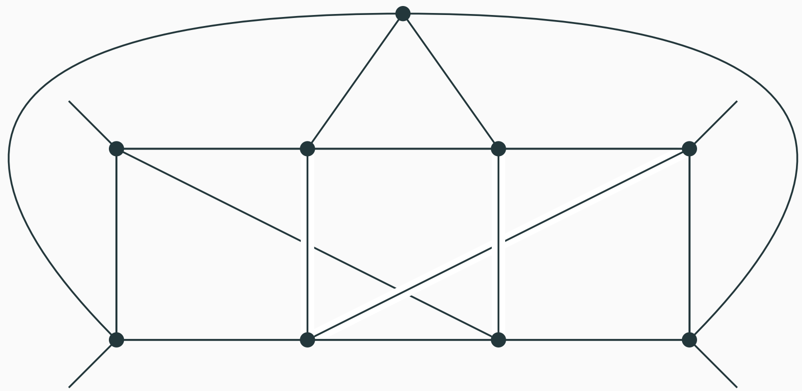

Figure 1: A non-generalized polylog/non-MZV 8-loop φ^4-graph.

$$\Gamma(\varepsilon)\int_{\mathbb{P}^{E-1}_{>0}} \frac{1}{\Psi_G(\boldsymbol{x})^{2-\varepsilon}}\left(\frac{\Psi_G}{\Phi_G}\right)^{\varepsilon}\Omega \approx \frac{1}{\epsilon}\, 422.9610\cdot(1\pm 10^{-6})+\dots$$

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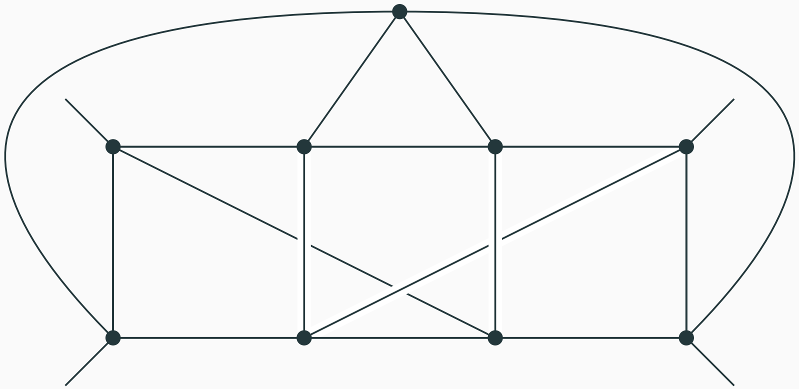

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- ~ 10 CPU secs to compute up to 10^{-3}-accuracy at 8 loops.

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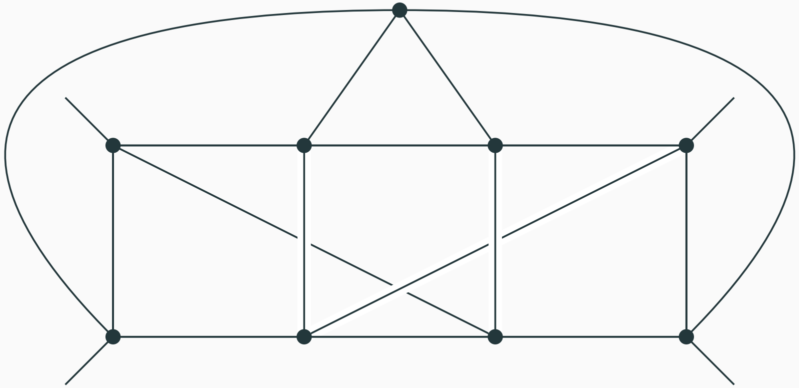

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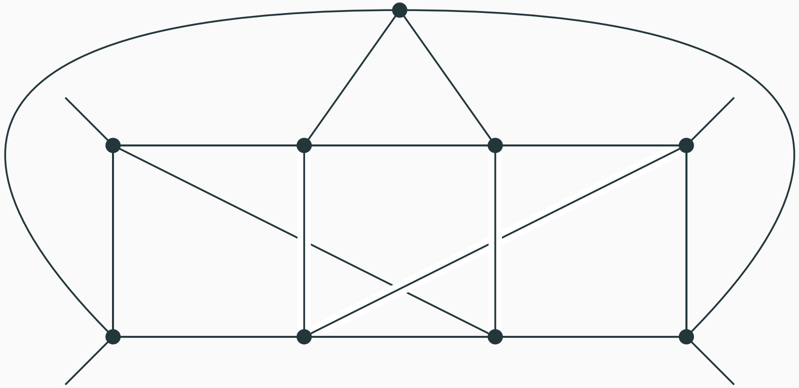

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- $\sim$ 10 CPU secs to compute up to 10^{-3}-accuracy at 8 loops.
- $\sim$ 30 CPU days to compute up to 10^{-6}-accuracy at 8 loops.
- Higher orders in ϵ can also be computed.

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Minimal runtime to evaluate a Euclidean Feynman integral with n edges up to δ-accuracy is *at most* $\mathcal{O}(n2^n + n^4\delta^{-2})$

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$\Rightarrow$ Improvements likely!

Tropicalized Feynman integrals

Algebraic geometric motivation

$$\int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G}$$

Algebraic geometric motivation

$$\int_{\mathbb{P}_{>0}^{E-1}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G}$$

Problem: Complicated geometry obstructs integration

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Problem: Complicated geometry obstructs integration

Solution: Simplify the geometry

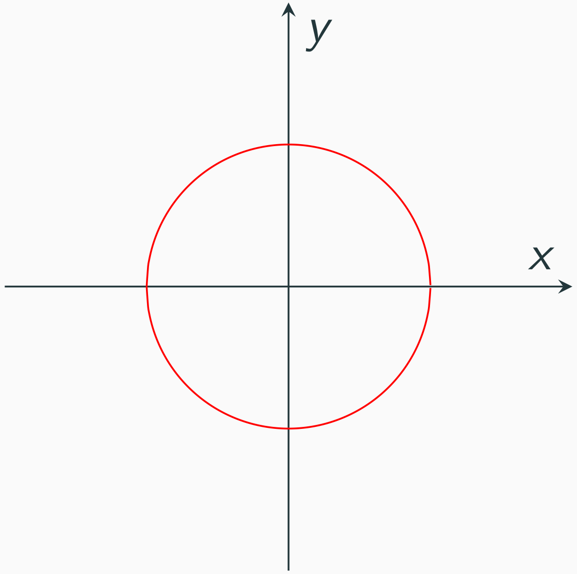
Tropical geometry

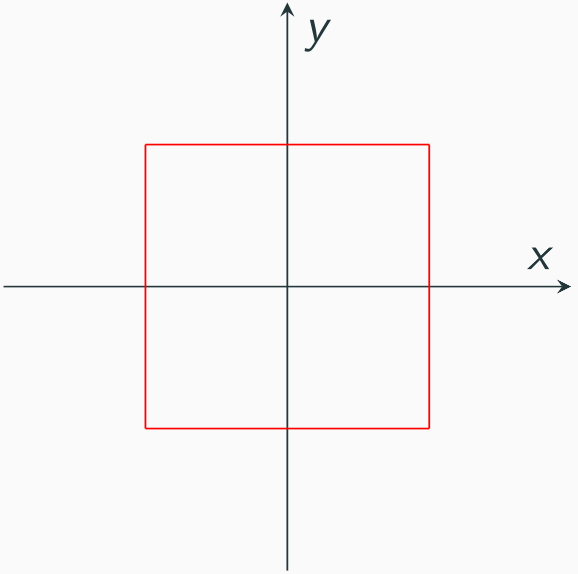
Philosophy

Deform geometry to sacrifice smoothness for simplicity.

Various applications in algebraic geometry.

$$1 = x^2 + y^2 \qquad \to \qquad 1 = (x^2 + y^2)^{\mathrm{tr}} = \max\{x^2, y^2\}$$





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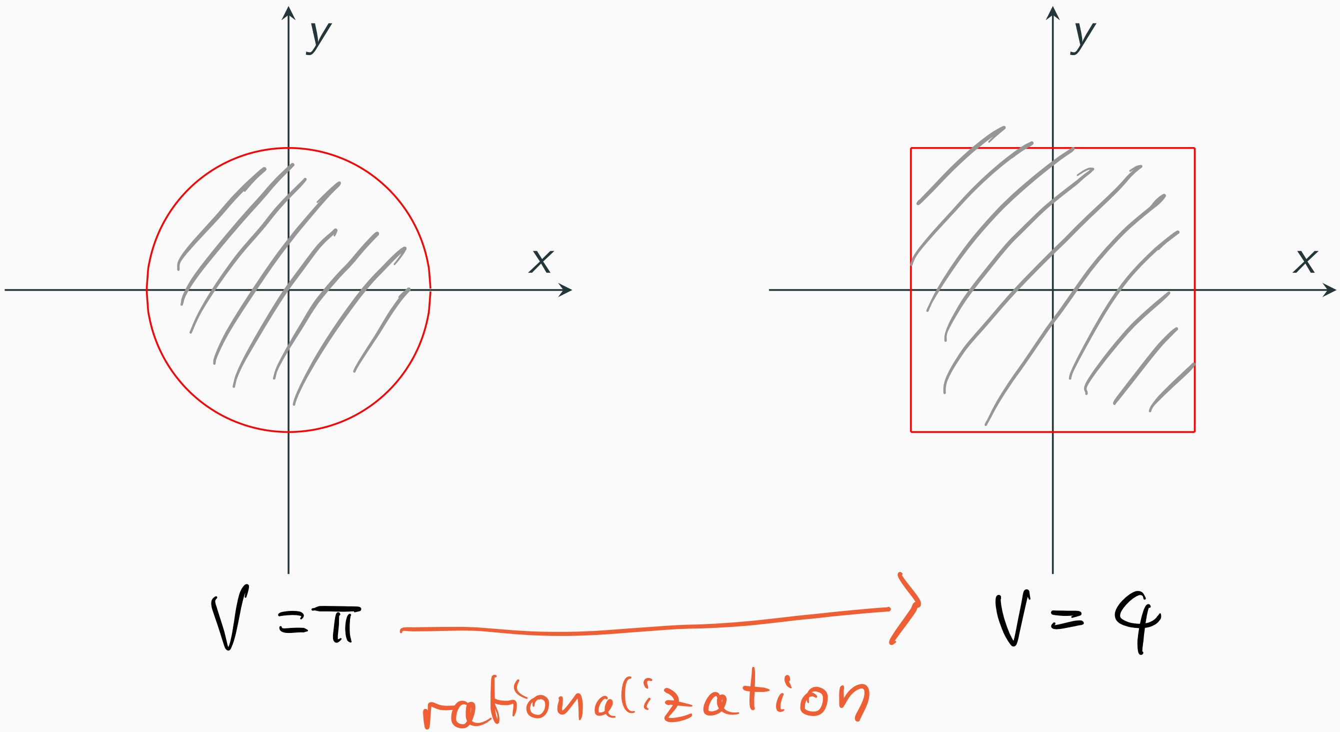

Tropical approximation of a polynomial

Let p be a homogeneous polynomial in n variables:

$$p(x_1, \ldots, x_n) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^n} a_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}}$$

$$\to \prod_{i=1}^{n} x_i^{k_i}$$

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'tropicalize'

$$p^{\mathrm{tr}}(x_1, \ldots, x_n) = \max_{\substack{\boldsymbol{k} \in \mathbb{N}_0^n \\ \text{s.t. } a_{\boldsymbol{k}} \neq 0}} \boldsymbol{x}^{\boldsymbol{k}}$$

p^{tr} is the tropical approximation of p.

The approximation property of the tropicalization

$$p(x_1, \ldots, x_n) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^n} a_{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}}; \qquad p^{\mathrm{tr}}(x_1, \ldots, x_n) = \max_{\substack{\boldsymbol{k} \in \mathbb{N}_0^n \\ \text{s.t. } a_{\boldsymbol{k}} \neq 0}} \boldsymbol{x}^{\boldsymbol{k}}$$

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Theorem (MB 2020)

If $p(\boldsymbol{x})$ is completely non-vanishing on $\mathbb{P}^n_{>0}$, then

$$C_1 p^{\mathrm{tr}}(\boldsymbol{x}) \leq |p(\boldsymbol{x})| \leq C_2 p^{\mathrm{tr}}(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{P}^n_{>0}$$

for some positive constants $C_1, C_2 > 0$.

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Proof:

Obvious if $p(\boldsymbol{x})$ has only positive coefficients. Otherwise not...

Application to Feynman graph polynomials

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QFT tropicalization

Replace all instances of Ψ and Φ with their tropicalized versions.

Approximation property

There are constants $C_1, C_2 > 0$, such that

$$C_1 \frac{1}{(\Psi_G^{\mathrm{tr}})^{D/2}} \left(\frac{\Psi_G^{\mathrm{tr}}}{\Phi_G^{\mathrm{tr}}}\right)^{\omega_G} \leq \left| \frac{1}{(\Psi_G)^{D/2}} \left(\frac{\Psi_G}{\Phi_G}\right)^{\omega_G} \right| \leq C_2 \frac{1}{(\Psi_G^{\mathrm{tr}})^{D/2}} \left(\frac{\Psi_G^{\mathrm{tr}}}{\Phi_G^{\mathrm{tr}}}\right)^{\omega_G}$$

for all $\boldsymbol{x} \in \mathbb{P}_{>0}^{E-1}$, if Ψ_G and Φ_G are *completely non-vanishing*.

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for all $\boldsymbol{x} \in \mathbb{P}_{>0}^{E-1}$, if Ψ_G and Φ_G are *completely non-vanishing*.

$$\Rightarrow C_1 I_G^{\mathrm{tr}} \leq |I_G| \leq C_2 I_G^{\mathrm{tr}}$$

$\Rightarrow$ The tropicalized integral gives both an upper and a lower bound

Tropical approach

Evaluating tropicalized Feynman integrals is easy

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- At low order ($n = 1, \ldots, 20$):
  Tropicalized Feynman integrals are easily calculated exactly
  All observables are rational numbers/functions.
  Panzer 2019; MB 2020

Tropical approach

Evaluating tropicalized Feynman integrals is easy

- At low order ($n = 1, \ldots, 20$):
  Tropicalized Feynman integrals are easily calculated exactly
  All observables are rational numbers/functions.
  Panzer 2019; MB 2020
- When the tropical version is known exactly, numerical integration of the original integrals is just an extra step.
  MB 2020

Tropical Numerical integration

It is convenient to generalize first (and specify again later):

$$\int_{\mathbb{P}^{E-1}_{>0}} \frac{\Omega}{\Psi_G^{D/2}(\boldsymbol{x})} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G} \qquad \rightarrow \qquad \int_{\mathbb{P}^{n-1}_{>0}} \frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}} \Omega$$

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Warm-Up

If q_j are all completely non-vanishing, then

$$\int_{\mathbb{P}^{n-1}_{>0}} \frac{\prod_i p_i^{\mathrm{tr}}(\boldsymbol{x})^{\mu_i}}{\prod_j q_j^{\mathrm{tr}}(\boldsymbol{x})^{\nu_i}} \Omega < \infty \quad \Leftrightarrow \quad \int_{\mathbb{P}^{n-1}_{>0}} \frac{\prod_i p_i(\boldsymbol{x})^{\mu_i}}{\prod_j q_j(\boldsymbol{x})^{\nu_i}} \Omega < \infty,$$

i.e. tropical convergence is equivalent to ordinary convergence.

Tropical numerical integration

Strategy

Treat the exact integral as perturbation around the tropical one.

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$$\int_{\mathbb{P}^{n-1}_{>0}} \frac{\prod_i p_i(\boldsymbol{x})}{\prod_j q_j(\boldsymbol{x})} \Omega$$

Tropical numerical integration

Strategy

Treat the exact integral as perturbation around the tropical one.

$$\int_{\mathbb{P}^{n-1}_{>0}} \frac{\prod_i p_i(\boldsymbol{x})}{\prod_j q_j(\boldsymbol{x})}\Omega = \int_{\mathbb{P}^{n-1}_{>0}} \Omega \, \frac{\prod_i p_i^{\mathrm{tr}}(\boldsymbol{x})}{\prod_j q_j^{\mathrm{tr}}(\boldsymbol{x})} \underbrace{\frac{\prod_i \frac{p_i(\boldsymbol{x})}{p_i^{\mathrm{tr}}(\boldsymbol{x})}}{\prod_j \frac{q_j(\boldsymbol{x})}{q_j^{\mathrm{tr}}(\boldsymbol{x})}}}_{\text{bounded}}$$

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tropicalized integrand

bounded

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Tropicalization solves the blowup problem!

The form $\widetilde{\mu}^{\mathrm{tr}}$ is the canonical measure on the common refinement of the normal fans of the Newton polytopes of the p_i, q_j.

The tropical form

$$\widetilde{\mu}^{\mathrm{tr}} = \frac{\prod_i p_i^{\mathrm{tr}}(\boldsymbol{x})}{\prod_j q_j^{\mathrm{tr}}(\boldsymbol{x})} \Omega$$

Sampling from this measure allows numerical integration.

Theorem (MB 2020)

If the Newton polytopes of p_i and q_j are 'not too complicated', then there is a (reasonably) fast algorithm to sample the measure

$$\mu^{\mathrm{tr}} = \frac{1}{Z} \frac{\prod_i p_i^{\mathrm{tr}}(\boldsymbol{x})}{\prod_j q_j^{\mathrm{tr}}(\boldsymbol{x})} \Omega \quad \text{on } \mathbb{P}_{>0}^{n-1}$$

with Z chosen such that $\int_{\mathbb{P}_{>0}^{n-1}} \mu^{\mathrm{tr}} = 1$.

$$\int_{\mathbb{P}^{n-1}_{>0}} \frac{\prod_i p_i(\boldsymbol{x})}{\prod_j q_j(\boldsymbol{x})}\Omega = Z \int_{\mathbb{P}^{n-1}_{>0}} \mu^{\mathrm{tr}} \frac{\prod_i \frac{p_i(\boldsymbol{x})}{p_i^{\mathrm{tr}}(\boldsymbol{x})}}{\prod_j \frac{q_j(\boldsymbol{x})}{q_j^{\mathrm{tr}}(\boldsymbol{x})}}$$

$\Rightarrow$ Can be evaluated by sampling from the measure μ^{tr}.

(Monte Carlo)

runtime depends on the shape of the polytopes: $\mathcal{O}(V^2 + V\delta^{-2})$.

Generalized permutahedra

Back to Feynman integrals and QFT

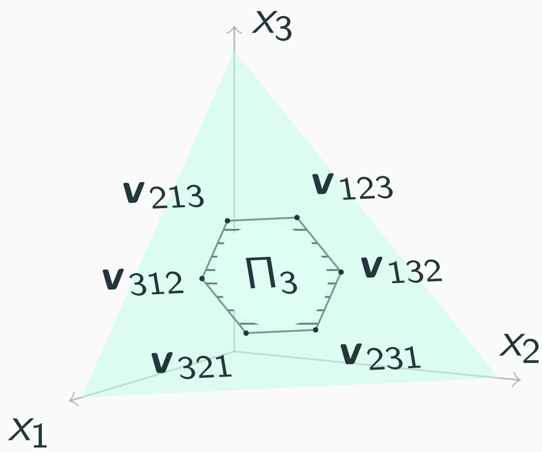
The relevant polytopes that appear in QFT have a special shape.

They are generalized permutahedra **Postnikov 2008**.

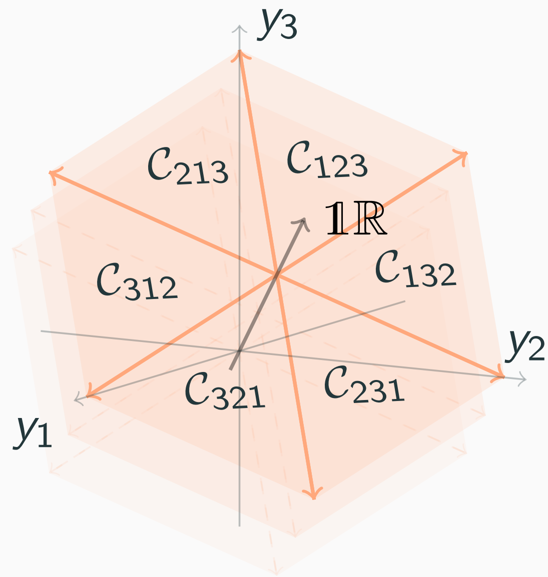
Theorem (Schultka 2018) using results from (Brown 2015)

The Newton polytopes of the graph polynomials Ψ_G and Φ_G are generalized permutahedra.

(Φ_G only if the momenta are Euclidean.)



(a) The permutahedron $\Pi_3 \subset \mathbb{R}^3$.



(b) Dual of Π_3: The corresponding braid arrangement fan.

This structure follows form the factorization properties of Ψ and Φ.

A direct and simple proof

Recall that

$$\Psi_G = \sum_T \prod_{e \notin T} x_e \text{ (sum over spanning trees)}$$

$$\Phi_G = \sum_F \|p(F)\|^2 \prod_{e \notin F} x_e \text{ (sum over 2-forests)}$$

Classical combinatorial arguments show that,

a given ordering of the edges fixes

a maximal spanning tree $\Rightarrow \Psi_G$ is a gen. permutahedron

a maximal 2-forest* $\Rightarrow \Phi_G$ is a gen. permutahedron

(Also slightly more general than **Brown** and **Schultka**)

Consequences of the generalized permutahedron property

Theorem (MB 2020)

If the Newton polytopes of p_i and q_j are gen. permutahedra, then there is a (very) fast algorithm to sample from the measure

$$\mu^{\mathrm{tr}} = \frac{1}{Z} \frac{\prod_i p_i^{\mathrm{tr}}(\boldsymbol{x})}{\prod_j q_j^{\mathrm{tr}}(\boldsymbol{x})} \Omega \quad \text{on } \mathbb{P}^{n-1}_{>0},$$

with Z chosen such that $\int_{\mathbb{P}^{n-1}_{>0}} \mu^{\mathrm{tr}} = 1$.

Makes heavy use of tools from **Aguiar, Ardila 2017**.

Generalized permutahedron sampling algorithm

Algorithm 4 to generate a sample from μ^{tr} for generalized permutahedra

Set $A = [n]$ and $\kappa = 1$
while $A \neq \emptyset$ **do**
  Pick a random $e \in A$ with probability $p_e = \frac{1}{J_r(A)} \frac{J_r(A \backslash e)}{r(A \backslash e)}$.
  Remove e from A, i.e. set $A \leftarrow A \setminus e$.
  Set $\sigma(|A|) = e$.
  Set $x_e = \kappa$.
  Pick a uniformly distributed random number $\xi \in [0, 1]$.
  Set $\kappa \leftarrow \kappa \xi^{1/r(A)}$.
end while
Return $\boldsymbol{x} = [x_1, \ldots, x_n] \in \mathrm{Exp}(\mathcal{C}_\sigma) \subset \mathbb{P}^{n-1}_{>0}$ and $\sigma = (\sigma(1), \ldots, \sigma(n)) \in S_n$.

→ easy to implement

→ code available on my website

Outlook

Minkowski singularities

Problem: Non-Euclidean kinematic regions are not as fast, because

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- Φ_G can vanish in the integration domain ($\Rightarrow$ analytic continuation is necessary).

  But the approximation property still works.

Minkowski singularities

Problem: Non-Euclidean kinematic regions are not as fast, because

- The generalized permutahedron structure breaks down at singular momentum configurations (IR singularities).
- Φ_G can vanish in the integration domain (⇒ analytic continuation is necessary).

  But the approximation property still works.
- Vanishing locus of Φ_G for complex $\boldsymbol{x}$ important.

Q: What tropical geometries do appear?

Further outlook: Amplitudes in Outer space

The n-th order correction to the scattering amplitude is given by,

$$A_n = \sum_{\substack{\text{graphs } G \\ L_G = n}} \frac{I_G}{|\operatorname{Aut} G|}$$

Further outlook: Amplitudes in Outer space

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$$= \sum_{\substack{\text{graphs } G \\ L_G = n}} \frac{1}{|\operatorname{Aut} G|} \int_{\mathcal{T}_G} \mu_G^{\mathrm{tr}} \left(\frac{\Psi_G^{\mathrm{tr}}}{\Psi_G}\right)^{2-\omega_G} \left(\frac{\Phi_G^{\mathrm{tr}}}{\Phi_G}\right)^{\omega_G}$$

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Integral over version of Outer space/tropical moduli space

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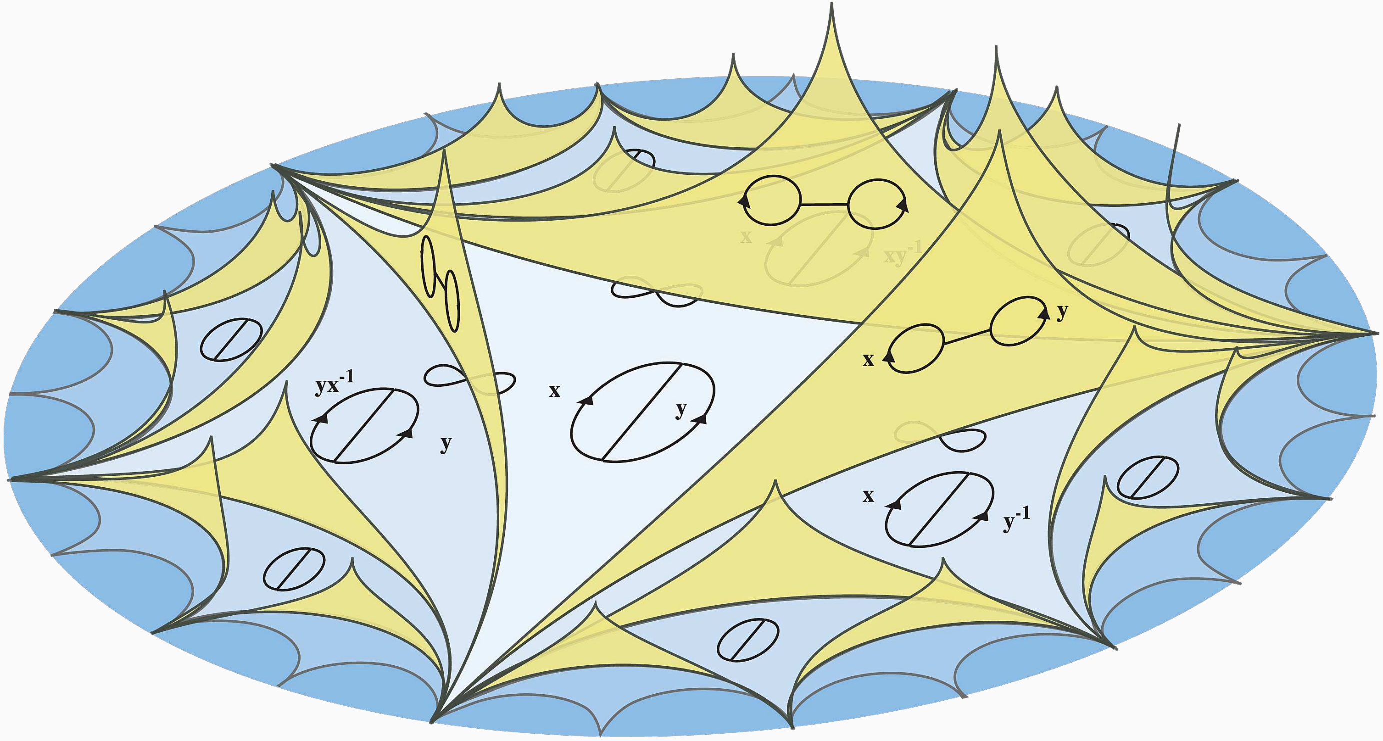

Figure 3: Outer space $\mathcal{O}_2$ which is a specific graph orbispace

Vogtmann 2018

Conclusions

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  Much left to explore:

  - Generalize generalized permutahedra

  $\Rightarrow$ IR singularities/Minkowski space

  - Tropical amplitudes
  - Relation to tropical moduli spaces/Outer space?
  - Gauge theory?
  - ...

Factorization properties

Generalized permutahedron property follows from factorizations:

For some $\gamma \subset \Gamma$, let $x'_e = \lambda_\gamma x_e$ if $e \in \gamma$ and else $x'_e = x_e$:

$$\Psi'_\Gamma = \lambda_\gamma^{h_1(\gamma)} \Psi_\gamma \Psi_{\Gamma/\gamma} + \mathcal{O}(\lambda_\gamma^{h_1(\gamma)+1}) \text{ as } \lambda_\gamma \to 0$$
$$\Phi'_\Gamma = \lambda_\gamma^{h_1(\gamma)} \Psi_\gamma \Phi_{\Gamma/\gamma} + \mathcal{O}(\lambda_\gamma^{h_1(\gamma)+1}) \text{ as } \lambda_\gamma \to 0$$

Degenerate case **Brown 2015**: If $\Phi_{\Gamma/\gamma} = 0$, then

$$\Phi'_\Gamma = \lambda_\gamma^{h_1(\gamma)+1} \Phi_\gamma \Psi_{\Gamma/\gamma} + \mathcal{O}(\lambda_\gamma^{h_1(\gamma)+2}) \text{ as } \lambda_\gamma \to 0.$$

(only with Euclidean momenta)

Open question

Generalized permutahedra are *universal* with respect to their Hopf monoid structure **Aguiar, Ardila 2017**.

How general are Feynman integrals within this class of polytopes?

Structures at play

A gen. permutahedron has a *facet presentation* (**Postnikov 2008**):

$$\mathcal{G}_z = \left\{ \boldsymbol{v} \in \mathbb{R}^n : \sum_{i \in [n]} v_i = z([n]) \text{ and } \sum_{i \in I} v_i \geq z(I) \text{ for all } I \subset [n] \right\},$$

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Complete geometric data is encoded in the boolean function

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$\Rightarrow$ good control over (tropical) geometry.

Important: Control over vertices

There is a surjective map from S_n to the vertices of $\mathcal{G}_z$:

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Given $\sigma \in S_n$, then $\boldsymbol{w}^{(\sigma,z)} \in \mathrm{Vert}_{\mathcal{G}_z}$ with

$$w_{\sigma(k)}^{(\sigma,z)} = z(A_k^\sigma) - z(A_{k-1}^\sigma) \text{ for all } k \in [n],$$

where $A_k^\sigma = \{\sigma(1), \ldots, \sigma(k)\} \subset [n] = \{1, \ldots, n\}$.

Implicit recursive structures

Computing the gen. permutahedral geometry (MB 2020)

For a boolean function $r : \mathbf{2}^{[n]} \to \mathbb{R}$ with $r(\emptyset) = 1$ and $r(A) > 0$ for all non-empty $A \subsetneq [n]$, we define the boolean function $J_r : \mathbf{2}^{[n]} \to \mathbb{R}_{>0}$ recursively as

$$J_r(A) = \sum_{e \in A} \frac{J_r(A \setminus e)}{r(A \setminus e)} \text{ for all non-empty } A \subset [n] \text{ where } J_r(\emptyset) = 1.$$

Tropicalized integral is a special case (MB 2020)

If $r(A) = z_{\mathcal{A}}(A) - z_{\mathcal{B}}(A)$ for all non-empty $A \subsetneq [n]$ and $r(\emptyset) = 1$, then $I^{\mathrm{tr}} = J_r([n])$.

Feynman integrand evaluation

$$\frac{1}{\Psi_G(\boldsymbol{x})^{D/2}} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G}$$

$$\Psi_G(\boldsymbol{x}) = \sum_{T} \prod_{e \notin T} x_e$$

$$\Phi_G(\boldsymbol{x}) = \sum_{F} \boldsymbol{p}(F)^2 \prod_{e \notin F} x_e$$

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$$\Psi_G(\boldsymbol{x}) = \sum_T \prod_{e \notin T} x_e \qquad = \left(\prod_e x_e \right) \det(\widetilde{L})$$

$$\Phi_G(\boldsymbol{x}) = \sum_F \boldsymbol{p}(F)^2 \prod_{e \notin F} x_e \qquad = \Psi_G \left(\mathrm{Tr}(P^T \widetilde{L}^{-1} P) \right)$$

where $P_{v,\mu} = p_\mu^{(v)}$ and $\widetilde{L}$ is the graph Laplacian.

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$$\frac{1}{\Psi_G(\boldsymbol{x})^{D/2}} \left(\frac{\Psi_G(\boldsymbol{x})}{\Phi_G(\boldsymbol{x})} \right)^{\omega_G}$$

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where $P_{v,\mu} = p_\mu^{(v)}$ and $\widetilde{L}$ is the graph Laplacian.

Very fast algorithms for such graph Laplacian computations exist (see **Spielman-Teng 2004**).