A novel theoretical approach to R(D(*)) through the Dispersive Matrix method

Work in collaboration with G. Martinelli, M. Naviglio and S. Simula

[PRD ’21 (2105.02497), PRD ’21 (2105.07851), PRD ’22 (2105.08674), 2109.15248, 2204.05925]

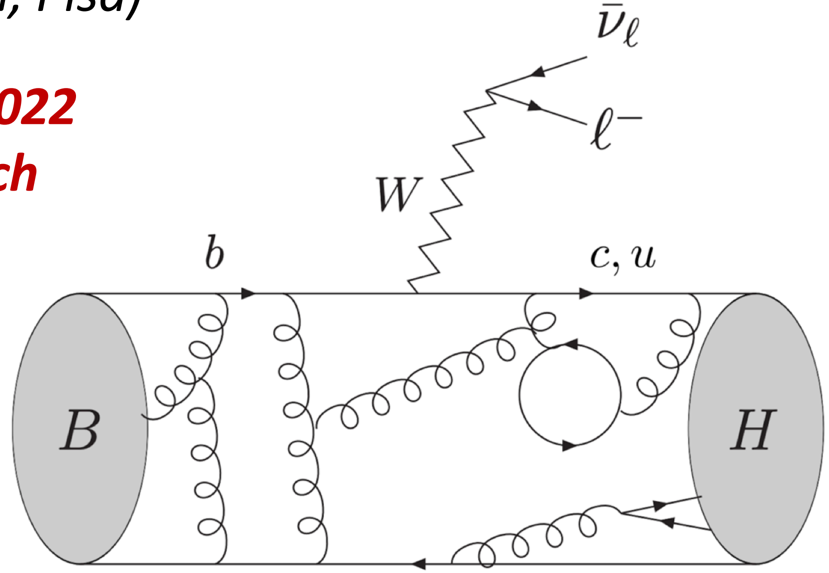
Ludovico Vittorio (SNS & INFN, Pisa)

LF(U)V Workshop 2022
University of Zurich

SCUOLA NORMALE SUPERIORE

INFN PISA
Istituto Nazionale di Fisica Nucleare
Sezione di Pisa

MINISTERO DELL' ISTRUZIONE, DELL'UNIVERSITÀ E DELLA RICERCA
PRIN “The consequences of flavor”



(from J.Phys.G 46 (2019) 2, 023001)

State-of-the-art of the semileptonic B → D(*) decays

Two critical issues:

- V_{cb} puzzle:

$2.8\,\sigma$ discrepancy!

EXCLUSIVE

FLAG Review 2021 [arXiv:2111.09849]

$$|V_{cb}| \times 10^3 = 39.36(68)$$

Fit only to the B → (D, D) channels*

VS

INCLUSIVE

$$|V_{cb}| \times 10^3 = 42.00(65)$$

$$|V_{cb}|_{\rm incl} \times 10^3 = 42.16 \pm 0.50$$

Bordone et al., Phys.Lett.B [2107.00604]

$$|V_{cb}|_{\rm incl} \times 10^3 = 41.69 \pm 0.63$$

Bernlochner et al., arXiv:2205.10274

State-of-the-art of the semileptonic B → D(*) decays

Two critical issues:

- V_{cb} puzzle:

2.8 σ discrepancy!

EXCLUSIVE

FLAG Review 2021 [arXiv:2111.09849]

$$|V_{cb}| \times 10^3 = 39.36(68)$$

Fit only to the B → (D, D) channels*

VS

INCLUSIVE

$$|V_{cb}| \times 10^3 = 42.00(65)$$

$$|V_{cb}|_{\rm incl} \times 10^3 = 42.16 \pm 0.50$$

Bordone et al., Phys.Lett.B [2107.00604]

$$|V_{cb}|_{\rm incl} \times 10^3 = 41.69 \pm 0.63$$

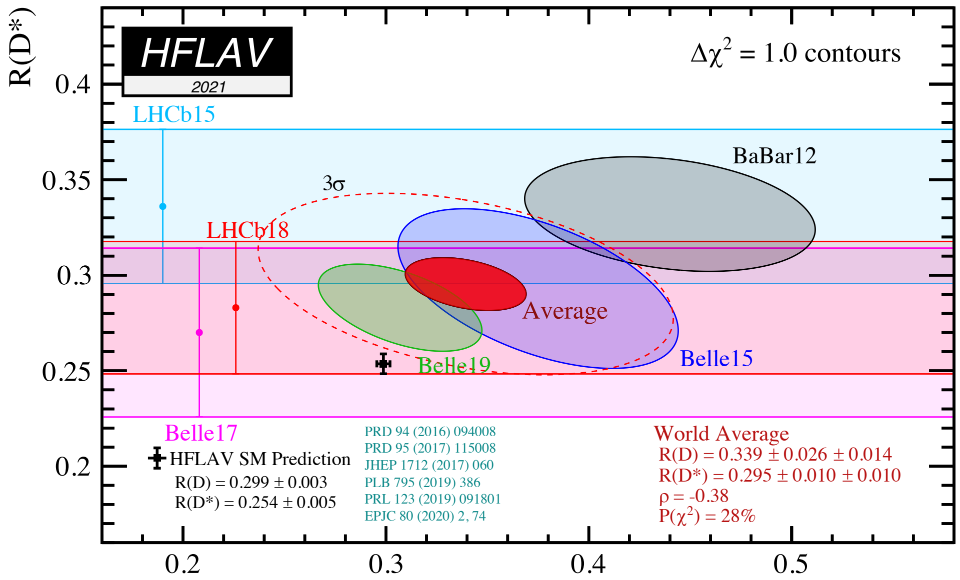
Bernlochner et al., arXiv:2205.10274

- $R_{D^{(*)}}$ anomalies:

$$\mathcal{R}(D) = \frac{\mathcal{B}(B \to D\tau\nu_\tau)}{\mathcal{B}(B \to D\ell\nu_\ell)},$$

$$\mathcal{R}(D^*) = \frac{\mathcal{B}(B \to D^*\tau\nu_\tau)}{\mathcal{B}(B \to D^*\ell\nu_\ell)}$$

3.4σ discrepancy!



HFLAV Collaboration, arXiv:2206.07501 [hep-ex]

State-of-the-art of the semileptonic B → D(*) decays

Two critical issues:

- V_{cb} puzzle:

EXCLUSIVE — *INCLUSIVE*

FLAG Review 2021 [arXiv:2111.09849]

$$|V_{cb}| \times 10^3 = 39.36(68) \quad \textbf{\textit{VS}} \quad |V_{cb}| \times 10^3 = 42.00(65)$$

Fit only to the B → (D, D) channels*

$$|V_{cb}|_{\rm incl} \times 10^3 = 42.16 \pm 0.50$$

Bordone et al., Phys.Lett.B [2107.00604]

$$|V_{cb}|_{\rm incl} \times 10^3 = 41.69 \pm 0.63$$

Bernlochner et al., arXiv:2205.10274

2.8 σ discrepancy!

- $R_{D^{(*)}}$ anomalies:

$$\mathcal{R}(D) = \frac{\mathcal{B}(B \to D\tau\nu_\tau)}{\mathcal{B}(B \to D\ell\nu_\ell)},$$

$$\mathcal{R}(D^*) = \frac{\mathcal{B}(B \to D^*\tau\nu_\tau)}{\mathcal{B}(B \to D^*\ell\nu_\ell)}$$

$$R(D) = 0.298 \pm 0.003$$

$$R(D)|_{\rm exp} = 0.339 \pm 0.026 \pm 0.014$$

and

$$R(D^*) = 0.252 \pm 0.005$$

$$R(D^*)|_{\rm exp} = 0.295 \pm 0.010 \pm 0.010$$

3.4σ discrepancy!

HFLAV Coll. (https://hflav-eos.web.cern. ch/hflav-eos/semi/spring21/html/RDsDsstar/RDRDs.html)

The Dispersive Matrix (DM) method completely changes the picture!

V_{cb}

1σ compatibility

EXCLUSIVE

$$|V_{cb}| \times 10^3 = 41.4 \pm 0.8$$

Combination of the $B_{(s)} \to (D, D^*, D_s, D_s^*)$ channels

VS

INCLUSIVE

$$|V_{cb}| \times 10^3 = 42.00(65)$$

$$|V_{cb}|_{\rm incl} \times 10^3 = 42.16 \pm 0.50$$

Bordone et al., Phys.Lett.B [2107.00604]

$$|V_{cb}|_{\rm incl} \times 10^3 = 41.69 \pm 0.63$$

Bernlochner et al., arXiv:2205.10274

$R_{D^{(*)}}$

The **DM method** allows to lighten both the problems!

$$R(D) = 0.296 \pm 0.008$$

$$R(D)|_{\rm exp} = 0.339 \pm 0.026 \pm 0.014$$

and

$$R(D^*) = 0.275 \pm 0.008$$

$$R(D^*)|_{\rm exp} = 0.295 \pm 0.010 \pm 0.010$$

1.3σ compatibility

The central role of the Form Factors (FFs) in excl. semil. *B* decays

- Production of a pseudoscalar meson (*i.e. D,π*) in case of massless lepton:

$$\frac{d\Gamma}{dq^2} \simeq \frac{G_F^2|V_{cb}|^2\eta_{EW}^2}{24\pi^3}|\vec{p}_D|^3|\boxed{f^+(q^2)}|^2$$

- Production of a vector meson (*i.e. D**) in case of massless leptons:

$$\frac{d\Gamma(B\to D^*(\to D\pi)\ell\nu)}{dw\,d\cos\theta_\ell d\cos\theta_v d\chi} = \frac{G_F^2|V_{cb}|^2\eta_{EW}^2}{4(4\pi)^4}3m_B m_{D^*}^2\sqrt{w^2-1}$$

$$\begin{aligned}&\times B(D^*\to D\pi)\{(1-\cos\theta_\ell)^2\sin^2\theta_v|H_+|^2\\&+(1+\cos\theta_\ell)^2\sin^2\theta_v|H_-|^2+4\sin^2\theta_\ell\cos^2\theta_v|H_0|^2\\&-2\sin^2\theta_\ell\sin^2\theta_v\cos 2\chi H_+H_-\\&-4\sin\theta_\ell(1-\cos\theta_\ell)\sin\theta_v\cos\theta_v\cos\chi H_+H_0\\&+4\sin\theta_\ell(1+\cos\theta_\ell)\sin\theta_v\cos\theta_v\cos\chi H_-H_0\},\end{aligned}$$

$$H_\pm(w) = \boxed{f(w)} \mp m_B m_{D^*}\sqrt{w^2-1}\,\boxed{g(w)}$$

$$H_0(w) = \frac{\boxed{\mathcal{F}_1(w)}}{\sqrt{m_B^2+m_D^2-2m_Bm_Dw}}$$

Relation between the momentum transfer and the recoil:

$$\boxed{q^2 = m_B^2 + m_P^2 - 2m_Bm_P\,w}$$

The central role of the Form Factors (FFs) in excl. semil. *B* decays

- Production of a pseudoscalar meson (*i.e. D,π*) in case of **massive** lepton:

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2|V_{cb}|^2\eta_{EW}^2}{24\pi^3}\left(1-\frac{m_\ell^2}{q^2}\right)^2$$
$$\times\left[|\vec{p}_D|^3\left(1+\frac{m_\ell^2}{2q^2}\right)|f^+(q^2)|^2 + m_B^2|\vec{p}_D|\left(1-\frac{m_D^2}{m_B^2}\right)^2\frac{3m_\ell^2}{8q^2}|f^0(q^2)|^2\right]$$

- Production of a vector meson (*i.e. D**) in case of **massive** leptons:

$$\frac{d\Gamma_\tau}{dw} = \frac{d\Gamma_{\tau,1}}{dw} + \frac{d\Gamma_{\tau,2}}{dw}$$

$$\frac{d\Gamma_{\tau,1}}{dw} = \left(1-\frac{m_\tau^2}{q(w)^2}\right)^2\left(1+\frac{m_\tau^2}{2q(w)^2}\right)\times\frac{d\Gamma}{dw} \qquad [r = m_{D^*}/m_B]$$

$$\frac{d\Gamma}{dw} = \frac{\eta_{EW}^2 G_F^2 m_{D^*}^2|V_{cb}|^2}{48\pi^3 m_B}\sqrt{w^2-1}\left[2\,q^2(w)\left(f(w)^2 + m_B^2 m_{D^*}^2\,(w^2-1)\,g(w)^2\right) + \mathcal{F}_1(w)^2\right]$$

$$\frac{d\Gamma_{\tau,2}}{dw} = \frac{\eta_{EW}^2|V_{cb}|^2 G_F^2 m_B^5}{32\pi^3}\frac{m_\tau^2(m_\tau^2-q(w)^2)^2 r^3(1+r)^2(w^2-1)^{3/2}P_1(w)^2}{q(w)^6}$$

Relation between the momentum transfer and the recoil:

$$q^2 = m_B^2 + m_P^2 - 2m_B m_P\, w$$

The Dispersive Matrix (DM) method

Our goal is to describe the FFs using a novel, non-perturbative and model independent approach: starting from the available LQCD computations of the FFs in the high-q^2 (or low-w) regime, we extract the FFs behaviour in the low-q^2 (or high-w) region!

Original proposal from L. Lellouch: NPB, 479 (1996)
New developments in PRD '21 (2105.02497)

The Dispersive Matrix (DM) method

Our goal is to describe the FFs using a novel, non-perturbative and model independent approach: starting from the available LQCD computations of the FFs in the high-q^2 (or low-w) regime, we extract the FFs behaviour in the low-q^2 (or high-w) region!

Original proposal from L. Lellouch: NPB, 479 (1996)
New developments in PRD '21 (2105.02497)

The resulting description of the FFs

- **is entirely based on first principles (LQCD evaluation of 2- and 3-point Euclidean correlators)**
- **is independent of any assumption on the functional dependence of the FFs on the momentum transfer**
- **can be applied to theoretical calculations of the FFs, but also to experimental data**
- **keep theoretical calculations and experimental data separated**
- **is universal: it can be applied to any exclusive semileptonic decays of mesons and baryons**

The Dispersive Matrix (DM) method

Our goal is to describe the FFs using a novel, non-perturbative and model independent approach: starting from the available LQCD computations of the FFs in the high-q^2 (or low-w) regime, we extract the FFs behaviour in the low-q^2 (or high-w) region!

Original proposal from L. Lellouch: NPB, 479 (1996)
New developments in PRD '21 (2105.02497)

The resulting description of the FFs

- **is entirely based on first principles (LQCD evaluation of 2- and 3-point Euclidean correlators)**
- **is independent of any assumption on the functional dependence of the FFs on the momentum transfer**
- **can be applied to theoretical calculations of the FFs, but also to experimental data**
- **keep theoretical calculations and experimental data separated**
- **is universal: it can be applied to any exclusive semileptonic decays of mesons and baryons**

→ No HQET, no series expansion, no perturbative bounds with respect to the well-known other parametrizations

The Dispersive Matrix (DM) method

Our goal is to describe the FFs using a novel, non-perturbative and model independent approach: starting from the available LQCD computations of the FFs in the high-q^2 (or low-w) regime, we extract the FFs behaviour in the low-q^2 (or high-w) region!

Original proposal from L. Lellouch: NPB, 479 (1996)
New developments in PRD '21 (2105.02497)

The resulting description of the FFs

- **is entirely based on first principles (LQCD evaluation of 2- and 3-point Euclidean correlators)**
- **is independent of any assumption on the functional dependence of the FFs on the momentum transfer**
- **can be applied to theoretical calculations of the FFs, but also to experimental data**
- **keep theoretical calculations and experimental data separated**
- **is universal: it can be applied to any exclusive semileptonic decays of mesons and baryons**

→ No HQET, no series expansion, no perturbative bounds with respect to the well-known other parametrizations

How does it work?

The DM method

Let us focus on a generic FF *f*: we can define

$$\mathbf{M} = \begin{pmatrix} \chi & \phi f & \phi_1 f_1 & \phi_2 f_2 & \dots & \phi_N f_N \\ \phi f & \frac{1}{1-z^2} & \frac{1}{1-z z_1} & \frac{1}{1-z z_2} & \dots & \frac{1}{1-z z_N} \\ \phi_1 f_1 & \frac{1}{1-z_1 z} & \frac{1}{1-z_1^2} & \frac{1}{1-z_1 z_2} & \dots & \frac{1}{1-z_1 z_N} \\ \phi_2 f_2 & \frac{1}{1-z_2 z} & \frac{1}{1-z_2 z_1} & \frac{1}{1-z_2^2} & \dots & \frac{1}{1-z_2 z_N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \phi_N f_N & \frac{1}{1-z_N z} & \frac{1}{1-z_N z_1} & \frac{1}{1-z_N z_2} & \dots & \frac{1}{1-z_N^2} \end{pmatrix}$$

$$\phi_i f_i \equiv \phi(z_i) f(z_i) \;\; (\text{with } i = 1, 2, \dots N)$$

$$z(t) = \frac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1}$$

$$t_\pm \equiv (m_B \pm m_D)^2$$

t: momentum transfer

The DM method

Non-perturbative values of the susceptibilities from the dispersion relations (see PRD '21 (2105.07851) and 2202.10285)

Estimates of the FFs, computed on the lattice

$$\mathbf{M} = \begin{pmatrix} \chi & \phi f & \phi_1 f_1 & \phi_2 f_2 & \dots & \phi_N f_N \\ \phi f & \frac{1}{1-z^2} & \frac{1}{1-z z_1} & \frac{1}{1-z z_2} & \dots & \frac{1}{1-z z_N} \\ \phi_1 f_1 & \frac{1}{1-z_1 z} & \frac{1}{1-z_1^2} & \frac{1}{1-z_1 z_2} & \dots & \frac{1}{1-z_1 z_N} \\ \phi_2 f_2 & \frac{1}{1-z_2 z} & \frac{1}{1-z_2 z_1} & \frac{1}{1-z_2^2} & \dots & \frac{1}{1-z_2 z_N} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \phi_N f_N & \frac{1}{1-z_N z} & \frac{1}{1-z_N z_1} & \frac{1}{1-z_N z_2} & \dots & \frac{1}{1-z_N^2} \end{pmatrix}$$

$$\phi_i f_i \equiv \phi(z_i) f(z_i) \ \ (\text{with } i = 1, 2, \dots N)$$

$$z(t) = \frac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1}$$

$$t_\pm \equiv (m_B \pm m_D)^2$$

t: momentum transfer

One can show that

$$\det \mathbf{M} \geq 0$$

⬇

$$f_{\rm lo}(z) \leq f(z) \leq f_{\rm up}(z)$$

Values of the momentum transfer @ which FFs are computed on the lattice

The "problematic" semileptonic B → D* channel

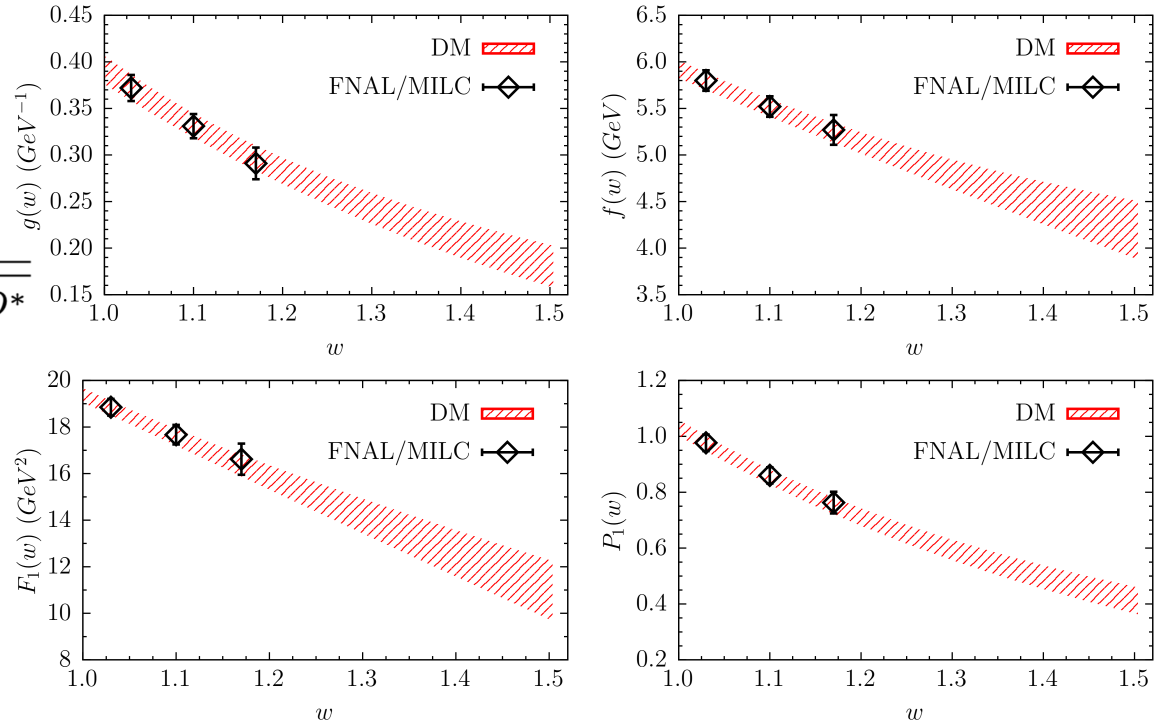
In **arXiv:2109.15248**, we have studied the final results of the FNAL/MILC computations of the FFs

- **3 FNAL/MILC data (diamonds) for each FF**: final results contained in **arXiv:2105.14019 [hep-lat]**

Two kinematical constraints (KCs):

$$\mathcal{F}_1(1) = (m_B - m_{D^*}) f(1)$$

$$P_1(w_{max}) = \frac{\mathcal{F}_1(w_{max})}{(1 + w_{max})(m_B - m_{D^*})\sqrt{m_B m_{D^*}}}$$



The “problematic” semileptonic B → D* channel

In **arXiv:2109.15248**, we have studied the final results of the FNAL/MILC computations of the FFs

- **3 FNAL/MILC data (diamonds) for each FF**: final results contained in **arXiv:2105.14019 [hep-lat]**

Two kinematical constraints (KCs):

$$\mathcal{F}_1(1) = (m_B - m_{D^*}) f(1)$$

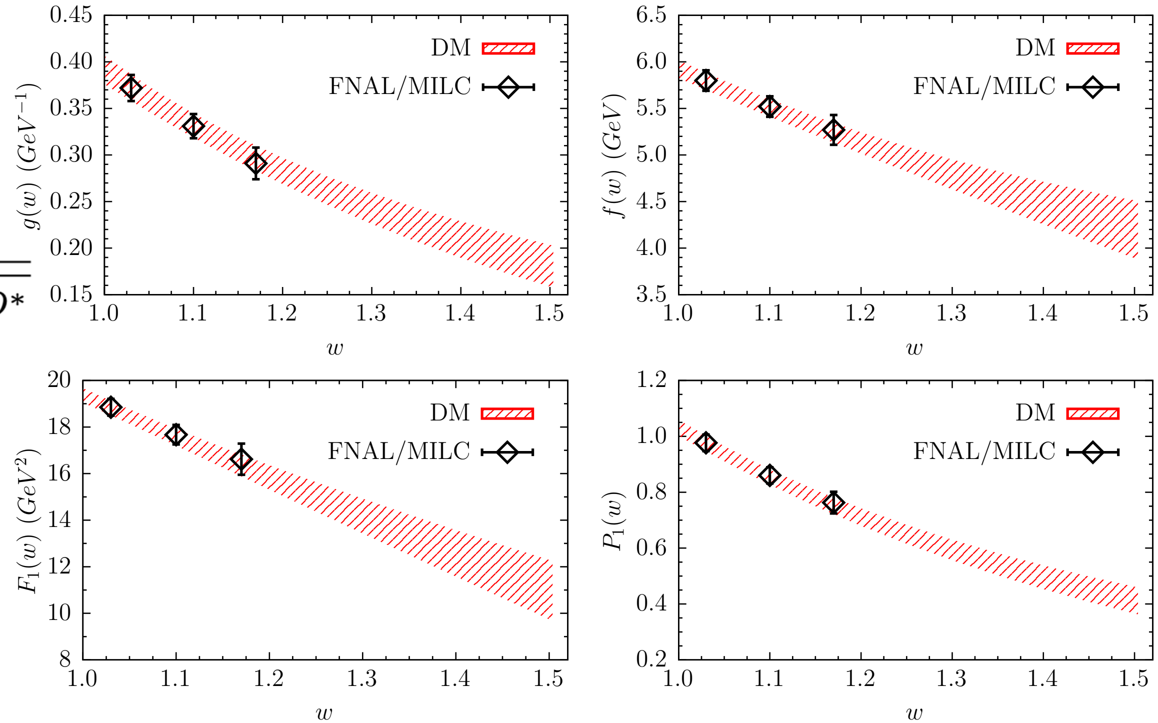
$$P_1(w_{max}) = \frac{\mathcal{F}_1(w_{max})}{(1+w_{max})(m_B - m_{D^*})\sqrt{m_B m_{D^*}}}$$

$$f(w_{max}) = 4.19 \pm 0.31 \text{ GeV} ,$$

$$g(w_{max}) = 0.180 \pm 0.023 \text{ GeV}^{-1} ,$$

$$\mathcal{F}_1(w_{max}) = 11.0 \pm 1.3 \text{ GeV}^2 ,$$

$$P_1(w_{max}) = 0.411 \pm 0.048 .$$



The “problematic” semileptonic **B → D*** channel

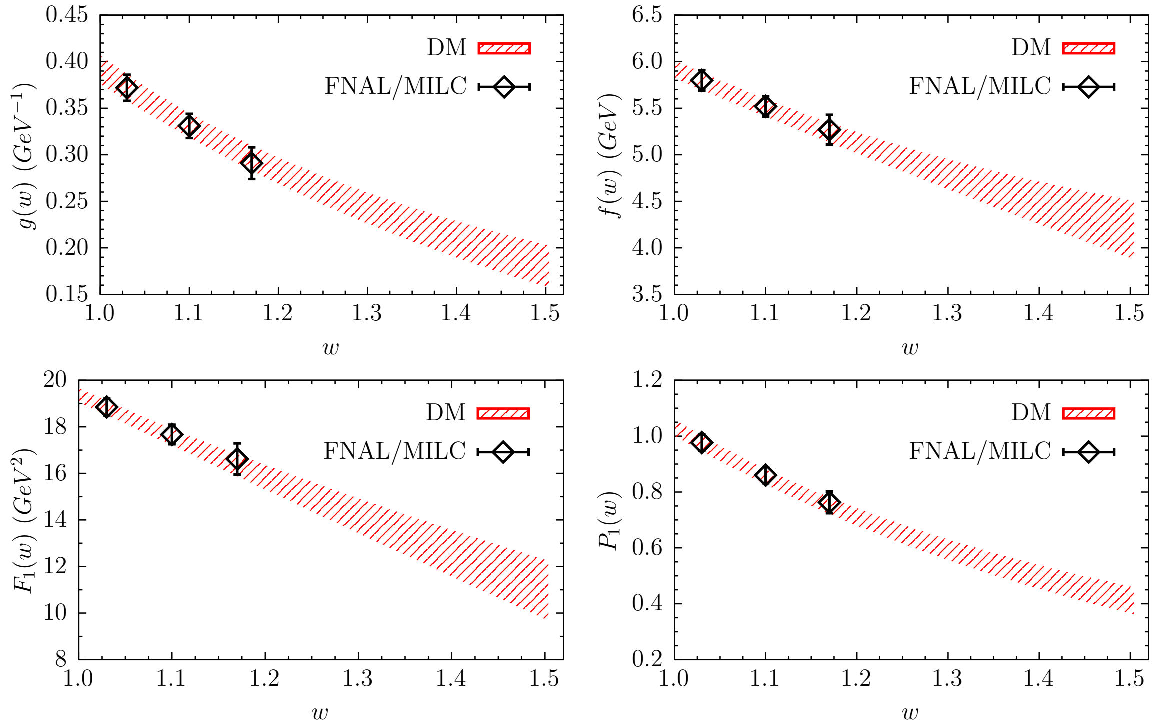
In **arXiv:2109.15248**, we have studied the final results of the FNAL/MILC computations of the FFs

- **3 FNAL/MILC data (diamonds) for each FF**: final results contained in **arXiv:2105.14019 [hep-lat]**

Using the unitarity bands of the FFs, we can compute new fully-theoretical expectation values of the anomaly ***R(D*):***

DM prediction

$$R(D^*) = 0.275 \pm 0.008$$



$$R(D^*)|_{\rm exp} = 0.295 \pm 0.010 \pm 0.010$$

HFLAV Coll. (https://hflav-eos.web.cern. ch/hflav-eos/semi/spring21/html/RDsDsstar/RDRDs.html)

The "problematic" semileptonic B → D* channel

In **arXiv:2109.15248**, we have studied the final results of the FNAL/MILC computations of the FFs

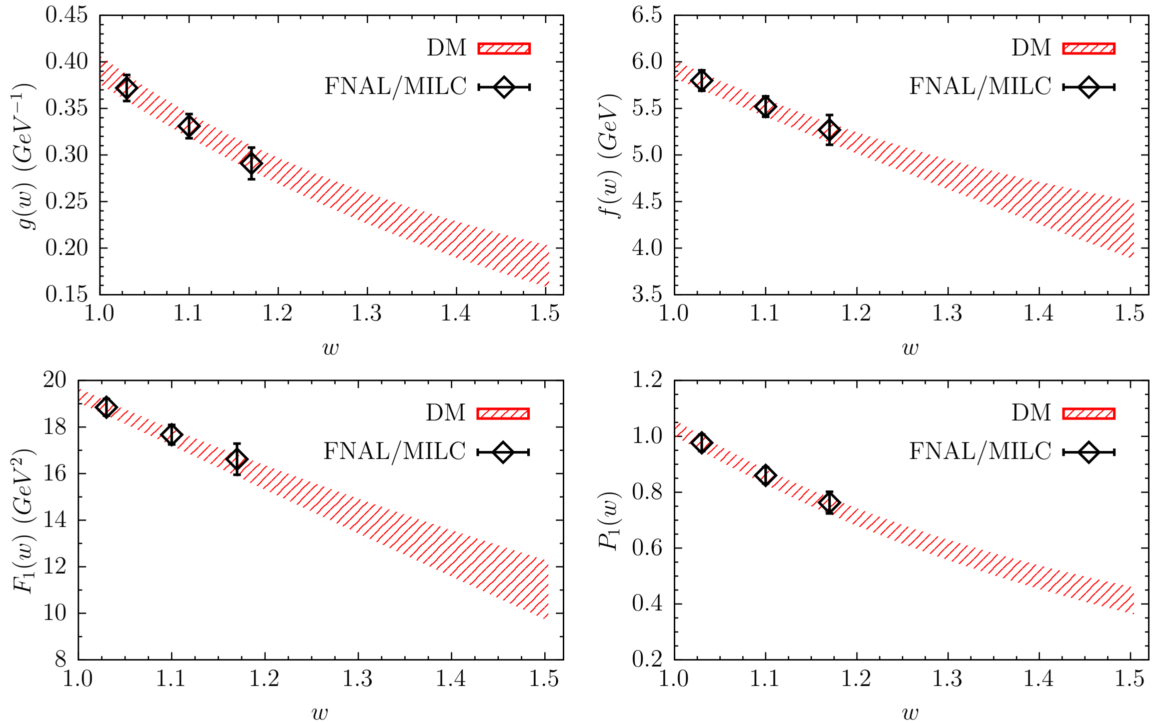
- **3 FNAL/MILC data (diamonds) for each FF**: final results contained in **arXiv:2105.14019 [hep-lat]**

*Using the unitarity bands of the FFs, we can compute new fully-theoretical expectation values of the anomaly **R(D*)**:*

DM prediction

$$R(D^*) = 0.275 \pm 0.008$$

1.3σ compatibility



$$R(D^*)|_{\rm exp} = 0.295 \pm 0.010 \pm 0.010$$

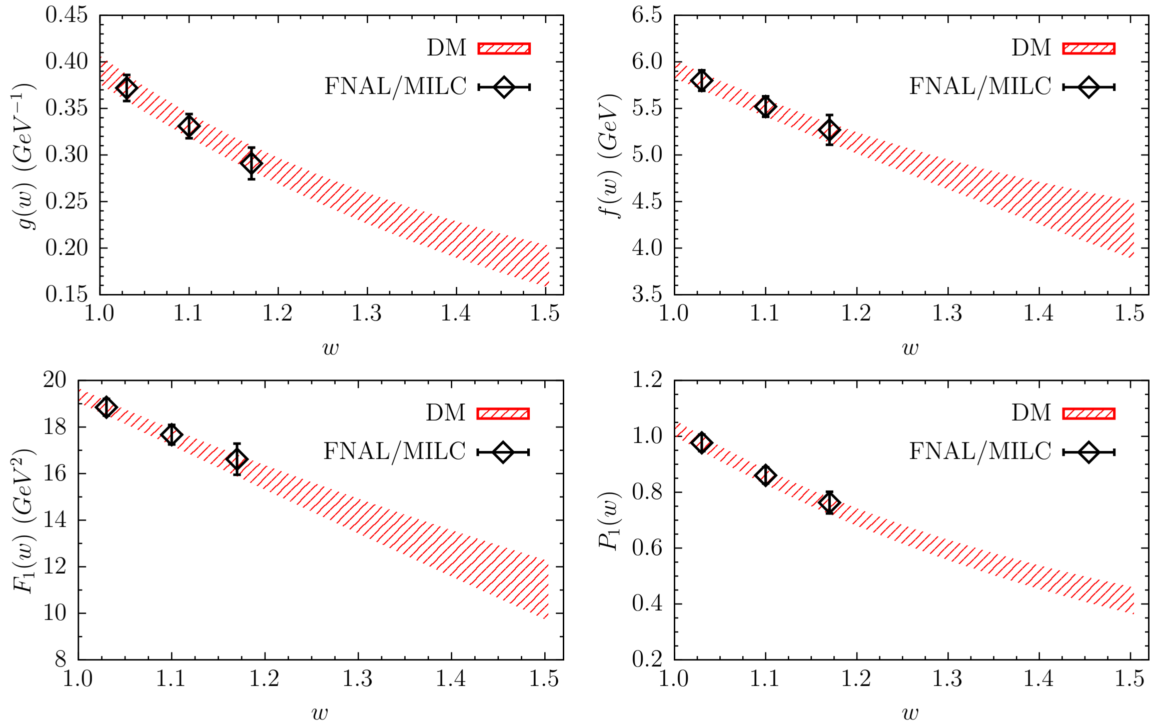
HFLAV Coll. (https://hflav-eos.web.cern.ch/hflav-eos/semi/spring21/html/RDsDsstar/RDRDs.html)

The “problematic” semileptonic **B → D*** channel

In **arXiv:2109.15248**, we have studied the final results of the FNAL/MILC computations of the FFs

- **3 FNAL/MILC data (diamonds) for each FF**: final results contained in **arXiv:2105.14019 [hep-lat]**

*Using the unitarity bands of the FFs, we can compute new fully-theoretical expectation values of the anomaly **R(D*)**:*



DM prediction

$$R(D^*) = 0.275 \pm 0.008$$

VS

HFLAV ‘21 SM prediction

$$R(D^*) \quad = \quad 0.252 \pm 0.005$$

$$R(D^*)|_{\rm exp} = 0.295 \pm 0.010 \pm 0.010$$

HFLAV Coll. (https://hflav-eos.web.cern. ch/hflav-eos/semi/spring21/html/RDsDsstar/RDRDs.html)

The "problematic" semileptonic B → D* channel

In **arXiv:2109.15248**, we have studied the final results of the FNAL/MILC computations of the FFs

- **3 FNAL/MILC data (diamonds) for each FF**: final results contained in **arXiv:2105.14019 [hep-lat]**

*Using the unitarity bands of the FFs, we can compute new fully-theoretical expectation values of the anomaly **R(D*)**:*

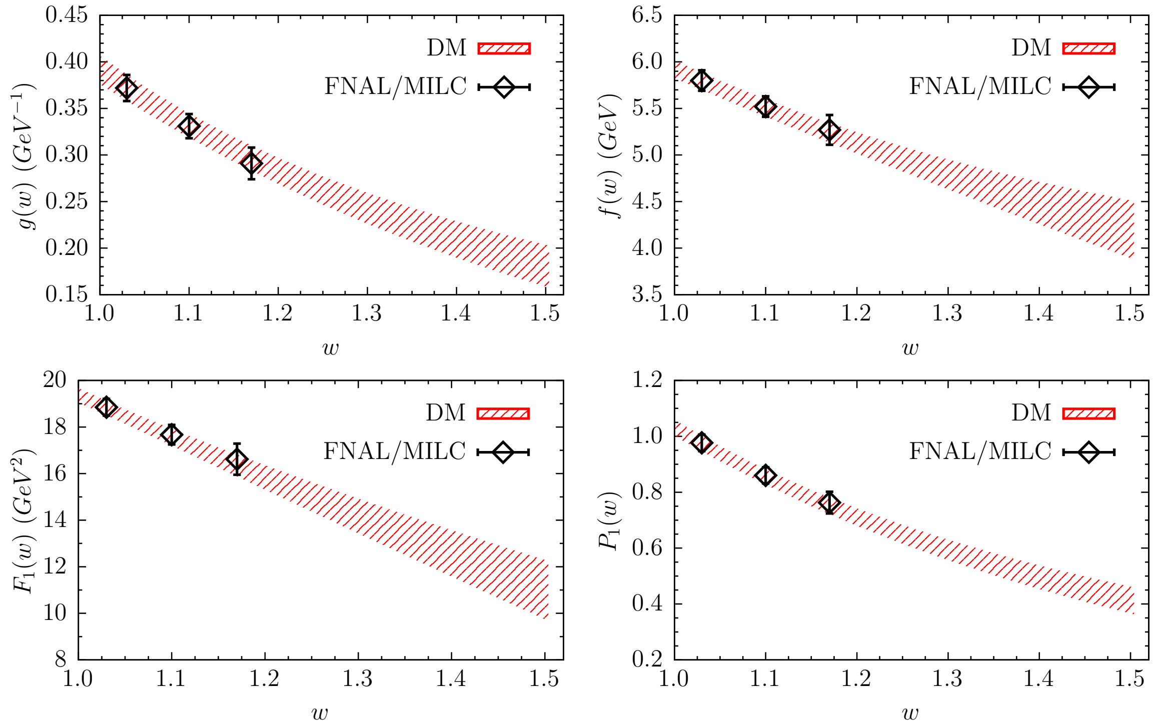
DM prediction

$$R(D^*) = 0.275 \pm 0.008$$

VS

HFLAV '21 SM prediction

$$R(D^*) = 0.252 \pm 0.005$$



WHY?

How to deal with experimental data?

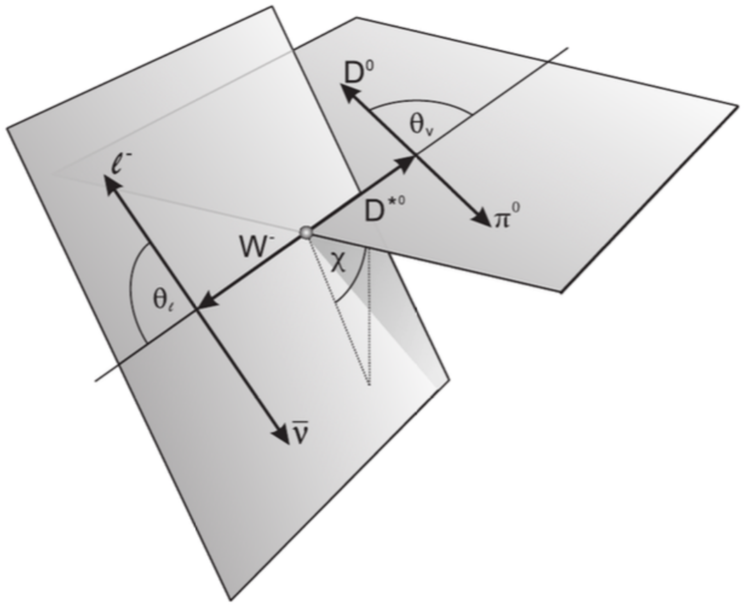
Starting from the FFs bands, we propose to use the experimental data **only** to compute bin-per-bin estimates of Vcb.

NB: the experimental data do NOT enter in the determination of the bands of the FFs

To do it, it is sufficient to compare the two sets of measurements of the differential decay widths

$$d\Gamma/dx, \quad x = w, \cos\theta_l, \cos\theta_v, \chi$$

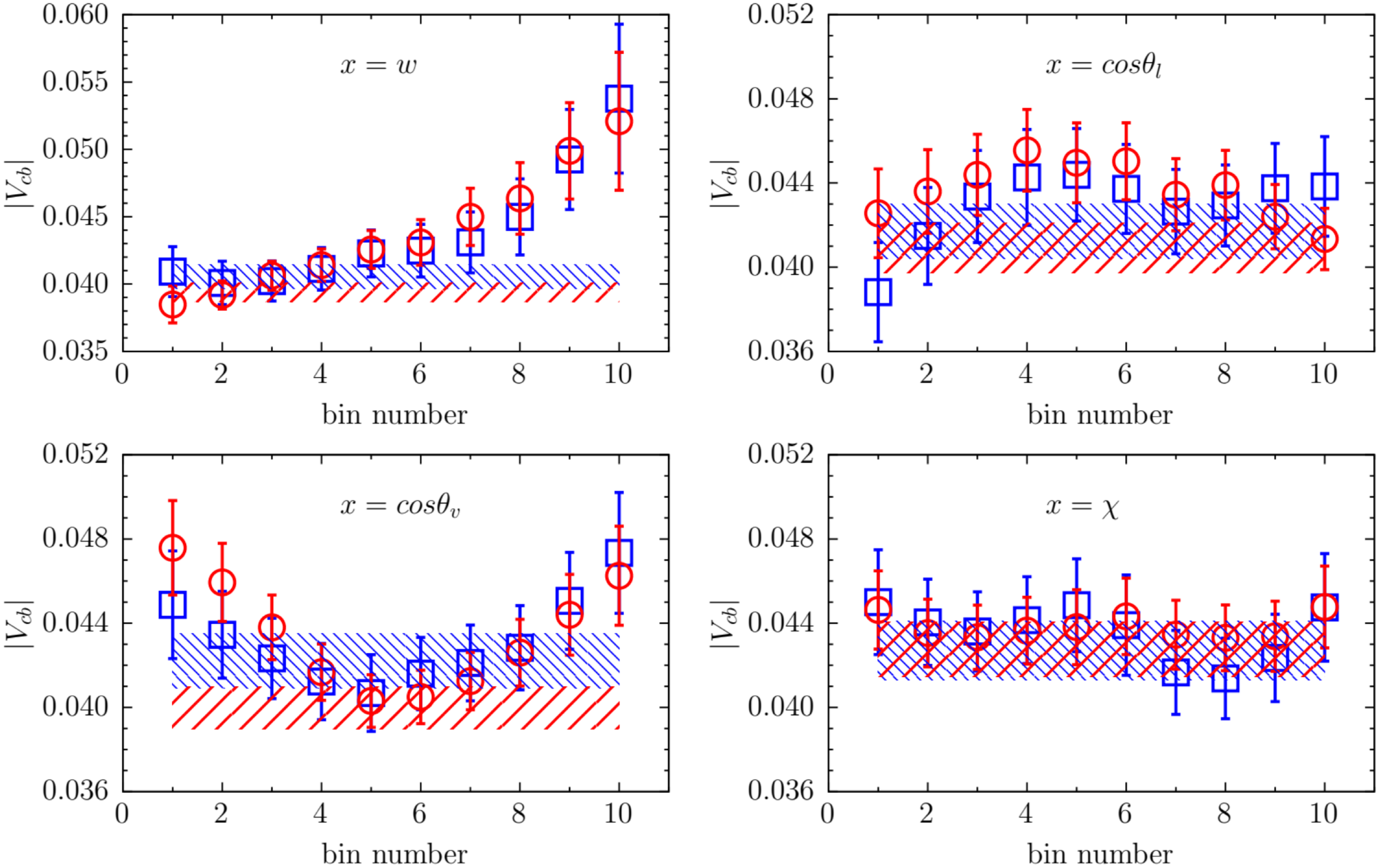
by the Belle Collaboration **(arXiv:1702.01521, arXiv:1809.03290)** with their *theoretical estimate*, computed through the unitarity bands shown before.



$$|V_{cb}|_i \equiv \sqrt{\frac{(d\Gamma/dx)_i^{exp.}}{(d\Gamma/dx)_i^{th.}}}$$

$$\frac{d\Gamma(B \to D^*(\to D\pi)\ell\nu)}{dw\, d\cos\theta_\ell\, d\cos\theta_v\, d\chi} = \frac{G_F^2 |V_{cb}|^2 \eta_{EW}^2}{4(4\pi)^4} 3 m_B m_{D^*}^2 \sqrt{w^2-1}$$
$$\times B(D^* \to D\pi)\{(1-\cos\theta_\ell)^2 \sin^2\theta_v |H_+|^2$$
$$+ (1+\cos\theta_\ell)^2 \sin^2\theta_v |H_-|^2 + 4\sin^2\theta_\ell \cos^2\theta_v |H_0|^2$$
$$- 2\sin^2\theta_\ell \sin^2\theta_v \cos 2\chi H_+ H_-$$
$$- 4\sin\theta_\ell (1-\cos\theta_\ell) \sin\theta_v \cos\theta_v \cos\chi H_+ H_0$$
$$+ 4\sin\theta_\ell (1+\cos\theta_\ell) \sin\theta_v \cos\theta_v \cos\chi H_- H_0\},$$

Exclusive Vcb determination through unitarity



Blue squares:
arXiv:1702.01521

Red points:
arXiv:1809.03290

Exclusive Vcb determination through unitarity

To compute the final average of these Vcb estimates:

$$\mu_x = \frac{1}{N}\sum_{k=1}^{N} x_k,$$

$$\sigma_x^2 = \frac{1}{N}\sum_{k=1}^{N}\sigma_k^2 + \frac{1}{N}\sum_{k=1}^{N}(x_k - \mu_x)^2$$

experiment	$\|V_{cb}\|(x = w)$	$\|V_{cb}\|(x = \cos\theta_l)$	$\|V_{cb}\|(x = \cos\theta_v)$	$\|V_{cb}\|(x = \chi)$
Ref. [14]	0.0405 (9)	0.0417 (13)	0.0422 (13)	0.0427 (14)
χ^2/(d.o.f.)	1.01	0.89	0.66	0.72
Ref. [15]	0.0394 (7)	0.0409 (12)	0.0400 (10)	0.0427 (13)
χ^2/(d.o.f.)	1.21	1.36	1.99	0.38

MOST IMPORTANT MESSAGE OF THE TALK: we want to avoid any bias by treating experimental and LQCD data differently in the DM approach to the hadronic FFs

Final DM estimate

$$|V_{cb}| \times 10^3 = 41.3 \pm 1.7$$

Compatible with the (most recent) **inclusive** values!

$$|V_{cb}|_{\rm incl} \times 10^3 = 42.16 \pm 0.50$$

Bordone et al., Phys.Lett.B [2107.00604]

$$|V_{cb}|_{\rm incl} \times 10^3 = 41.69 \pm 0.63$$

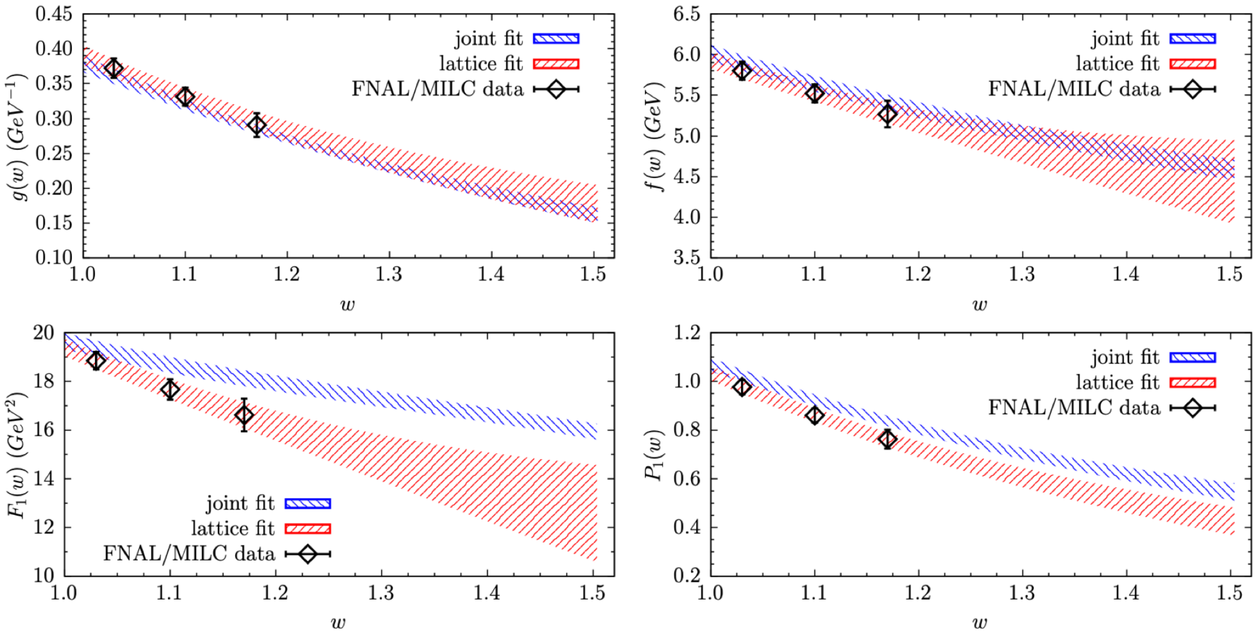
Bernlochner et al., arXiv:2205.10274

How to deal with experimental data? 2nd possibility

In principle, one can use **also** experimental data (in addition to the LQCD ones) to constrain the shape of the FFs…

VERY DANGEROUS: in this case, you are trying to fix both the normalization (*i.e.* |Vcb|) and the slopes of the FFs at the same time…

Let us see this in detail: *let us consider the BGL fits performed by FNAL/MILC Collaborations in arXiv:2105.14019 [hep-lat]*



joint fit:
BGL fit of LQCD points + Belle + BaBar exp. data

$|V_{cb}| \cdot 10^3 = 38.40 \pm 0.74$

$R(D^*) = 0.2483 \pm 0.0013$

lattice fit:
quadratic BGL fit of LQCD points only

$|V_{cb}| > |V_{cb}|^{\text{joint fit}}$?

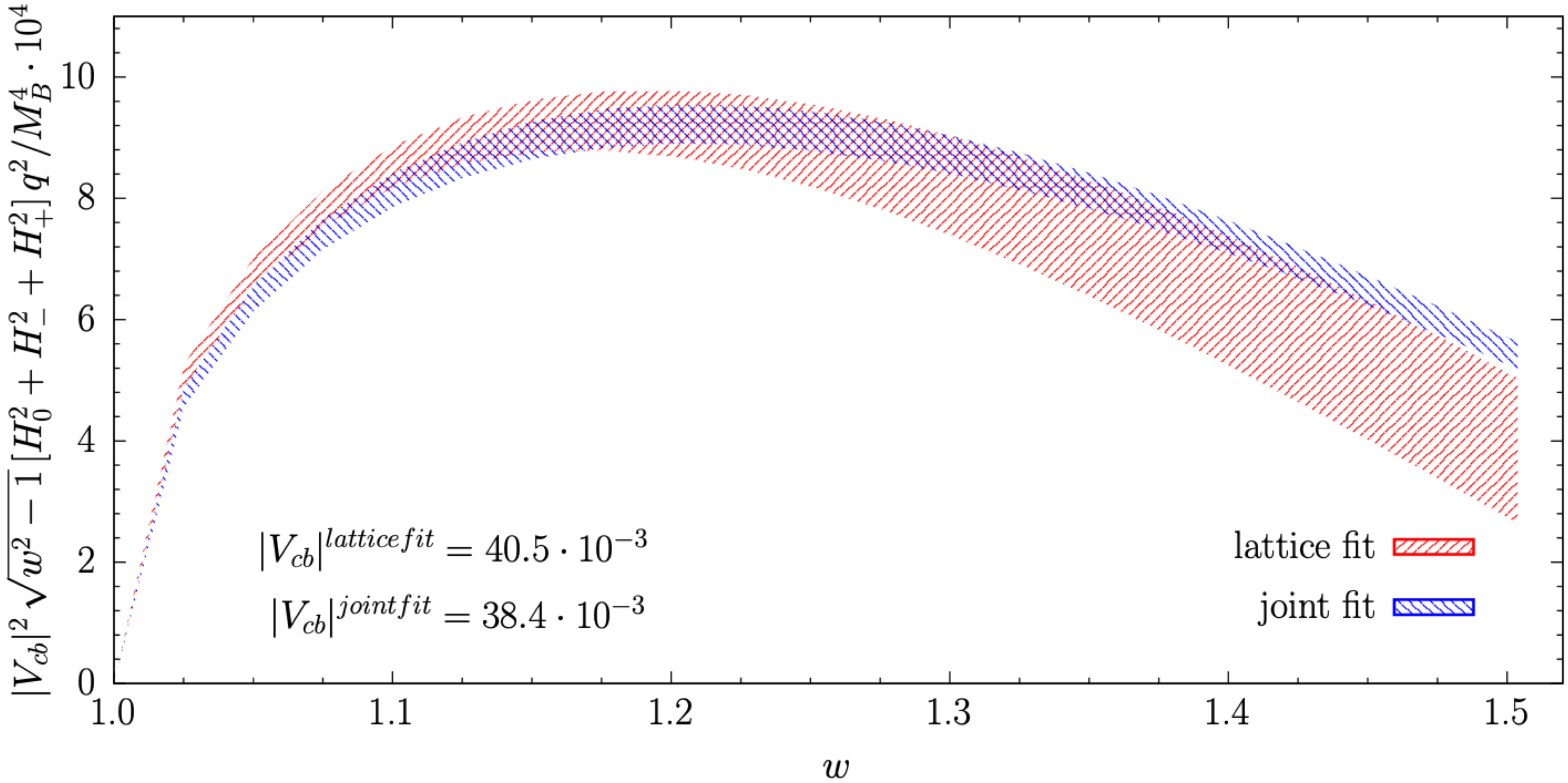
$R(D^*) = 0.265 \pm 0.013$

simultaneous fit of the lattice points and experimental data to determine the shape of the FFs and to extract $|V_{cb}|$

*** slope differences between exp's and theory $\rightarrow$ bias on $|V_{cb}|^{\text{joint fit}}$? ***

Comparison between the two different analysis strategies

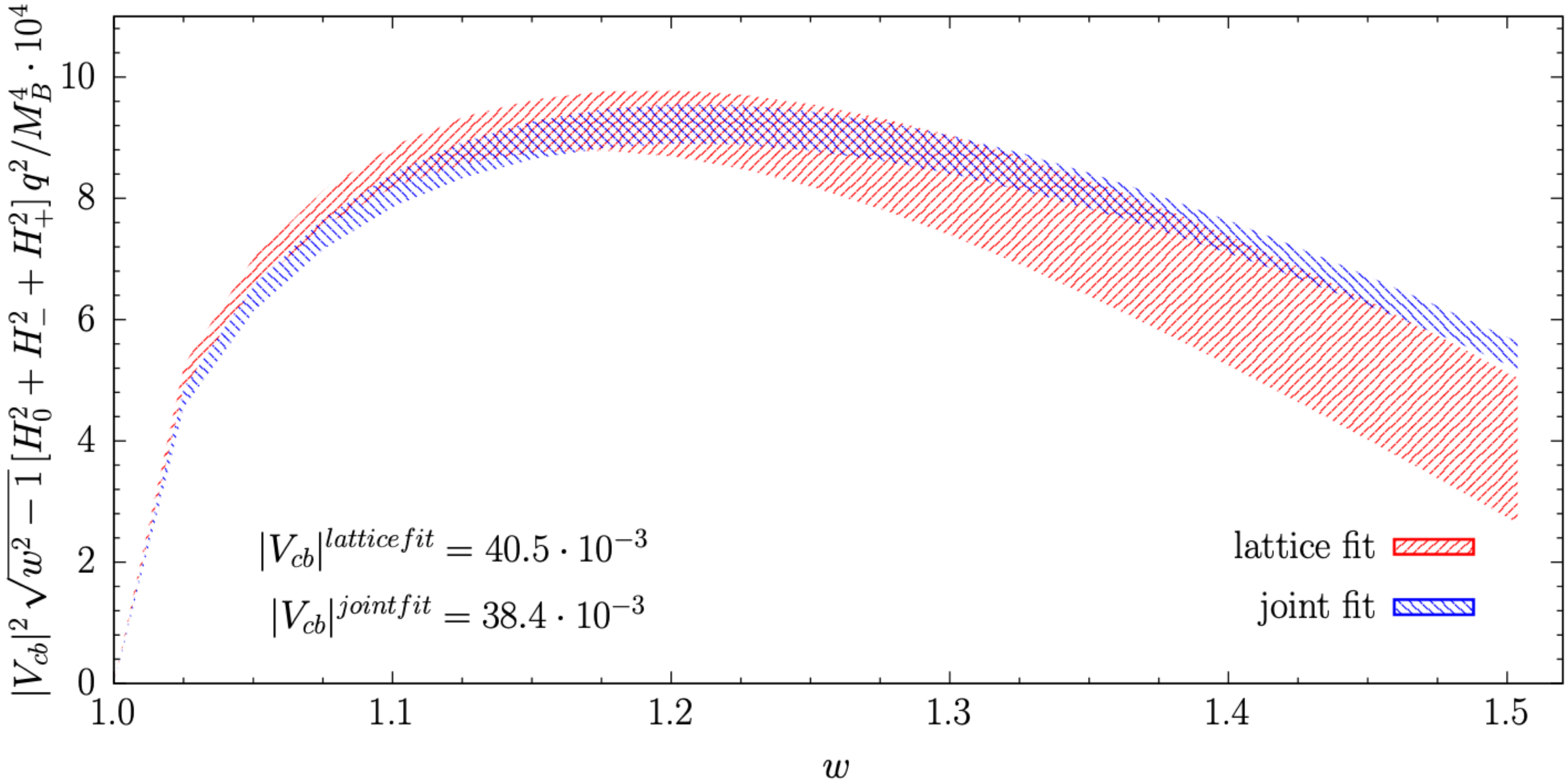
$$\frac{d\Gamma}{dw} \propto |V_{cb}|^2 \sqrt{w^2-1}\,\frac{q^2}{M_B^4}\left[H_0^2(w)+H_-^2(w)+H_+^2(w)\right] = |V_{cb}|^2\sqrt{w^2-1}\left\{\left(\frac{\mathscr{F}_1(w)}{M_B^2}\right)^2 + 2\frac{q^2}{M_B^2}\left[\left(\frac{f(w)}{M_B}\right)^2 + r^2(w^2-1)\,m_B^2\,g^2(w)\right]\right\} \qquad m_\ell = 0$$



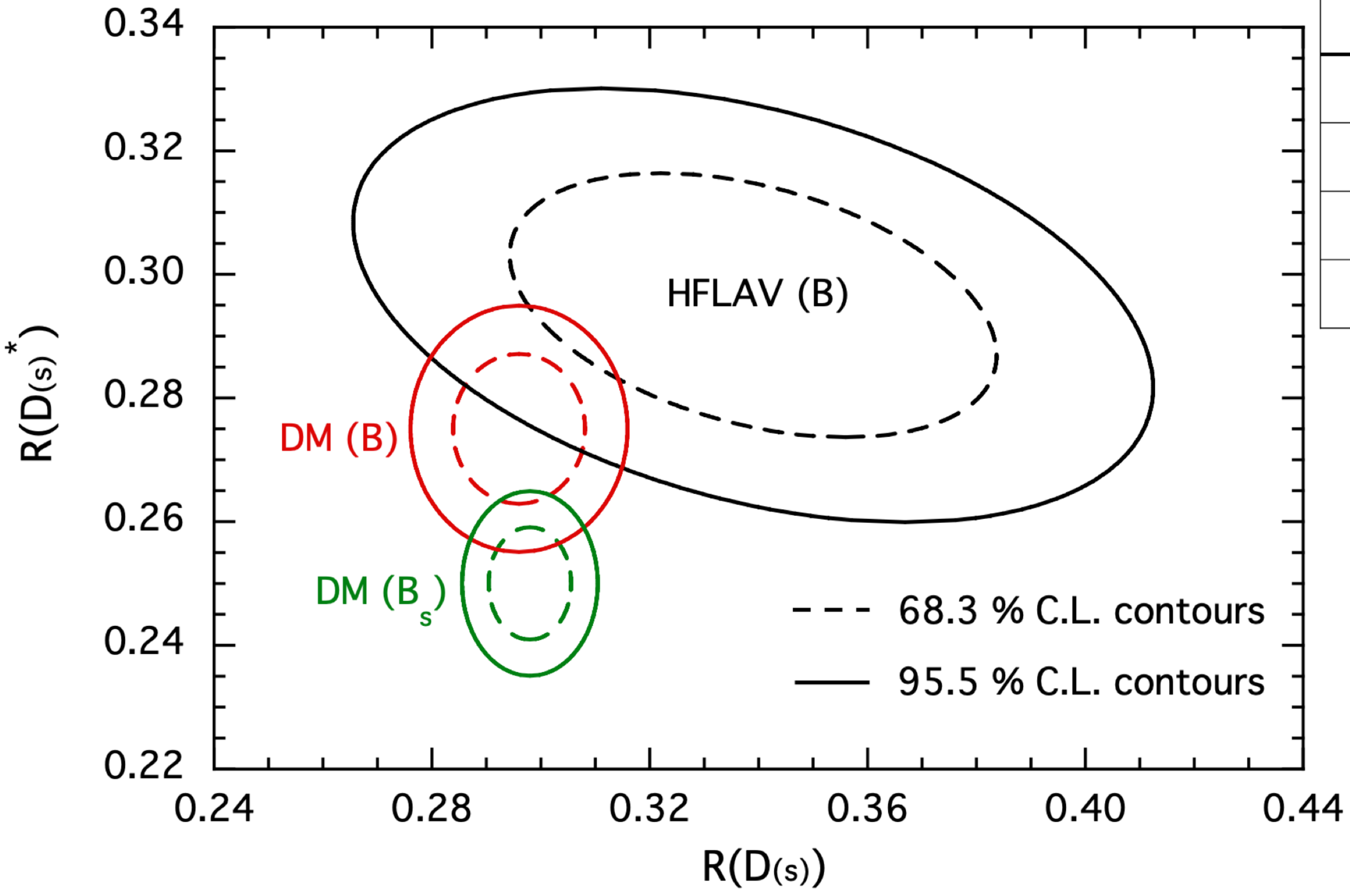
Comparison between the two different analysis strategies

Both the curves <u>reproduce the data</u>!

But they induce *different values of the (|Vcb|,R(D*)) couple*…



Summary of all the DM results



	DM	HFLAV '21
R(D)	0.296 (8)	0.339 (26) (14)
R(D*)	0.275 (8)	0.295 (10) (10)
$R(D_s)$	0.298 (5)	
$R(D_s^*)$	0.250 (6)	

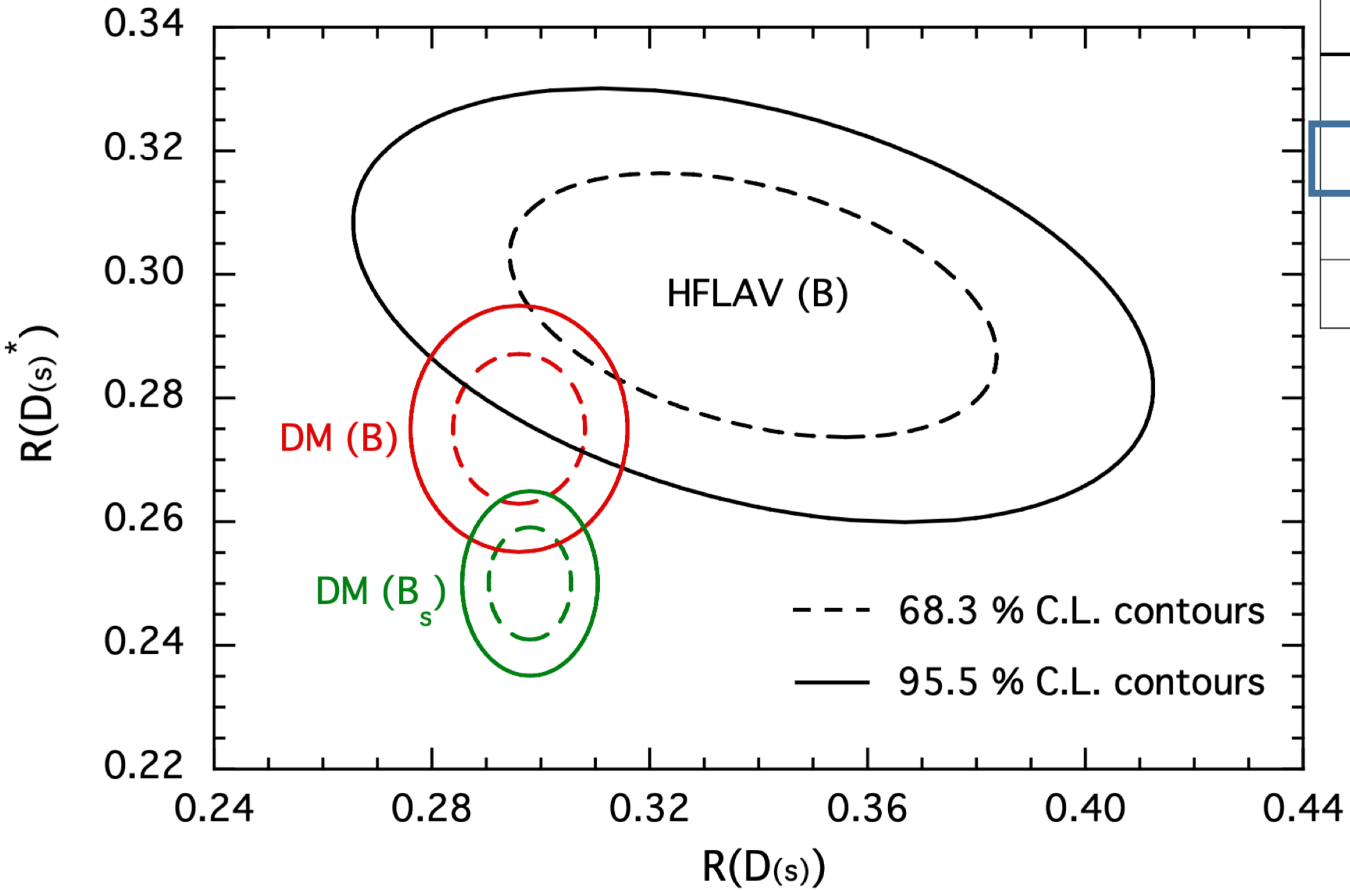
B → D case (**PRD '22 (2105.08674)**):

- **3 FNAL/MILC data for each FF (see PRD '15 (arXiv:1503.07237))**

$B_s \to D_s^*$ case (***arXiv:2204.05925)***

- **3 data points for each FF from HPQCD fits (see PRD '20 (arXiv:1906.00701) and PRD '22 (arXiv:2105.11433))**

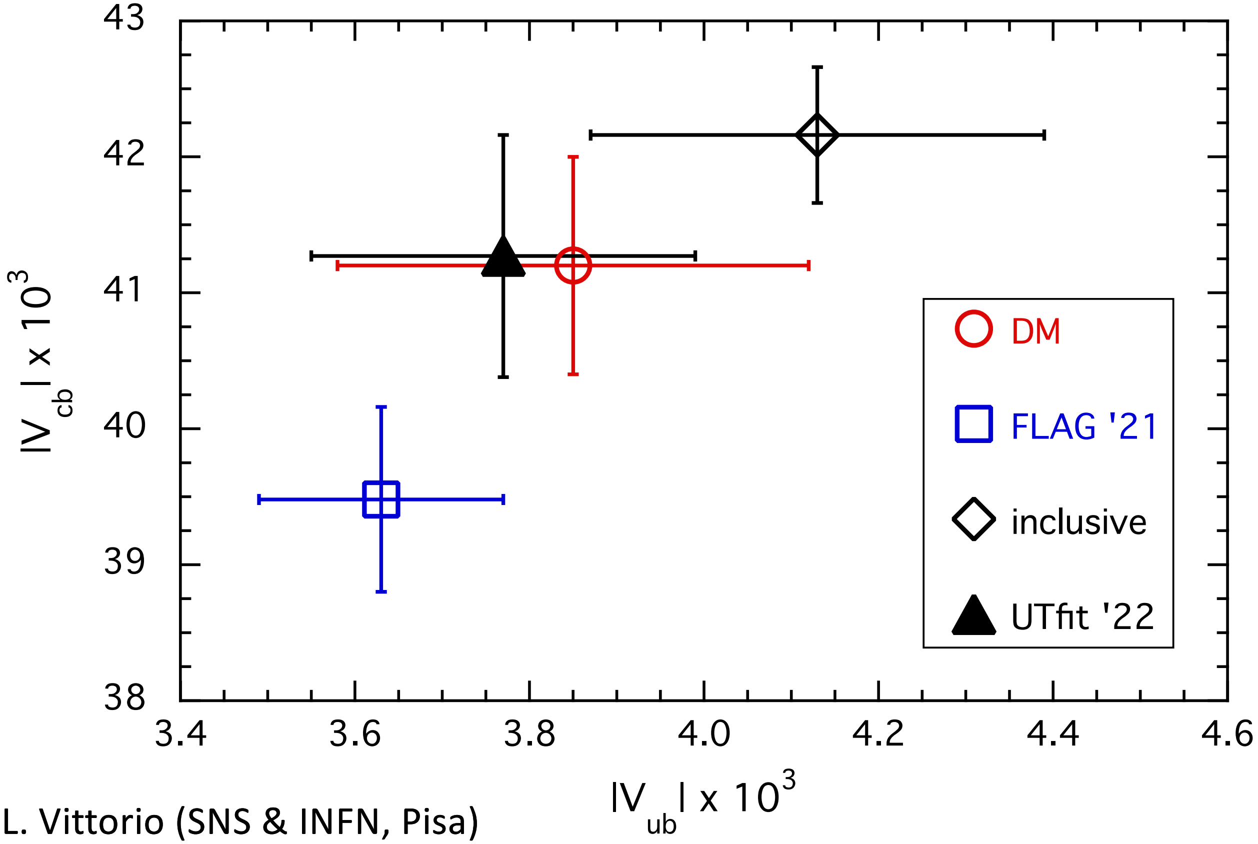
Summary of all the DM results



	DM	HFLAV '21
R(D)	0.296 (8)	0.339 (26) (14)
R(D*)	0.275 (8)	0.295 (10) (10)
$R(D_s)$	0.298 (5)	
$R(D_s^*)$	0.250 (6)	

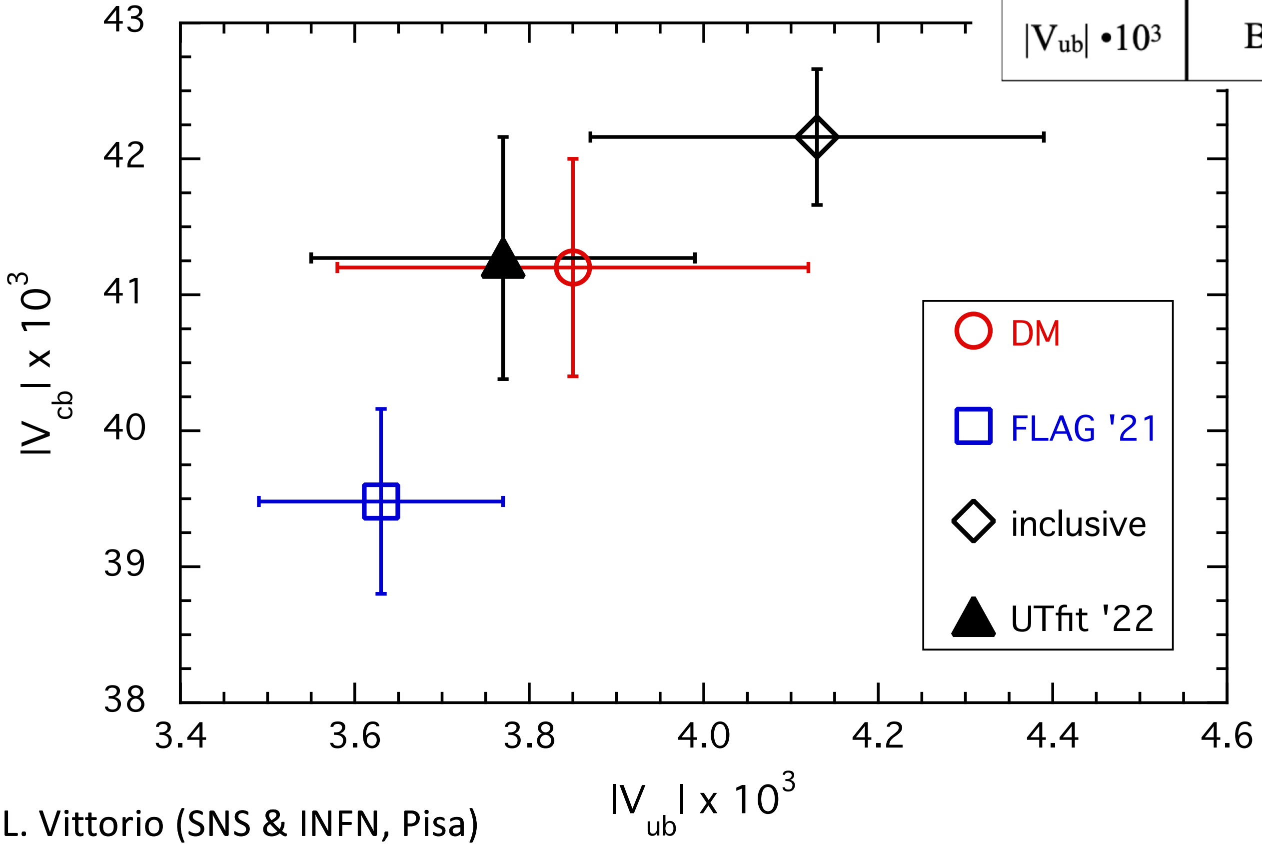
For the first time, this discrepancy is reduced at the 1.3σ level

Summary of all the DM results



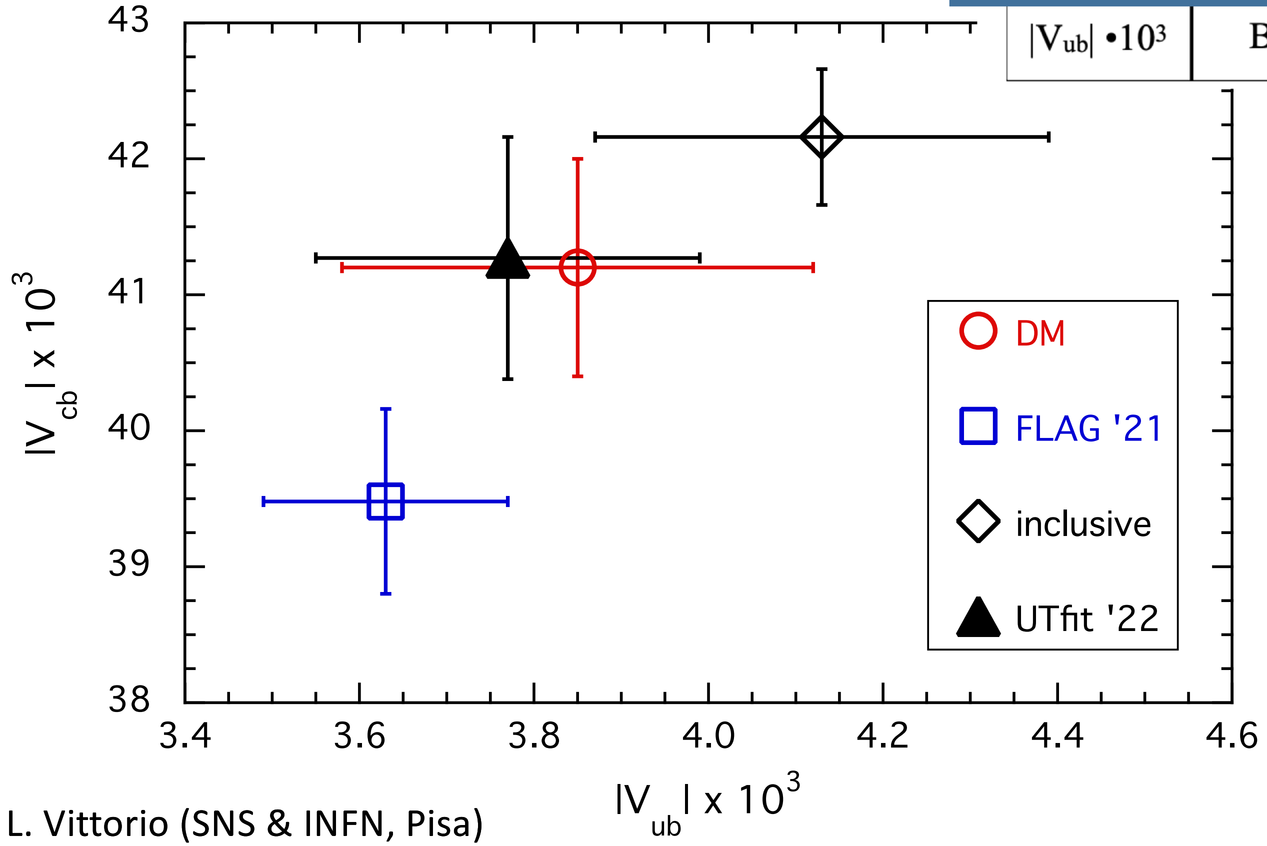
Summary of all the DM results

	decays	DM	FLAG '21	inclusive
$\|V_{cb}\| \cdot 10^3$	$B_{(s)} \to D_{(s)}^{(*)}$	41.2 (8)	39.48 (68)	42.16 (50)
$\|V_{ub}\| \cdot 10^3$	$B_{(s)} \to \pi, K$	3.85 (27)	3.63 (14)	4.13 (26)



Summary of all the DM results

	decays	DM	FLAG '21	inclusive
$\lvert V_{cb}\rvert \cdot 10^3$	$B_{(s)} \to D_{(s)}^{(*)}$	41.2 (8)	39.48 (68)	42.16 (50)
$\lvert V_{ub}\rvert \cdot 10^3$	$B_{(s)} \to \pi, K$	3.85 (27)	3.63 (14)	4.13 (26)



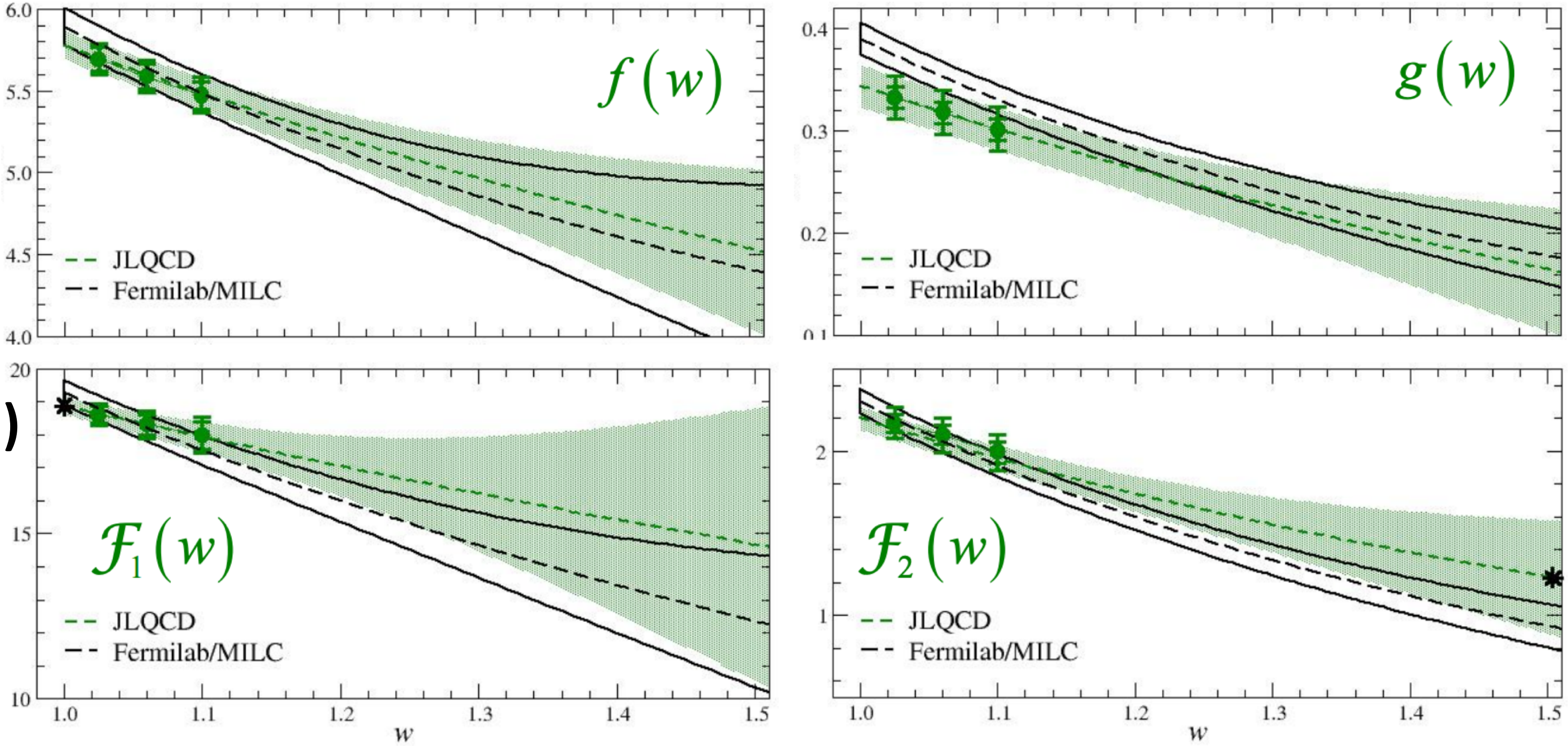
For the first time, there is a very precise exclusive determination of $\lvert V_{cb}\rvert$, which is compatible with the inclusive determination

THANKS FOR YOUR ATTENTION!

BACK-UP SLIDES

Future perspectives for LQCD data

JLQCD (prelim.)

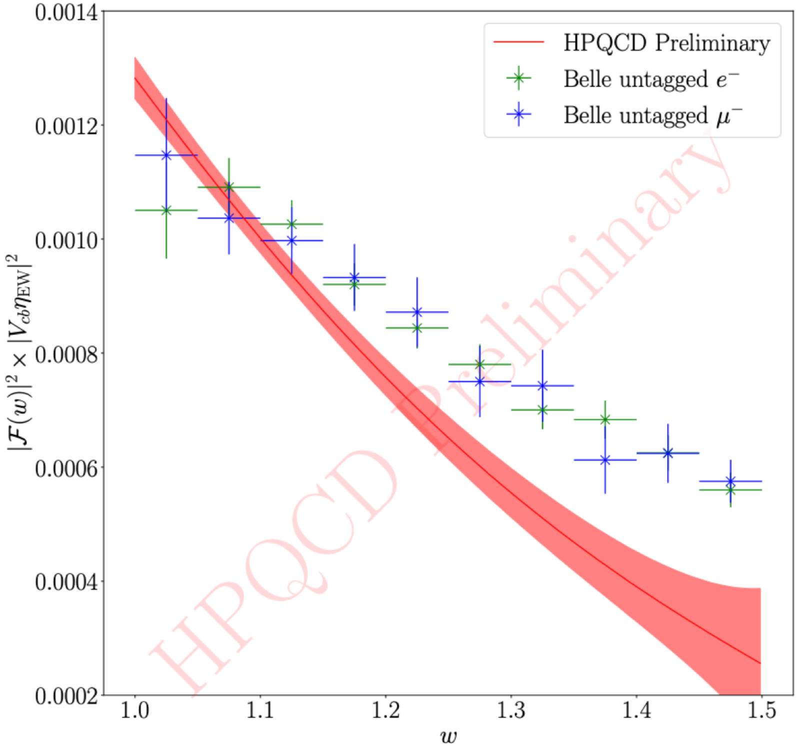


Kaneko's talk @ "Challenges in Semileptonic B decays 2022" Workshop

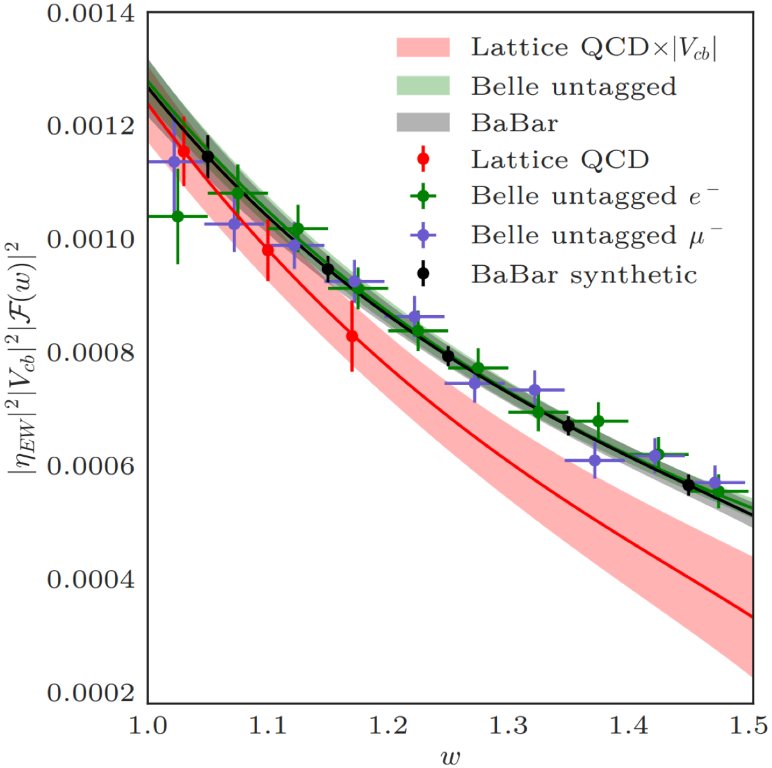
Future perspectives for LQCD data

Harrison's talk @ "Challenges in Semileptonic B decays 2022" Workshop

HPQCD (prelim.)



FNAL/MILC, arXiv:arXiv:2105.14019 [hep-lat]



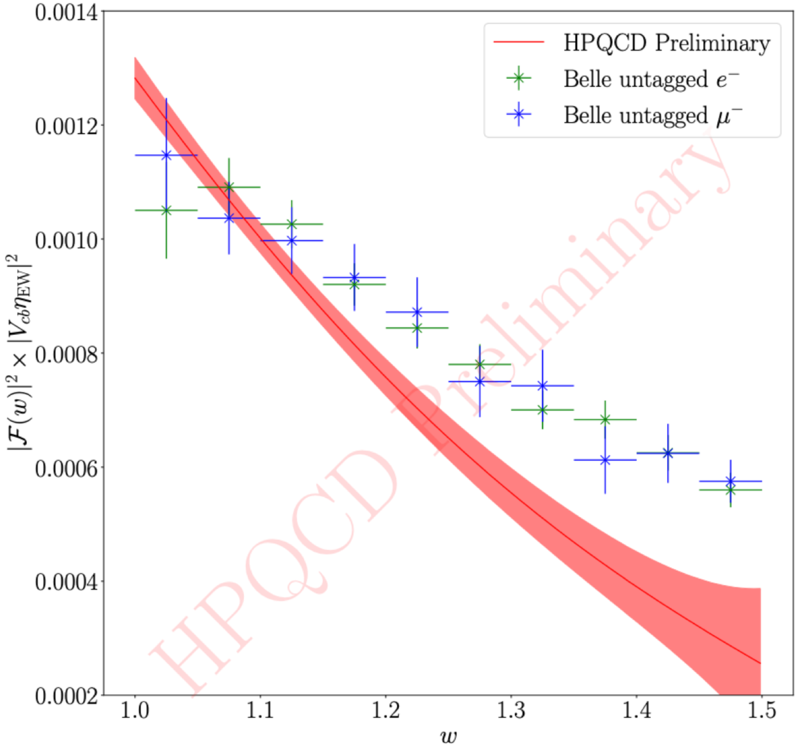
FNAL/MILC

$$\frac{d\Gamma(\bar{B} \to D^*\ell\,\bar{\nu})}{dw} = \frac{G_F^2|V_{cb}|^2}{48\pi^3}(m_B - m_{D^*})^2\, m_{D^*}^3 \sqrt{w^2-1}\,(w+1)^2 \times\left[1 + \frac{4w}{w+1}\frac{m_B^2 - 2w\, m_B m_{D^*} + m_{D^*}^2}{(m_B - m_{D^*})^2}\right]|\mathcal{F}(w)|^2$$

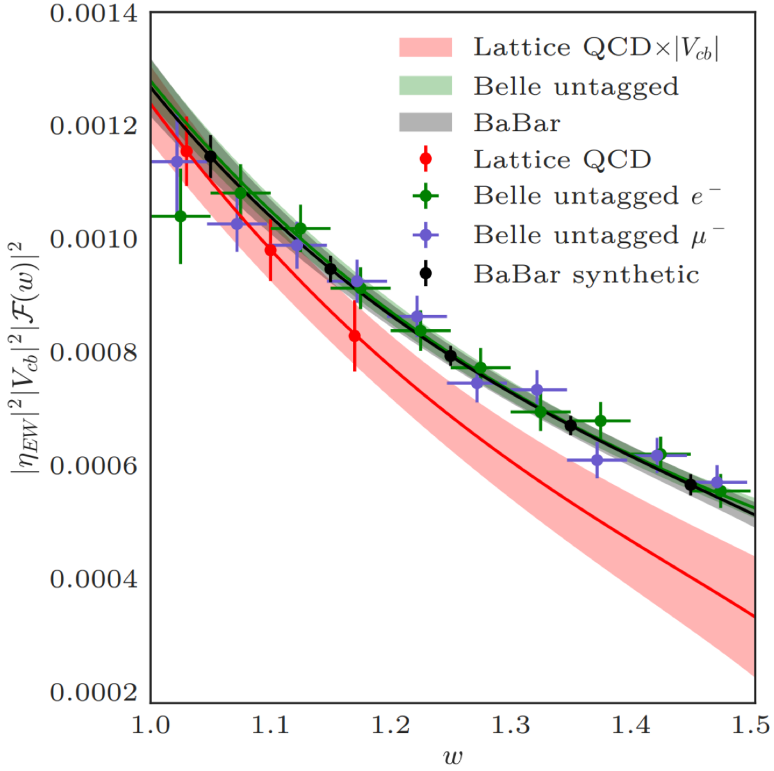
Future perspectives for LQCD data

Harrison's talk @ "Challenges in Semileptonic B decays 2022" Workshop

HPQCD (prelim.)



FNAL/MILC, arXiv:arXiv:2105.14019 [hep-lat]



FNAL/MILC

$$\frac{d\Gamma(\bar{B}\to D^*\ell\,\bar{\nu})}{dw} = \frac{G_F^2|V_{cb}|^2}{48\pi^3}(m_B - m_{D^*})^2\, m_{D^*}^3\sqrt{w^2-1}\,(w+1)^2 \times\left[1+\frac{4w}{w+1}\frac{m_B^2-2w\,m_B m_{D^*}+m_{D^*}^2}{(m_B-m_{D^*})^2}\right]|\mathcal{F}(w)|^2$$

<u>CONCLUSION</u>: FNAL/MILC and HPQCD have similar shape, which is different from Belle & different from JLQCD (which is affected by higher uncertainties…)

The DM method: explicit expressions

To be more specific, one can demonstrate that:

PRD '21 (2105.02497)

LOWER bound

$$\beta - \sqrt{\gamma} \le f(z) \le \beta + \sqrt{\gamma}$$

UPPER bound

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

The DM method: explicit expressions

To be more specific, one can demonstrate that:

PRD '21 (2105.02497)

LOWER bound $$\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$$ *UPPER bound*

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

The functions d_i are simply kinematical terms

The DM method: explicit expressions

To be more specific, one can demonstrate that:

PRD '21 (2105.02497)

LOWER bound

$$\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$$

UPPER bound

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

UNITARITY FILTER: **unitarity is satisfied if γ is semipositive definite, namely if**

$$\chi \geq \sum_{i,j=1} N f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}$$

The DM method: explicit expressions

To be more specific, one can demonstrate that:

PRD '21 (2105.02497)

LOWER bound $\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$ *UPPER bound*

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

UNITARITY FILTER: **unitarity is satisfied if γ is semipositive definite, namely if**

$$\chi \geq \sum_{i,j=1} N f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}$$

This is a **parametrization-independent unitarity test** of the LQCD input data

An instructive example: semileptonic **D → K** decays

The semileptonic D → K decays have a great advantage. A computation on the lattice of the FFs in the whole kinematical range has been performed by V. Lubicz et al in **PRD '17 [arXiv:1706.03017 [hep-lat]].**

An instructive example: semileptonic **D → K** decays

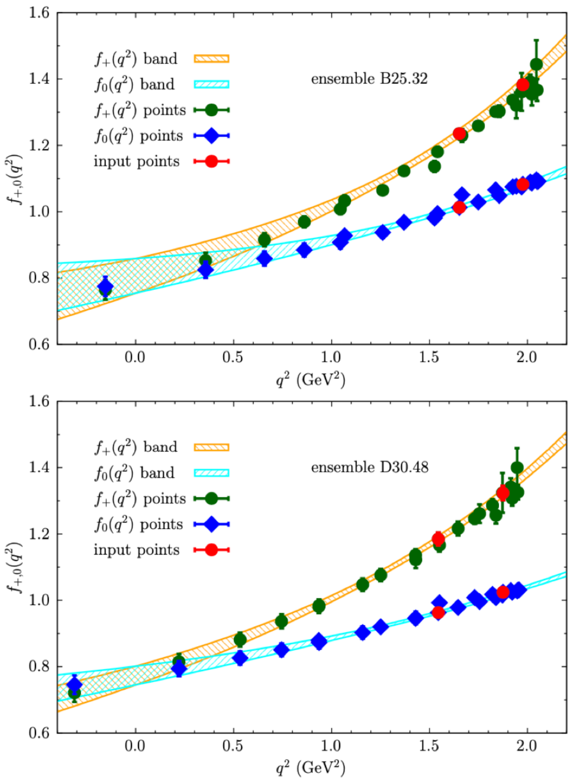
The semileptonic D → K decays have a great advantage. A computation on the lattice of the FFs in the whole kinematical range has been performed by V. Lubicz et al in **PRD ’17 [arXiv:1706.03017 [hep-lat]].**

Thus, we can compare the DM extrapolation of the FFs at zero momentum transfer (obtained by using as inputs only LQCD values at high momentum transfer) with the direct computation on the lattice!

An instructive example: semileptonic **D → K** decays

The semileptonic D → K decays have a great advantage. A computation on the lattice of the FFs in the whole kinematical range has been performed by V. Lubicz et al in **PRD '17 [arXiv:1706.03017 [hep-lat]].**

Thus, we can compare the DM extrapolation of the FFs at zero momentum transfer (obtained by using as inputs only LQCD values at high momentum transfer) with the direct computation on the lattice!



The red points are the only data used as input for the DM method!!

The figures show the bands obtained by using as inputs only the red points and the rest of the lattice points that are not used as input in our analysis in the case of the ETMC ensembles B25.32 and D30.48.

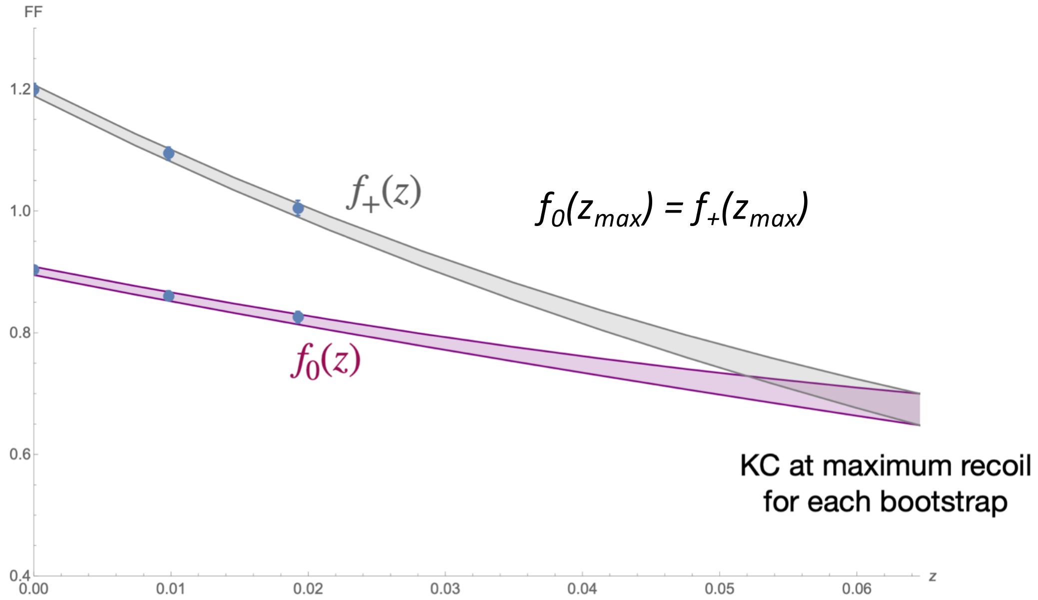
The agreement is excellent!

These results suggest that it will be possible to obtain quite precise determinations of the form factors for B decays by combining form factors at large q^2 with the non perturbative calculation of the susceptibilities.

The simplest example: semileptonic **B → D** decays

In **PRD '21 (arXiv:2105.08674)**, our DM method has been applied to B → D decays:

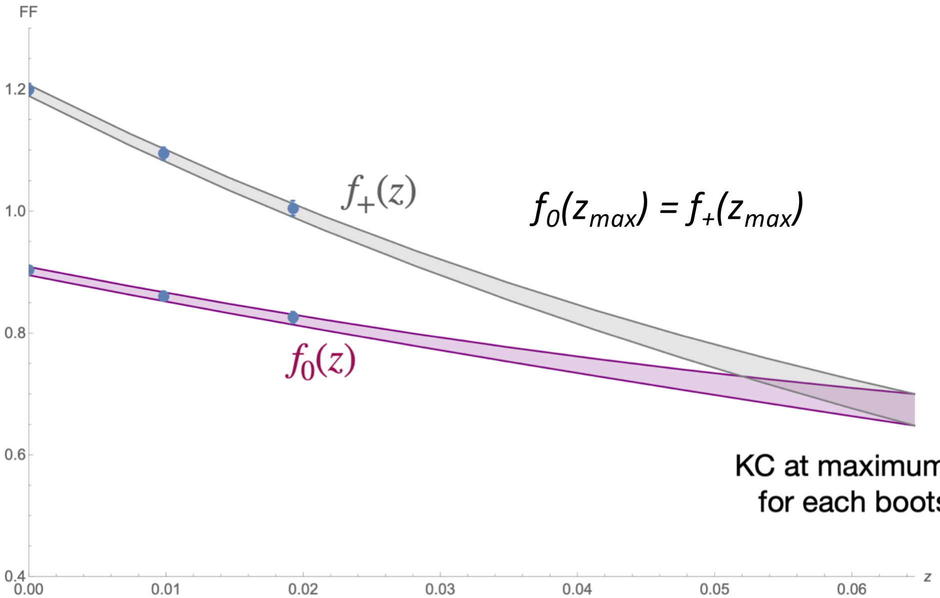
- **3 FNAL/MILC data for each FF**: final results contained in **PRD '15 (arXiv:1503.07237)**



The simplest example: semileptonic **B → D** decays

In **PRD '21 (arXiv:2105.08674)**, our DM method has been applied to B → D decays:

- **3 FNAL/MILC data for each FF**: final results contained in **PRD '15 (arXiv:1503.07237)**



*Recalling that for production of a pseudoscalar meson (i.e. D, π) in case of **massive** lepton:*

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2 |V_{cb}|^2 \eta_{EW}^2}{24\pi^3} \left(1 - \frac{m_\ell^2}{q^2}\right)^2 \times \left[|\vec{p}_D|^3 \left(1 + \frac{m_\ell^2}{2q^2}\right) |f^+(q^2)|^2 + m_B^2 |\vec{p}_D| \left(1 - \frac{m_D^2}{m_B^2}\right)^2 \frac{3m_\ell^2}{8q^2} |f^0(q^2)|^2 \right]$$

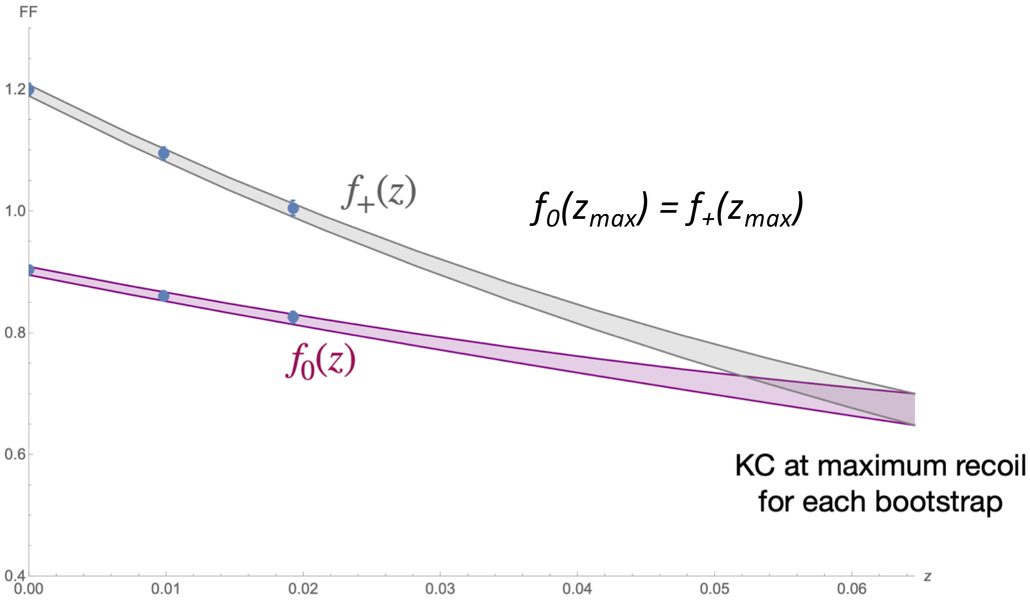
FULLY-THEORETICAL ESTIMATE!

$$R(D) = 0.296 \pm 0.008$$

The simplest example: semileptonic **B → D** decays

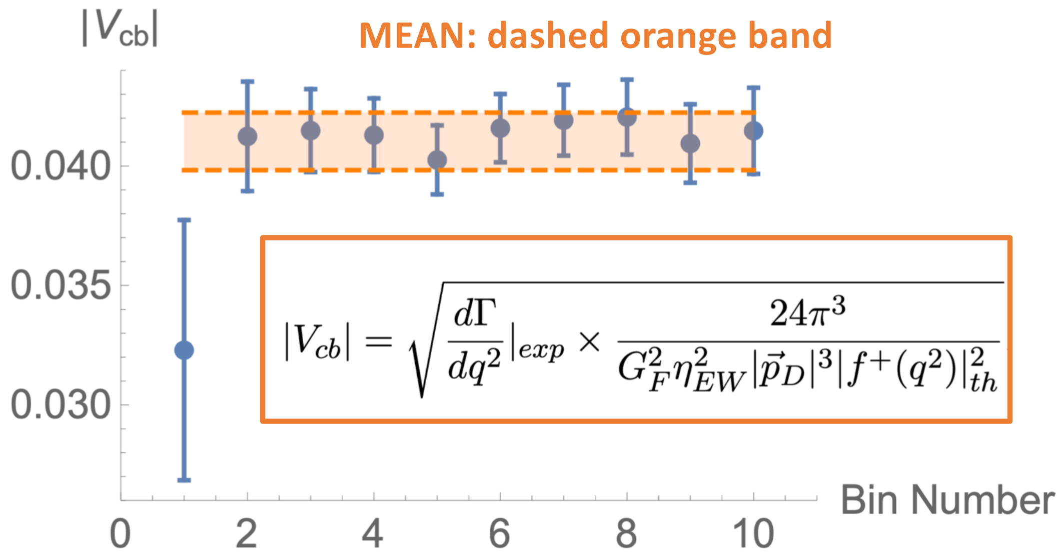
In **PRD ’21 (arXiv:2105.08674)**, our DM method has been applied to B → D decays:

- **3 FNAL/MILC data for each FF**: final results contained in **PRD ‘15 (arXiv:1503.07237)**



Blue points from Belle data: arXiv:1510.03657

MEAN: dashed orange band



$$|V_{cb}| = \sqrt{\frac{d\Gamma}{dq^2}|_{exp} \times \frac{24\pi^3}{G_F^2 \eta_{EW}^2 |\vec{p}_D|^3 |f^+(q^2)|^2_{th}}}$$

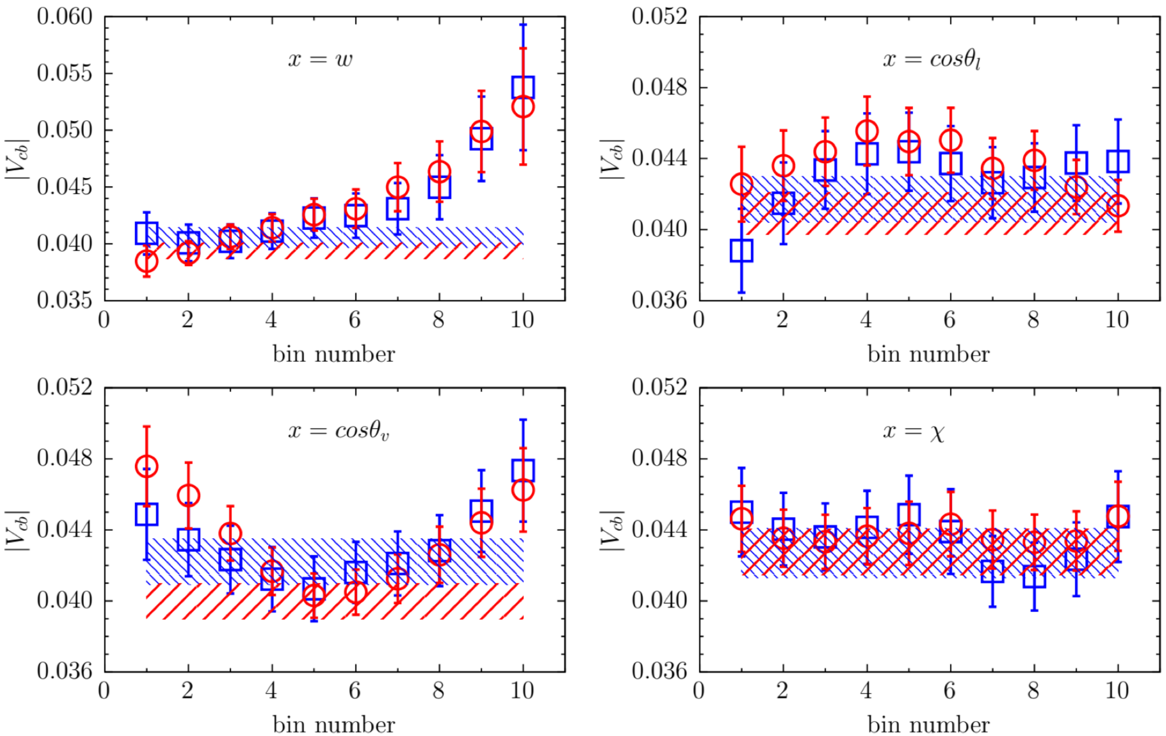
$$R(D) = 0.296 \pm 0.008$$

$$|V_{cb}| \times 10^3 = 41.0 \pm 1.2$$

Summary of the DM study of LQCD and exps. data

According to our prescription, **the shape of the FFs have to be constrained by using *only* the results of the LQCD computations** on the lattice. In this way:

- the estimate of R(D*) is fully-theoretical
- |Vcb| can be exctracted by a direct comparison with the experimental data, which do not introduce any bias



Remark 1

The value of $|V_{cb}|$ exhibits some dependence on the specific w-bin. The value obtained adopting a constant fit is dominated by the bins at small values of the recoil w, where direct lattice data are available and the lenght of the momentum extrapolation is limited.

Remark 2

The value of $|V_{cb}|$ deviates from a constant fit for $x = \cos(\theta_v)$. If we try a quadratic fit of the form

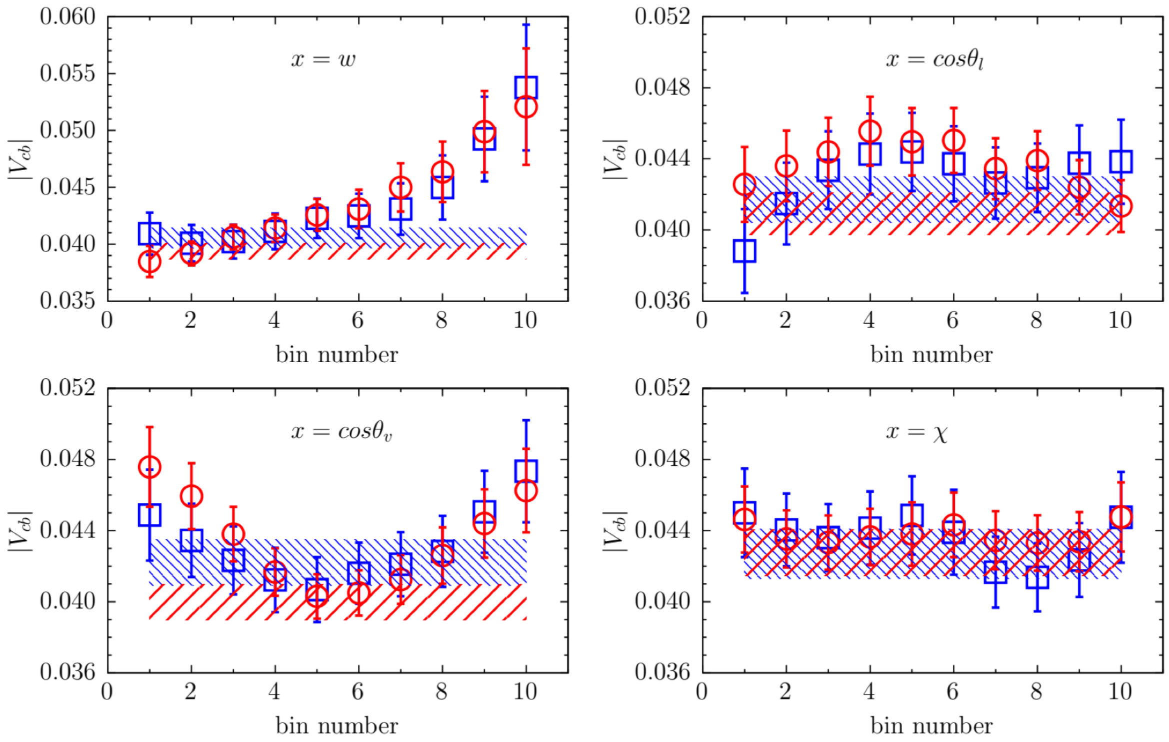
$$|V_{cb}|\left[1 + \delta B \cos^2(\theta_v)\right]$$

we get $\delta B \neq 0$ (2-3σ level) and $|V_{cb}|$ more consistent between the two sets of Belle data, but still in agreement with the value of $|V_{cb}|$ obtained with a constant fit

Summary of the DM study of LQCD and exps. data

According to our prescription, **the shape of the FFs have to be constrained by using *only* the results of the LQCD computations** on the lattice. In this way:

- the estimate of R(D*) is fully-theoretical
- |Vcb| can be exctracted by a direct comparison with the experimental data, which do not introduce any bias



Remark 1

The value of $|V_{cb}|$ exhibits some dependence on the specific w-bin. The value obtained adopting a constant fit is dominated by the bins at small values of the recoil w, where direct lattice data are available and the lenght of the momentum extrapolation is limited.

Remark 2

The value of $|V_{cb}|$ deviates from a constant fit for $x = \cos(\theta_v)$. If we try a quadratic fit of the form

$$|V_{cb}|\left[1 + \delta B\ \cos^2(\theta_v)\right]$$

we get $\delta B \neq 0$ (2-3σ level) and $|V_{cb}|$ more consistent between the two sets of Belle data, but still in agreement with the value of $|V_{cb}|$ obtained with a constant fit

OTHER DATA ARE FUNDAMENTAL!

Super-quick parenthesis: semileptonic $B_s \to D_s^{(*)}$ decays

In **arXiv:2204.05925**, our DM method has been applied to $B_s \to D_s^{(*)}$ decays.

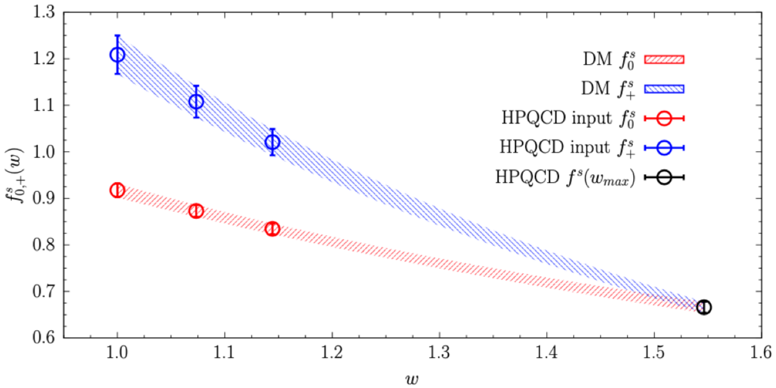
LQCD form factors taken from the results of the fits preformed by the HPQCD Collaboration in **arXiv:1906.00701** ($B_s \to D_s$) and **arXiv:2105.11433** ($B_s \to D_s^*$):we extract 3 data points for the FFs at small values of the recoil and apply the DM approach

Super-quick parenthesis: semileptonic $\mathbf{B_s} \rightarrow \mathbf{D_s^{(*)}}$ decays

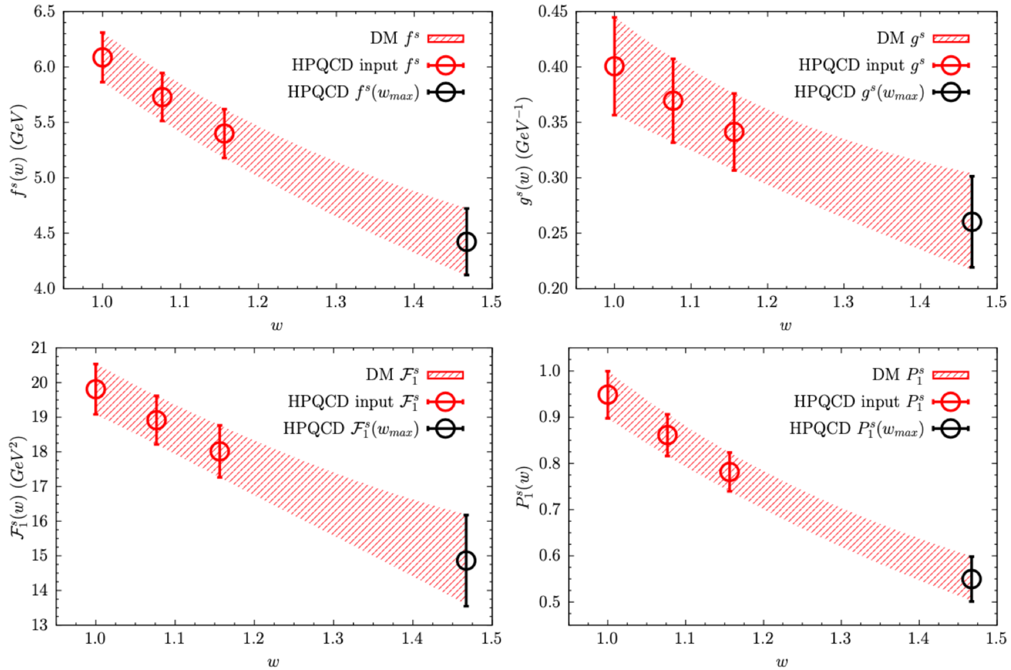
In **arXiv:2204.05925**, our DM method has been applied to $B_s \rightarrow D_s^{(*)}$ decays.

LQCD form factors taken from the results of the fits preformed by the HPQCD Collaboration in **arXiv:1906.00701** ($B_s \rightarrow D_s$) and **arXiv:2105.11433** ($B_s \rightarrow D_s$*):we extract 3 data points for the FFs at small values of the recoil and apply the DM approach

$B_s \rightarrow D_s \ell \nu_\ell$



$B_s \rightarrow D_s^* \ell \nu_\ell$



* nice agreement in the whole kinematical range

Super-quick parenthesis: semileptonic $B_s \to D_s^{(*)}$ decays

Without entering in the details of this analysis, phenomenological applications give the results

$$|V_{cb}|^{\rm DM} \cdot 10^3 = 41.7 \pm 1.9 \qquad \text{from } B_s \to D_s \ell \nu_\ell \text{ decays}$$

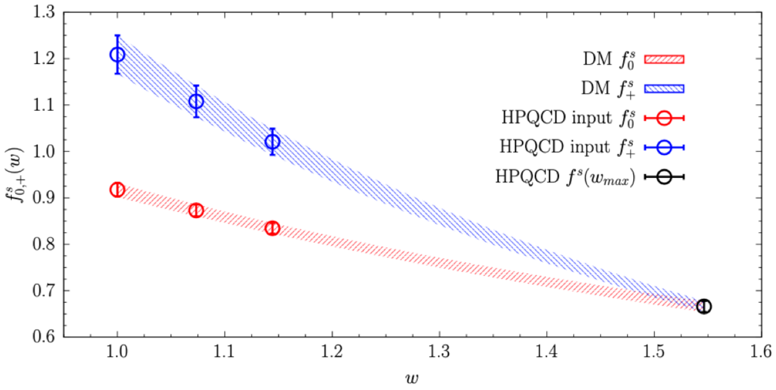
$$= 40.7 \pm 2.4 \qquad \text{from } B_s \to D_s^* \ell \nu_\ell \text{ decays}$$

through available exps. data

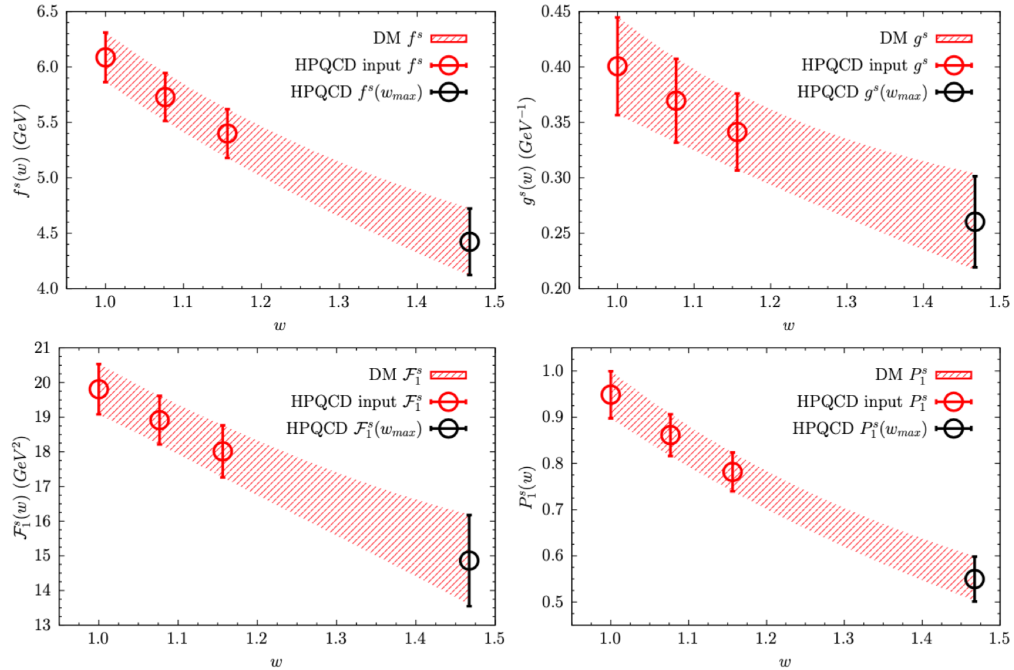
$$R(D_s) = 0.298\ (5)$$
$$R(D_s^*) = 0.250\ (6)$$

fully-theoretical

$B_s \to D_s \ell \nu_\ell$



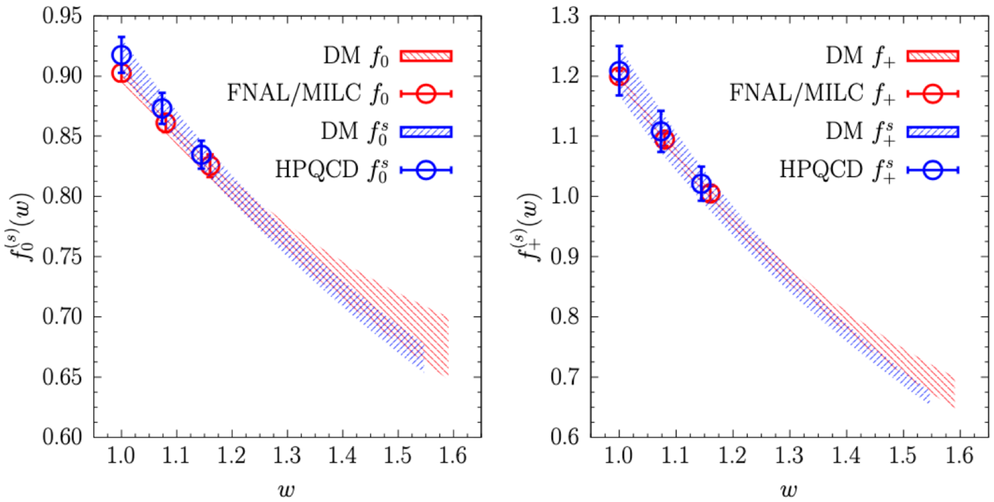
$B_s \to D_s^* \ell \nu_\ell$



* nice agreement in the whole kinematical range

Super-quick parenthesis: semileptonic $B_s \rightarrow D_s^{(*)}$ decays

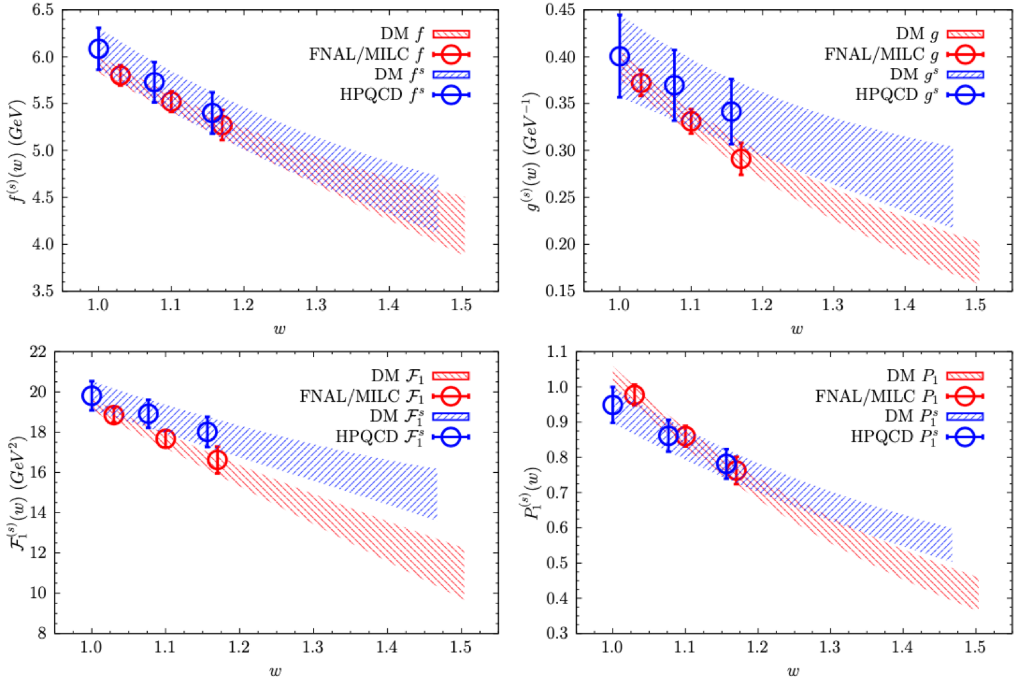
$B_{(s)} \rightarrow D_{(s)} \ell \nu_\ell$



red: u/d spectator quark

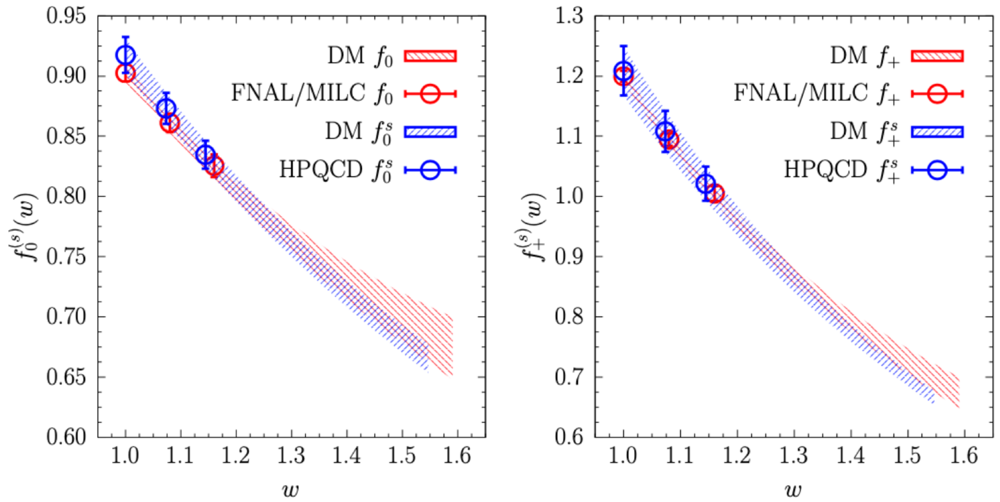
blue: strange spectator quark

$B_{(s)} \rightarrow D^*_{(s)} \ell \nu_\ell$



Super-quick parenthesis: semileptonic $B_s \rightarrow D_s^{(*)}$ decays

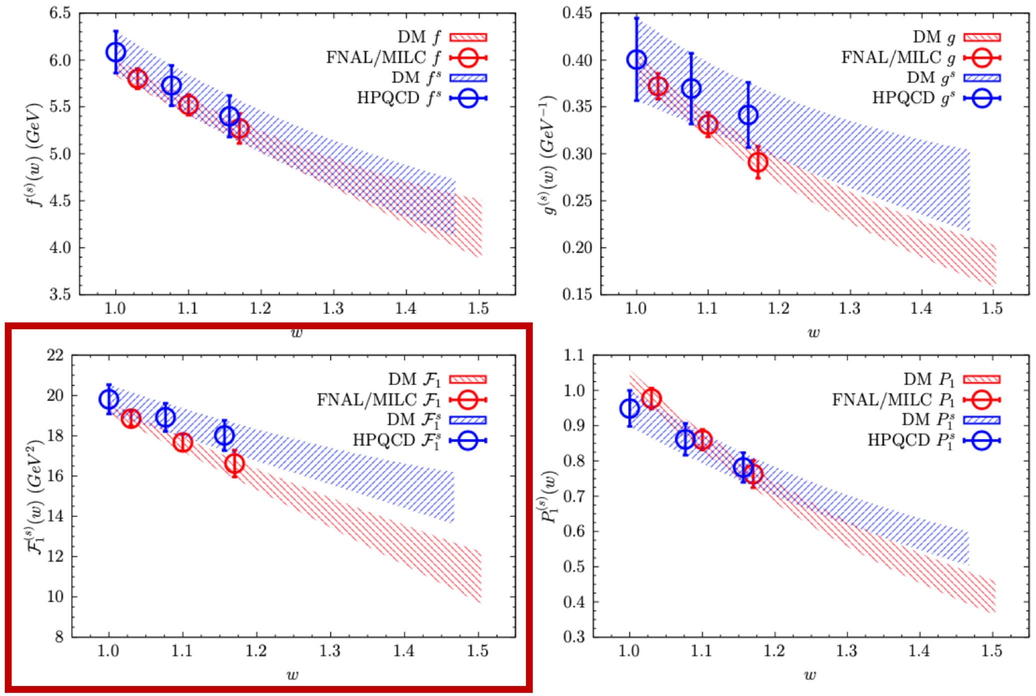
$B_{(s)} \rightarrow D_{(s)} \ell \nu_\ell$



red: u/d spectator quark

blue: strange spectator quark

$B_{(s)} \rightarrow D^*_{(s)} \ell \nu_\ell$



$SU(3)_F$ symmetry breaking effects?

Interesting question for the future!

The Dispersive Matrix (DM) method

Let us examine the case of the production of a pseudoscalar meson (as for the $B \to D$ case). Supposing to have *n* LQCD data for the FFs at the quadratic momenta $\{t_1, \cdots, t_n\}$ (hereafter $t \equiv q^2$), we define

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

$$\langle h_1 | h_2 \rangle = \int_{|z|=1} \frac{dz}{2\pi i z} \bar{h}_1(z) h_2(z)$$

$$g_t(z) \equiv \frac{1}{1 - \bar{z}(t) z}$$

CENTRAL REQUIREMENT:

$$\det \mathbf{M} \geq 0$$

The conformal variable z is related to the momentum transfer as:

$$z(t) = \frac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1}$$

$$t_\pm \equiv (m_B \pm m_D)^2$$

Two advantages:

1. *z is real*
2. *1-to-1 correspondence:*

$[0, t_{max}=t_-] \Rightarrow [z_{max}, 0]$

A lot of work in the past:

L. Lellouch, NPB, 479 (1996), p. 353-391

C. Bourrely, B. Machet, and E. de Rafael, NPB, 189 (1981), pp. 157 – 181

E. de Rafael and J. Taron, PRD, 50 (1994), p. 373-380

The DM method

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

We also have to define the kinematical functions

$$\phi_0(z, Q^2) = \sqrt{\frac{2n_I}{3}} \sqrt{\frac{3t_+ t_-}{4\pi}} \frac{1}{t_+ - t_-} \frac{1+z}{(1-z)^{5/2}} \left(\beta(0) + \frac{1+z}{1-z} \right)^{-2} \left(\beta(-Q^2) + \frac{1+z}{1-z} \right)^{-2},$$

$$\phi_+(z, Q^2) = \sqrt{\frac{2n_I}{3}} \sqrt{\frac{1}{\pi(t_+ - t_-)} \frac{(1+z)^2}{(1-z)^{9/2}}} \left(\beta(0) + \frac{1+z}{1-z} \right)^{-2} \left(\beta(-Q^2) + \frac{1+z}{1-z} \right)^{-3}, \quad \beta(t) \equiv \sqrt{\frac{t_+ - t}{t_+ - t_-}}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, @ *{t_1,…,t_n}*: from Cauchy's theorem (for generic *m*)

$$\langle g_{t_m} | \phi f \rangle = \phi(t_m, Q^2) f(t_m)$$

LQCD data!

$$\langle g_{t_m} | g_{t_l} \rangle = \frac{1}{1 - \bar{z}(t_l) z(t_m)}$$

- non-perturbative values of the susceptibilities, since from the dispersion relations (calling Q^2 the Euclidean quadratic momentum)

$$\chi(Q^2) \geq \langle \phi f | \phi f \rangle$$

Since the susceptibilities are computed on the lattice, we can in principle use whatever value of Q^2 !

The DM method

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

In the presence of **poles @** $t_{P1}, t_{P2}, \cdots ..., t_{PN}$**:**

$$\phi(z, q^2) \to \phi_P(z, q^2) \equiv \phi(z, q^2) \times \frac{z - z(t_{P1})}{1 - \bar{z}(t_{P1}) z} \times \cdots \times \frac{z - z(t_{PN})}{1 - \bar{z}(t_{PN}) z}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, @ $\{t_1,...,t_n\}$: from Cauchy's theorem (for generic *m*)

$$\langle g_{t_m} | \phi f \rangle = \phi(t_m, Q^2) f(t_m)$$

LQCD data!

$$\langle g_{t_m} | g_{t_l} \rangle = \frac{1}{1 - \bar{z}(t_l) z(t_m)}$$

- non-perturbative values of the susceptibilities, since from the dispersion relations (calling Q^2 the Euclidean quadratic momentum)

$$\chi(Q^2) \geq \langle \phi f | \phi f \rangle$$

Statistical and systematic uncertainties

How can we finally combine all the N_U lower and upper bounds of both the FFs??

One bootstrap event case:

after a single extraction, we have one value of the lower bound f_L and one value of the upper one f_U for each FF. Assuming that the true value of each FF can be **everywhere inside the range (f_U - f_L) with equal probability**, we associate to the FFs a *flat* distribution

$$P(f_{0(+)}) = \frac{1}{f_{U,0(+)} - f_{L,0(+)}} \Theta(f_{0(+)} - f_{L,0(+)}) \Theta(f_{U,0(+)} - f_{0(+)})$$

Many bootstrap events case:

how to mediate over the whole set of bootstrap events? Since the lower and the upper bounds of a generic FF are deeply correlated, we will assume a multivariate Gaussian distribution:

$$P(f_L, f_U) = \frac{\sqrt{\det \rho}}{2\pi} \exp\left[-\frac{\rho_{up,up}(f_U - \langle f_U \rangle)^2 + \rho_{lo,lo}(f_L - \langle f_L \rangle)^2 + 2\rho_{lo,up}(f_U - \langle f_U \rangle)(f_L - \langle f_L \rangle)}{2}\right]$$

In conclusion, we can combine the bounds of each FF in a final mean value and a final standard deviation, defined as

$$\langle f \rangle = \frac{\langle f_L \rangle + \langle f_U \rangle}{2},$$

$$\sigma_f = \frac{1}{12}(\langle f_U \rangle - \langle f_L \rangle)^2 + \frac{1}{3}(\sigma^2_{f_{lo}} + \sigma^2_{f_{up}} + \rho_{lo,up}\sigma_{f_{lo}}\sigma_{f_{up}})$$

NO PARAMETRIZATION ADOPTED!!!

Kinematical Constraints (KCs)

REMINDER: after the unitarity filter we were left with $N_U < N$ *survived events!!!*

Let us focus on the pseudoscalar case. Since by construction the following *kinematical constraint* holds

$$f_0(0) = f_+(0)$$

we will filter only the $N_{KC} < N_U$ events for which the two bands of the FFs intersect each other @ *t = 0*. Namely, for each of these events we also define

$$\phi_{lo} = \max[F_{+,lo}(t=0), F_{0,lo}(t=0)]$$

$$\phi_{up} = \min[F_{+,up}(t=0), F_{0,up}(t=0)]$$

From WE theorem

$$\langle D(p_D)|\, V^\mu \,|B(p_B)\rangle \;=\; f_+(p_B+p_D)^\mu + f_-(p_B-p_D)^\mu$$

One then defines

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_B^2 - m_D^2} f_-(q^2)$$

$$\langle D(p_D)\,|\,V^\mu\,|\,B(p_B)\rangle = f^+(q^2)\left(p_B^\mu + p_D^\mu - \frac{m_B^2 - m_D^2}{q^2} q^\mu\right) + f^0(q^2)\frac{m_B^2 - m_D^2}{q^2} q^\mu$$

Kinematical Constraints (KCs)

We then consider a **modified matrix**

$$\mathbf{M_C} = \begin{pmatrix} \langle\phi f|\phi f\rangle & \langle\phi f|g_t\rangle & \langle\phi f|g_{t_1}\rangle & \cdots & \langle\phi f|g_{t_n}\rangle & \langle\phi f|g_{t_{n+1}}\rangle \\ \langle g_t|\phi f\rangle & \langle g_t|g_t\rangle & \langle g_t|g_{t_1}\rangle & \cdots & \langle g_t|g_{t_n}\rangle & \langle g_t|g_{t_{n+1}}\rangle \\ \langle g_{t_1}|\phi f\rangle & \langle g_{t_1}|g_t\rangle & \langle g_{t_1}|g_{t_1}\rangle & \cdots & \langle g_{t_1}|g_{t_n}\rangle & \langle g_{t_1}|g_{t_{n+1}}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n}|\phi f\rangle & \langle g_{t_n}|g_t\rangle & \langle g_{t_n}|g_{t_1}\rangle & \cdots & \langle g_{t_n}|g_{t_n}\rangle & \langle g_{t_n}|g_{t_{n+1}}\rangle \\ \langle g_{t_{n+1}}|\phi f\rangle & \langle g_{t_{n+1}}|g_t\rangle & \langle g_{t_{n+1}}|g_{t_1}\rangle & \cdots & \langle g_{t_{n+1}}|g_{t_n}\rangle & \langle g_{t_{n+1}}|g_{t_{n+1}}\rangle \end{pmatrix}$$

with $t_{n+1} = 0$. Hence, we compute the new lower and upper bounds of the FFs in this way. For each of the N_{KC} events, we extract $N_{KC,2}$ values of $f_0(0) = f_+(0) \equiv f(0)$ with uniform distribution defined in the range $[\phi_{lo}, \phi_{up}]$. Thus, for both the FFs and for each of the N_{KC} events we define

$$\begin{aligned} F_{lo}(t) &= \min[F_{lo}^1(t), F_{lo}^2(t), \cdots, F_{lo}^{N_{KC,2}}(t)], \\ F_{up}(t) &= \max[F_{up}^1(t), F_{up}^2(t), \cdots, F_{up}^{N_{KC,2}}(t)] \end{aligned}$$

Non-perturbative computation of the susceptibilities

In **arXiv:2105.07851**, we have presented the results of the first computation on the lattice of the susceptibilities for the $b \to c$ quark transition, using the N_f=2+1+1 gauge ensembles generated by ETM Collaboration.

How are they defined? The starting point is the HVP tensor:

$$\Pi^V_{\mu\nu}(Q) = \int d^4x \; e^{-iQ\cdot x} \langle 0 | T \left[\bar{b}(x)\gamma^E_\mu c(x) \; \bar{c}(0)\gamma^E_\nu b(0) \right] |0\rangle$$
$$= -Q_\mu Q_\nu \Pi_{0^+}(Q^2) + (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi_{1^-}(Q^2)$$

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2 \Pi_{0^+}(Q^2)\right] = \int_0^\infty dt \; t^2 j_0(Qt) \; C_{0^+}(t) \; , \qquad C_{0^+}(t) = \int d^3x \langle 0|T\left[\bar{b}(x)\gamma_0 c(x) \; \bar{c}(0)\gamma_0 b(0)\right]|0\rangle \; ,$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2 \Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt \; t^4 \frac{j_1(Qt)}{Qt} \; C_{1^-}(t) \qquad C_{1^-}(t) = \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0|T\left[\bar{b}(x)\gamma_j c(x) \; \bar{c}(0)\gamma_j b(0)\right]|0\rangle \; ,$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2 \Pi_{0^-}(Q^2)\right] = \int_0^\infty dt \; t^2 j_0(Qt) \; C_{0^-}(t) \; , \qquad C_{0^-}(t) = \int d^3x \langle 0|T\left[\bar{b}(x)\gamma_0\gamma_5 c(x) \; \bar{c}(0)\gamma_0\gamma_5 b(0)\right]|0\rangle \; ,$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2 \Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt \; t^4 \frac{j_1(Qt)}{Qt} \; C_{1^+}(t) \qquad C_{1^+}(t) = \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0|T\left[\bar{b}(x)\gamma_j\gamma_5 c(x) \; \bar{c}(0)\gamma_j\gamma_5 b(0)\right]|0\rangle$$

Non-perturbative computation of the susceptibilities

In **arXiv:2105.07851**, we have presented the results of the first computation on the lattice of the susceptibilities for the $b \to c$ quark transition, using the N_f=2+1+1 gauge ensembles generated by ETM Collaboration.

How are they defined? The starting point is the HVP tensor:

$$\Pi^V_{\mu\nu}(Q) = \int d^4x\ e^{-iQ\cdot x}\langle 0|T\left[\bar{b}(x)\gamma^E_\mu c(x)\ \bar{c}(0)\gamma^E_\nu b(0)\right]|0\rangle$$
$$= -Q_\mu Q_\nu \Pi_{0^+}(Q^2) + (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi_{1^-}(Q^2)$$

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\ t^2 j_0(Qt)\ C_{0^+}(t)\ , \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\, t'^4\, \frac{j_1(Qt')}{Qt'}\left[(m_b - m_c)^2 C_S(t') + Q^2 C_{0^+}(t')\right]$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\ t^4\frac{j_1(Qt)}{Qt}\ C_{1^-}(t)$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^-}(Q^2)\right] = \int_0^\infty dt\ t^2 j_0(Qt)\ C_{0^-}(t)\ , \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\, t'^4\, \frac{j_1(Qt')}{Qt'}\left[(m_b + m_c)^2 C_P(t') + Q^2 C_{0^-}(t')\right]$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\ t^4\frac{j_1(Qt)}{Qt}\ C_{1^+}(t)$$

Non-perturbative computation of the susceptibilities

The possibility to compute the χs
on the lattice allows us
to choose *whatever value of Q^2 !!!!*
(i.e. near the region of production
of the resonances)

NOT POSSIBLE IN PERTURBATION THEORY!!!

$$(m_b + m_c)\Lambda_{QCD} << (m_b + m_c)^2 - q^2$$

POSSIBLE IMPROVEMENT IN THE STUDY
OF THE FFs through our method!

Work in progress...

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

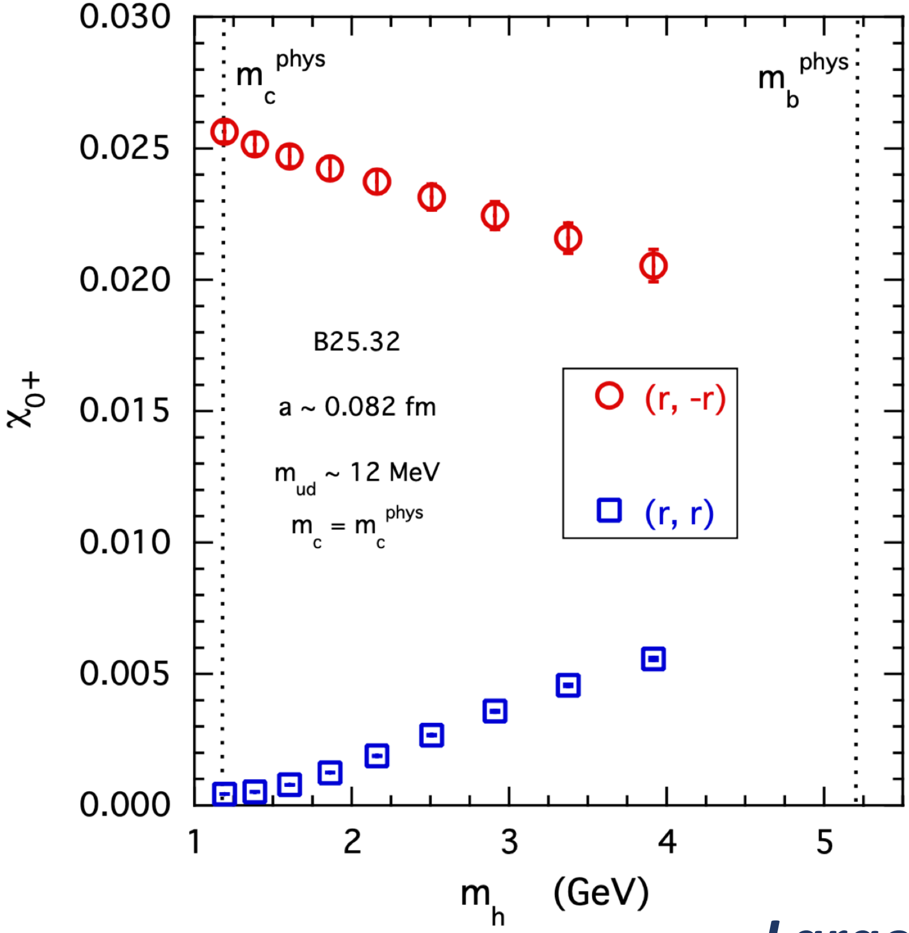
$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^+}(t)\;, \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\, t'^4\, \frac{j_1(Qt')}{Qt'}\left[(m_b - m_c)^2 C_S(t') + Q^2 C_{0^+}(t')\right]$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4 \frac{j_1(Qt)}{Qt}\; C_{1^-}(t)$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^-}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^-}(t)\;, \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\, t'^4\, \frac{j_1(Qt')}{Qt'}\left[(m_b + m_c)^2 C_P(t') + Q^2 C_{0^-}(t')\right]$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4 \frac{j_1(Qt)}{Qt}\; C_{1^+}(t)$$

Non-perturbative computation of the susceptibilities



Following set of masses:

$$m_h(n) = \lambda^{n-1}\, m_c^{phys} \qquad \text{for } n = 1, 2, \ldots$$

$$m_h = a\mu_h/(Z_P a)$$

$$\lambda \equiv [m_b^{phys}/m_c^{phys}]^{1/10} = [5.198/1.176]^{1/10} \simeq 1.1602$$

Nine masses values!

$$m_h(1) = m_c^{phys}$$

$$m_h(9) \simeq 3.9 \text{ GeV} \simeq 0.75\, m_b^{phys}$$

***r*: Wilson parameter**

Large discretisation effects and contact terms

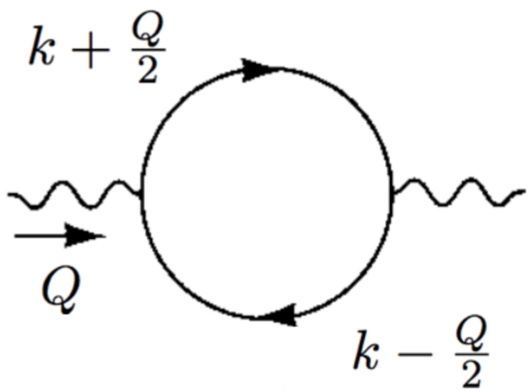
Contact terms & perturbative subtraction

In **twisted mass LQCD:**

$$\Pi_V^{\alpha\beta} = \int_{-\pi/a}^{+\pi/a} \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left[\gamma^\alpha G_1(k+\frac{Q}{2})\gamma^\beta G_2(k-\frac{Q}{2})\right],$$

$$G_i(p) = \frac{-i\gamma_\mu \mathring{p}_\mu + \mathcal{M}_i(p) - i r_i \mu_{q,i}\gamma_5}{\mathring{p}_\mu^2 + \mathcal{M}_i^2(p) + \mu_{q,i}^2}$$

$$\mathring{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu), \qquad \mathcal{M}_i(p) \equiv m_i + \frac{r_i}{2} a\hat{p}_\mu^2, \qquad \hat{p} \equiv \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right).$$



$$\begin{aligned}\Pi_V^{\alpha\beta} &= a^{-2}(Z_1^I + (r_1^2 - r_2^2)Z_2^I + (r_1^2 - r_2^2)(r_1^2 + r_2^2)Z_3^I)g^{\alpha\beta} \\ &+ (\mu_1^2 Z^{\mu_1^2} + \mu_2^2 Z^{\mu_2^2} + \mu_1\mu_2 Z^{\mu_1\mu_2})g^{\alpha\beta} + (Z_1^{Q^2} + (r_1^2 - r_2^2)Z_2^{Q^2})Q\cdot Q g^{\alpha\beta} \\ &+ (Z_1^{Q^\alpha Q^\beta} + (r_1^2 - r_2^2)Z_2^{Q^\alpha Q^\beta})Q^\alpha Q^\beta + r_1 r_2(a^{-2}Z_1^{r_1r_2}g^{\alpha\beta} + (Z_2^{r_1r_2} + (r_1^2 + r_2^2)Z_3^{r_1r_2} \\ &+ (r_1^4 + r_2^4)Z_4^{r_1r_2})Q\cdot Q g^{\alpha\beta} + (\mu_1^2 Z_5^{r_1r_2} + \mu_2^2 Z_6^{r_1r_2})g^{\alpha\beta}) + O(a^2),\end{aligned}$$

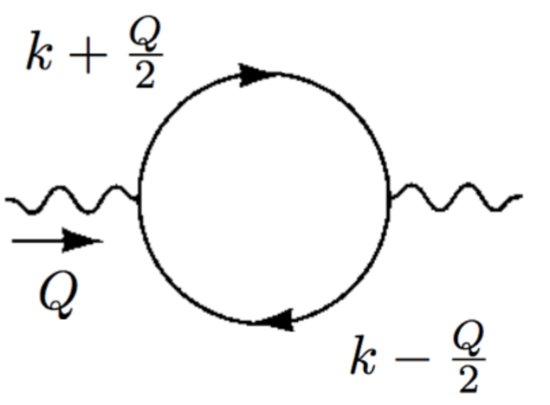
CONTACT TERMS!!!

F. Burger et al., ETM Coll., JHEP '15 [arXiv:1412.0546]

Contact terms & perturbative subtraction

In **twisted mass LQCD** (tmLQCD):

$$\Pi_V^{\alpha\beta} = \int_{-\pi/a}^{+\pi/a} \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma^\alpha G_1(k+\frac{Q}{2})\gamma^\beta G_2(k-\frac{Q}{2})\right],$$



Thus, by separating the *longitudinal* and the *transverse* contributions, we can compute the susceptibilities for all the spin-parity quantum numbers **in the free theory on the lattice**, *i.e.* at order $\mathcal{O}(\alpha_s^0)$ using twisted-mass fermions!

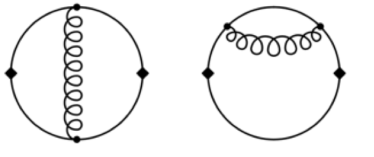
$$\chi_j^{free} = \boxed{\chi_j^{LO}} + \boxed{\chi_j^{discr}}$$

LO term of PT @ $\mathcal{O}(\alpha_s^0)$ contact terms and discretization effects @ $\mathcal{O}(\alpha_s^0 a^m)$ with $m \geq 0$

Perturbative subtraction:

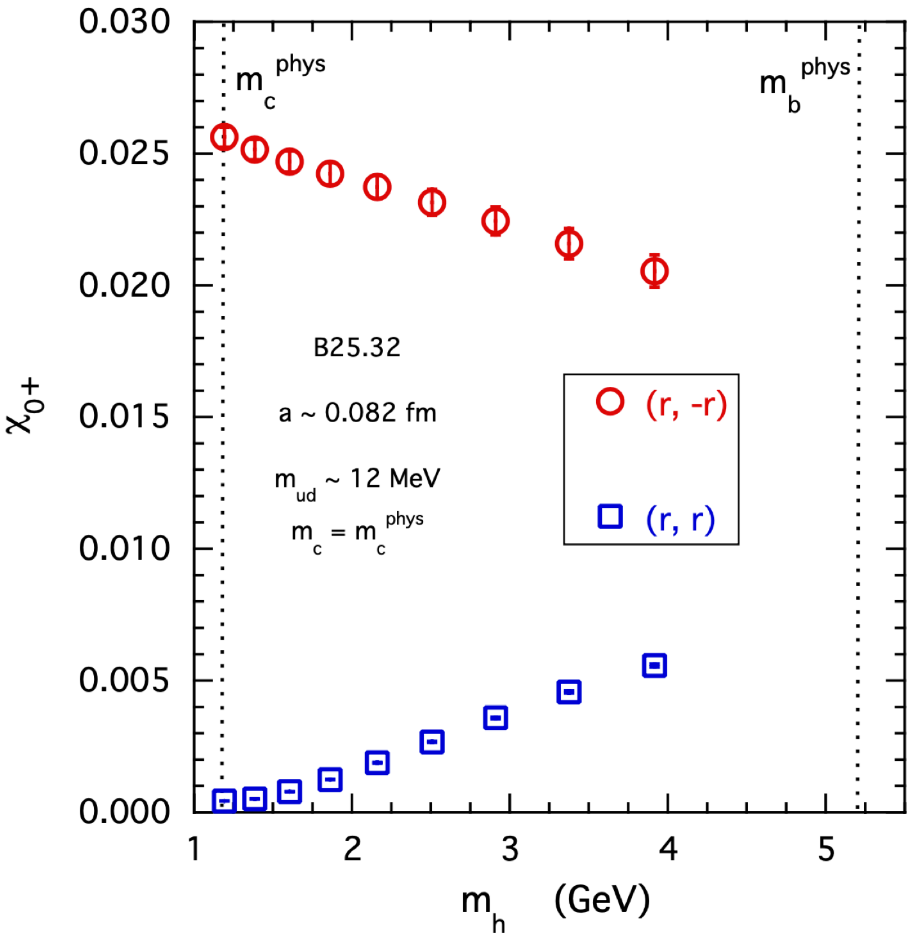
$$\chi_j \to \chi_j - \left[\chi_j^{free} - \chi_j^{LO}\right]$$

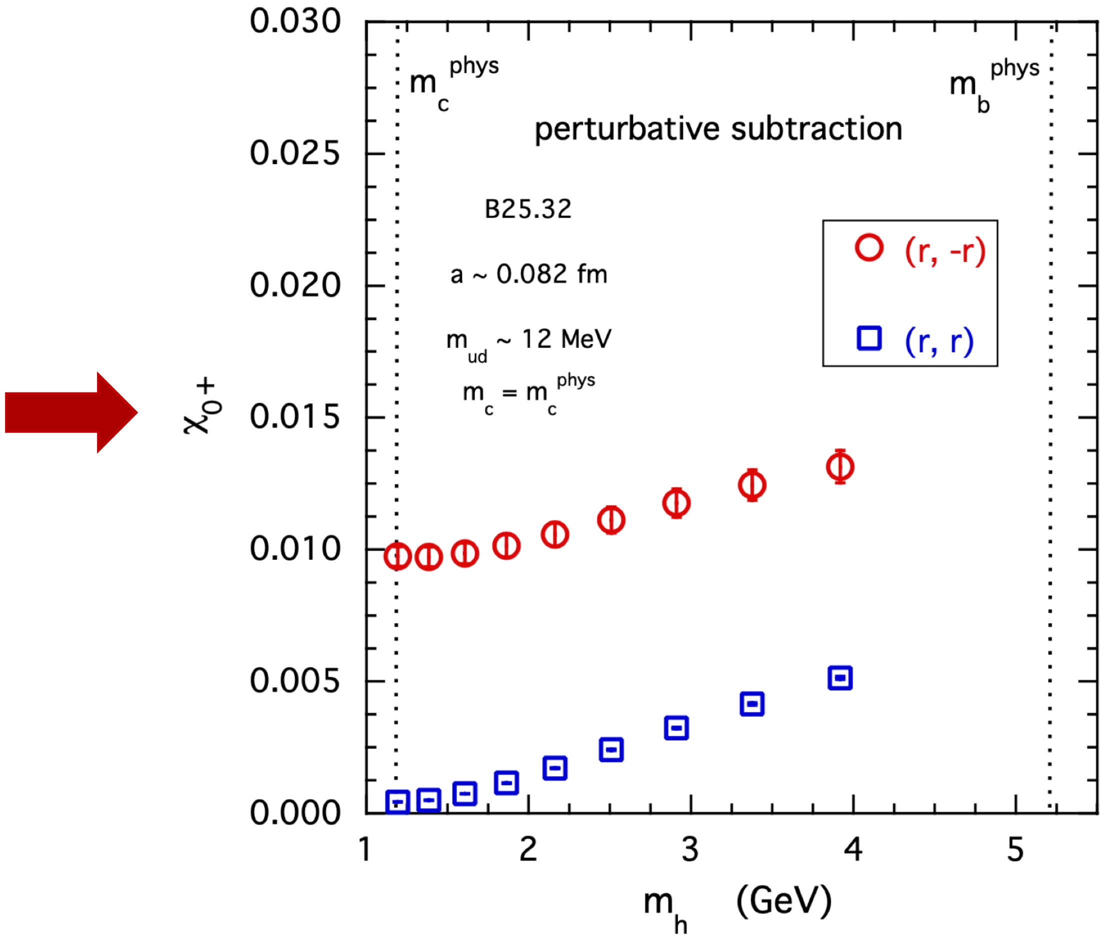
Higher order corrections?



Work in progress...

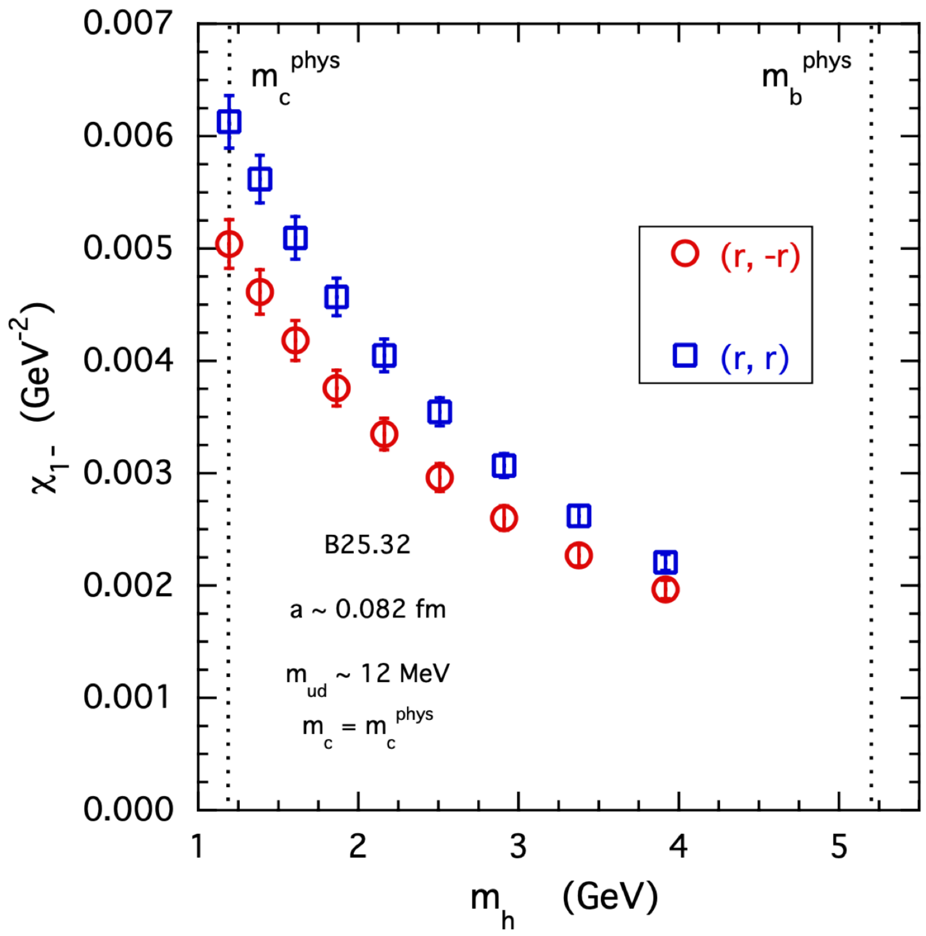
Contact terms & perturbative subtraction

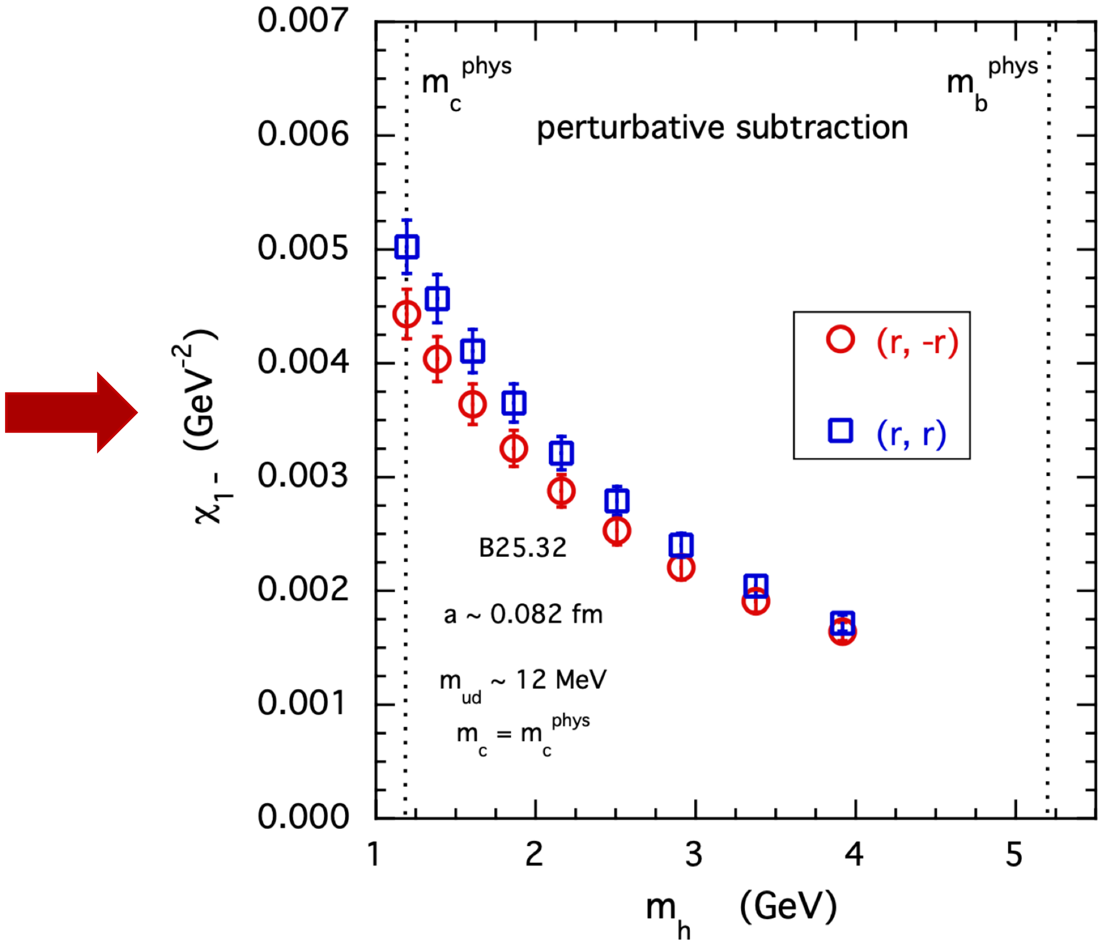




NOT ENOUGH…

Contact terms & perturbative subtraction





OK

ETMC ratio method & final results

For the extrapolation to the physical *b*-quark point we have used the ETMC ratio method:

$$R_j(n; a^2, m_{ud}) \equiv \frac{\chi_j[m_h(n); a^2, m_{ud}]}{\chi_j[m_h(n-1); a^2, m_{ud}]} \boxed{\frac{\rho_j[m_h(n)]}{\rho_j[m_h(n-1)]}}$$

to ensure that $\lim_{n\to\infty} R_j(n) = 1$

$$\rho_{0^+}(m_h) = \rho_{0^-}(m_h) = 1\ ,$$
$$\rho_{1^-}(m_h) = \rho_{1^+}(m_h) = (m_h^{pole})^2$$

All the details are deeply discussed in *arXiv:2105.07851*. In this way, we have obtained **the first lattice QCD determination of susceptibilities of heavy-to-heavy (and heavy-to-light, *in prep.*) transition current densities:**

b → c ***b → u***

	Perturbative ($b \to c$)	With subtraction ($b \to c$)	Non-perturbative ($b \to c$)	With subtraction ($b \to c$)	Non-perturbative ($b \to u$)	With subtraction ($b \to u$)
$\chi_{V_L}[10^{-3}]$	6.204(81)	—	7.58(59)	—	2.04(20)	—
$\chi_{A_L}[10^{-3}]$	24.1	19.4	25.8(1.7)	21.9(1.9)	2.34(13)	—
$\chi_{V_T}[10^{-4}\ \text{GeV}^{-2}]$	6.486(48)	5.131(48)	6.72(41)	5.88(44)	4.88(1.16)	4.45(1.16)
$\chi_{A_T}[10^{-4}\ \text{GeV}^{-2}]$	3.894	—	4.69(30)	—	4.65(1.02)	—

Bigi, Gambino PRD '16
Bigi, Gambino, Schacht PLB '17
Bigi, Gambino, Schacht JHEP '17

ETMC ratio method & final results

For the extrapolation to the physical *b*-quark point we have used the ETMC ratio method:

$$R_j(n; a^2, m_{ud}) \equiv \frac{\chi_j[m_h(n); a^2, m_{ud}]}{\chi_j[m_h(n-1); a^2, m_{ud}]} \frac{\rho_j[m_h(n)]}{\rho_j[m_h(n-1)]}$$

$$\rho_{0^+}(m_h) = \rho_{0^-}(m_h) = 1 \; , \qquad \rho_{1^-}(m_h) = \rho_{1^+}(m_h) = (m_h^{pole})^2$$

to ensure that $\lim_{n \to \infty} R_j(n) = 1$

All the details are deeply discussed in *arXiv:2105.07851*. In this way, we have obtained **the first lattice QCD determination of susceptibilities of heavy-to-heavy (and heavy-to-light,** *in prep.***) transition current densities:**

b → c | ***b → u***

	$b \to c$ Perturbative	$b \to c$ With subtraction	$b \to c$ Non-perturbative	$b \to c$ With subtraction	$b \to u$ Non-perturbative	$b \to u$ With subtraction
$\chi_{V_L}\,[10^{-3}]$	6.204(81)	—	7.58(59)	—	2.04(20)	—
$\chi_{A_L}\,[10^{-3}]$	24.1	19.4	25.8(1.7)	21.9(1.9)	2.34(13)	—
$\chi_{V_T}\,[10^{-4}\ \text{GeV}^{-2}]$	6.486(48)	5.131(48)	6.72(41)	5.88(44)	4.88(1.16)	4.45(1.16)
$\chi_{A_T}\,[10^{-4}\ \text{GeV}^{-2}]$	3.894	—	4.69(30)	—	4.65(1.02)	—

Differences with PT? **~4% for 1^-, ~7% for 0^-, ~20 % for 0^+ and 1^+**