The (squared) amplituhedron and its boundary structure

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SAGEX Scattering Amplitudes: from Geometry to Experiment closing meeting 21/6/2022

based on 2106.09372 with Gabriele Dian

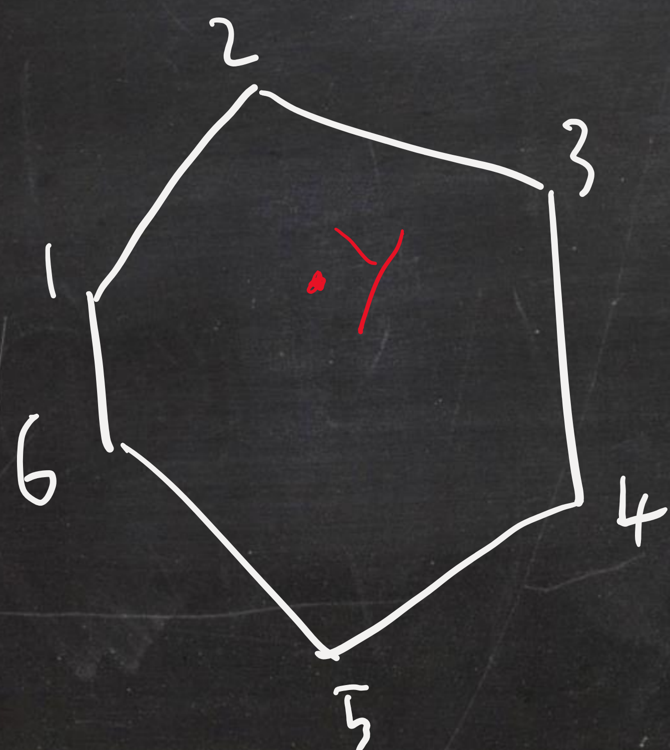
+ to appear with Gabriele Dian, Alastair Stewart

Outline

- Intro to Amplituhedron, max winding number
- Amplituhedron-like, arbitrary winding number =product of amplitudes
- Squared amplituhedron: non unit max residues
- loop amplituhedron also!! NOT a positive geometry!
- Reason? Internal boundaries
- weighted positive geometries and deepest cut, problem and solution

The amplituhedron

Toy model: polygons in $\mathbb{P}^2$, $\quad Y, Z_i \in \mathbb{P}^2$



Y in Polygon

in

$$\Downarrow$$

$$\boxed{\langle Y\ i\ i{+}1\rangle > 0}$$

Y to the right of line (i i+1)

$\langle Y Z_i Z_{i+1}\rangle$

note

$$\langle Y\ 1\ i\rangle = (+,\dots,+,-,\dots,-)$$

i=2,3,4,5

Flips once from + to - Flipping Number 1

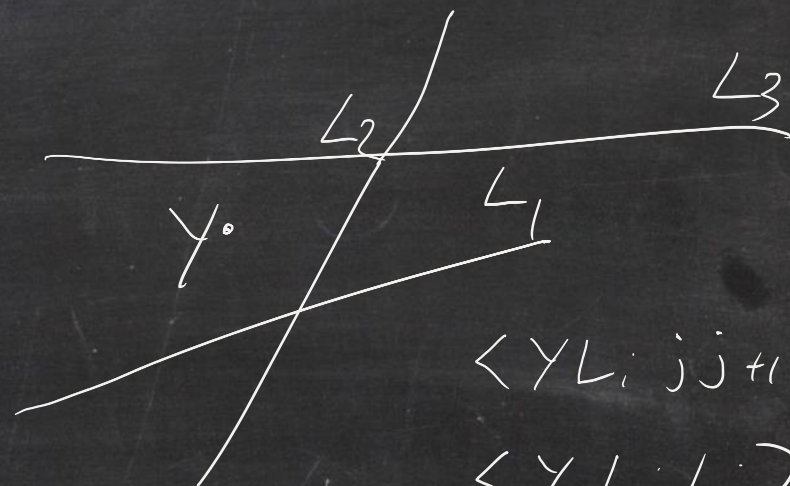
Natural Generalization [Arkani-Hamed, Trnka]

Toy model Polygons in P^2		Tree Amplituhedron	$A_{n,k,m}$	(physics m=4)
$Y \in P^2 = Gr(1,3)$	$\longrightarrow$	$Gr(k, k+m)$	K-planes in $\mathbb{R}^{k+m}$	
$Z_i \in P^2$	$\longrightarrow$	P^{m+k-1}	lines in $\mathbb{R}^{k+m}$	
$\langle Y\, i\; i{+}1\rangle > 0$ $i=1..n$	$\longrightarrow$	$\langle Y\, i\; i{+}1\; j\; j{+}1\rangle > 0$		(m=4)
$\langle Y\,1\,i\rangle$ has One sign flip	$\longrightarrow$	$\langle Y\,1\,2\,3\,i\rangle$ has k sign flips (max possible)		

(Polygons in P^2 = $A_{n,1,2}$ amplituhedron)

L-Loop amplituhedron = the amplituhedron Y together with L lines L_i, $i=1,\dots,L$ $\in Gr(2, m+k)$

in $\mathbb{P}^{m+k-1}$



$\langle Y L_i\, j\, j{+}1 \rangle > 0$

$\langle Y L_i L_j \rangle > 0$

loop flipping number.

Amplituhedron, Amplituhedron-like geometries

Amplituhedron: [Arkani-Hamed, Thomas, Trnka]

$$\mathcal{A}_{n,k} := \left\{ Y \in Gr(k,k+4) \;\middle|\; \begin{array}{ll} \langle Y i\, i+1\, j\, j+1\rangle > 0 & 1 \le i < j-1 \le n-2 \\ \langle Y i\, i+1\, 1\, n\rangle \, (-1)^k > 0 & 1 \le i < n-1 \\ \{\langle Y 123 i\rangle\} & \underline{\text{has } k \text{ sign flips}} \text{ as } i = 4,..,n \end{array} \right\}$$

for $Z \in Gr_>(k+4,n)$,

(tree level)

Natural generalization

Amplituhedron-like: [Dian, PH]

$$\mathcal{H}^{(f)}_{n,k} := \left\{ Y \in Gr(k,k+4) \;\middle|\; \begin{array}{ll} \langle Y i\, i+1\, j\, j+1\rangle > 0 & 1 \le i < j-1 \le n-2 \\ \langle Y i\, i+1\, 1\, n\rangle \, (-1)^{f} > 0 & 1 \le i < n-1 \\ \{\langle Y 123 i\rangle\} & \underline{\text{has } f \text{ sign flips}} \text{ as } i = 4,..,n \end{array} \right\}$$

for $Z \in Gr_+(k+4,n)$

$0 \le f \le k$

- We only consider $k=n-4$

$$\mathcal{A}_{n,k} = \mathcal{H}^{(k)}_{n,k}$$

What do these give?

Loop versions also $(k=n-4)$

Amplituhedron -> amplitudes (N=4 SYM planar, perturbative integrands)

Amplituhedron-like -> products of amplitudes

Main claim:

$$\Omega\left(\mathcal{H}^{(f)}_{n,n-4}\right) = \boxed{H^{(f)}_{n,n-4} = \underbrace{A_{n,f} * A_{n,n-f-4}}_{\text{products of superamplitudes}} \,.}$$

canonical form

M=4

- Loop version too!

*-product

K1+k2 plane

$$\left(\prod_{a=1}^{m}\langle I_a\rangle_{k_1+m}\right) * \left(\prod_{b=1}^{m}\langle J_b\rangle_{k_2+m}\right) = \frac{(-1)^{(k_1k_2+k_2)m}}{m!}\sum_{\sigma\in S_m}\prod_{a=1}^{m}\langle Y(I_a\cap J_{\sigma(a)})\rangle_{k_1+k_2+m}$$

$$\langle Y(I\cap J)\rangle = \sum_{i\in M(I)}\langle Y i\rangle\,\langle \bar{i}J\rangle\,\mathrm{sgn}(i\bar{i})$$

$M(I) = \binom{I}{m}$ = set of ordered m elements in I

Proofs (are hard!)

(amplituhedron-like -> products of amplitudes)

- Alternative definition of amplituhedron-like (analogue of original amplituhedron definition via a positive C matrix Y=C.Z but now C splits into two pieces)
- Partial proof of equivalence of definitions (prove defn1 contains defn2, assuming equivalence of amplituhedron definitions)
- Prove products of onshell diagrams give a tessellation of amplituhedron-like geometries (defn2)
- Tree level only. Similar proof should work for loops. Not done in general but interesting special cases.

"squared amplituhedron"

- Simpler object (to describe);
- physical inequalities
- no flipping constraint
- ie union over all flipping constraints

Squared amplituhedron: $\mathscr{H}_{n,n-4,l} := \mathscr{H}^{+}_{n,n-4,l} \cup \mathscr{H}^{-}_{n,n-4,l}$

Cyclic + twisted cyclic

$$\mathscr{H}^{\pm}_{n,k,l} := \left\{ Y, (AB)_1, .., (AB)_l \;\middle|\; \begin{array}{ll} \langle Y i i+1 j j+1 \rangle > 0 & 1 \leq i < j-1 \leq n-2 \\ \pm \langle Y i i{+}1 1 n \rangle > 0 & 1 \leq i < n-1 \\ \langle Y (AB)_j i i{+}1 \rangle > 0 & \forall\, j,\ \forall i = 1, .., n{-}1 \\ \pm \langle Y (AB)_j 1 n \rangle > 0 & \forall\, j \\ \langle Y (AB)_i (AB)_j \rangle > 0 & \forall i \neq j \end{array} \right\}$$

for $Z \in Gr_{>}(k+4, n)$.

physical inequalities only

squared amplituhedron = union of amplituhedron-like

squared amplituhedron :

$$\mathscr{H}_{n,n-4,l} = \bigcup_{f,l'} \mathscr{H}^{(f,l')}_{n,n-4,l}$$

Orientations match precisely!

$$\overset{?}{\Longrightarrow} \quad H_{n,n-4,l} = \sum_{f,l'} H^{(f,l')}_{n,n-4,l} = \sum_{f,l'} A_{n,f,l'} * A_{n,n-4-f,l-l'}$$

$$= \left(A_n^{*2}\right)_{n-4,l}$$

(agreeing with correlahedron)

Squared amplituhedron -> Square of amplitude!

Correlahedron [Eden, Mason, PH]

- Geometry

n+k-plane 2-planes

$$\boxed{Y \in Gr(n+k; n+k+4) \quad \langle Y X_i X_j \rangle > 0 \quad \equiv \quad G_{n;k}}$$

Large dimension but simple, no winding number constraint

$$\begin{tikzcd}
\textbf{correlahedron} \arrow[rr, "??"] \arrow[d, Rightarrow, "freeze/project"'] & & \textbf{correlator} \arrow[d, Rightarrow, "freeze/project"] \\
\textbf{amplituhedron} \arrow[rr, "cylindrical\ decomp"] & & \textbf{amplitude}^2
\end{tikzcd}$$

squared

light-like limit in amplituhedron space

- Correlahedron gives all half BPS single trace correlators [see SAGEX review!!]
- All correlators = new observation!! Consequence from [Caron-Huot, Coronado]
- previously thought just stress-tensor multiplet

Problem:

- Canonical form (amplitude from amplituhedron) means max residues = 0, +/-1

- the maximal residues of the squared amplituhedron are not only +/- 1

e.g. $$\boxed{(A^2)_{6,2} = 2A_{6,2} + A_{6,1} * A_{6,1}}$$

max residues = 0, +/-2, +/-4

….Even worse!

Loop amplituhedron has max residues different from +/- 1 !!!

eg.

$$\mathrm{MHV}(2) = \frac{\langle A_1 B_1 \mathrm{d}^2 A_1\rangle\langle A_1 B_1 \mathrm{d}^2 B_1\rangle\langle A_2 B_2 \mathrm{d}^2 A_2\rangle\langle A_2 B_2 \mathrm{d}^2 B_2\rangle\langle 1234\rangle^3}{\langle A_1 B_1 A_2 B_2\rangle\langle A_1 B_1 14\rangle\langle A_1 B_1 12\rangle\langle A_2 B_2 23\rangle\langle A_2 B_2 34\rangle} \times$$

$$\times \left[\frac{1}{\langle A_1 B_1 34\rangle\langle A_2 B_2 12\rangle} + \frac{1}{\langle A_1 B_1 23\rangle\langle A_2 B_2 14\rangle}\right] \quad + \quad A_1 B_1 \leftrightarrow A_2 B_2 \, .$$

- Parametrise 4×4 $Z = (Z_1 Z_2 Z_3 Z_4)$ as identity and the loops as

$$\begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} 1 & a_i & 0 & -b_i \\ 0 & c_i & 1 & d_i \end{pmatrix} \, .$$

Then (omitting the differential)

$$\mathrm{MHV}(2) = -\frac{a_2 d_1 + a_1 d_2 + b_2 c_1 + b_1 c_2}{a_1 a_2 b_1 b_2 c_1 c_2 d_1 d_2 \left(\left(a_1 - a_2\right)\left(d_1 - d_2\right) + \left(b_1 - b_2\right)\left(c_1 - c_2\right)\right)}$$

- Now we take the residues in $b_1 = 0, c_1 = 0, b_2 = 0, c_2 = 0$
- complicated pole factorises revealing new pole

$$-\frac{a_2 d_1 + a_1 d_2}{a_1 a_2 d_1 d_2 (a_1 - a_2)(d_1 - d_2)} .$$

- Now take the residue in a_1 at $a_1 = a_2$

$$-\frac{(d_1 + d_2)}{a_2 d_1 d_2 (d_1 - d_2)} .$$

- Now take residue in d_1 at $d_1 = d_2$,

max res=2

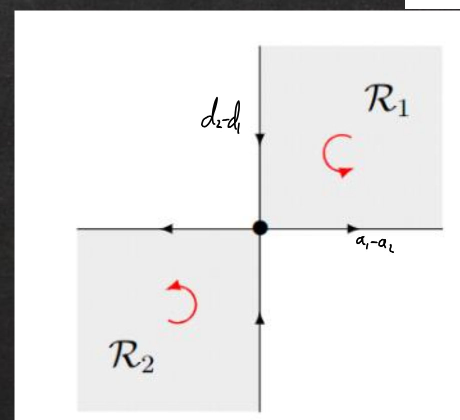
$$-\frac{2}{a_2 d_2} ,$$

Loop amplituhedron $\neq$ positive geometry!!??

- Examine the above residues geometrically.
- Start with amplituhedron. Carefully take boundaries corresponding to each of the above residues:

$$a_1, a_2, d_1, d_2 > 0 \quad \underbrace{(a_1 - a_2)(d_1 - d_2) < 0}_{=} \quad \longrightarrow \Omega = -\frac{a_2 d_1 + a_1 d_2}{a_1 a_2 d_1 d_2 (a_1 - a_2)(d_1 - d_2)}$$

$$\mathcal{R}_1 := \{a_1, a_2, d_1, d_2 \mid a_1 > a_2 > 0 \wedge d_2 > d_1 > 0\}$$
$$\mathcal{R}_2 := \{a_1, a_2, d_1, d_2 \mid a_2 > a_1 > 0 \wedge d_1 > d_2 > 0\}$$



R_1, R_2 same orientation

$d_2 - d_1 = 0$ boundary.

$a_1 - a_2$

"internal boundary" separating two regions of opposite orientation

Previously unnoticed feature:

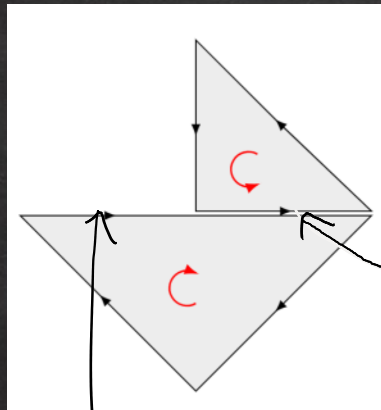
The (loop) amplituhedron contains internal boundaries!

Propose generalized canonical form with recursive def:

$$\mathrm{Res}_f \Omega = \boxed{\lim_{f \to 0} f\Omega \;=\; df \wedge (\omega_{\text{ext}} + 2\omega_{\text{int}})}$$

canonical form of standard (external) boundary region

canonical form of internal boundary region eg.



$$\Omega(R_1) = \frac{dx\, dy}{xy(x+y-1)} + \frac{2dx\, dy}{y(x+y+1)(x-y-1)}$$

(just subtract the two triangles)

Internal boundary

External boundary

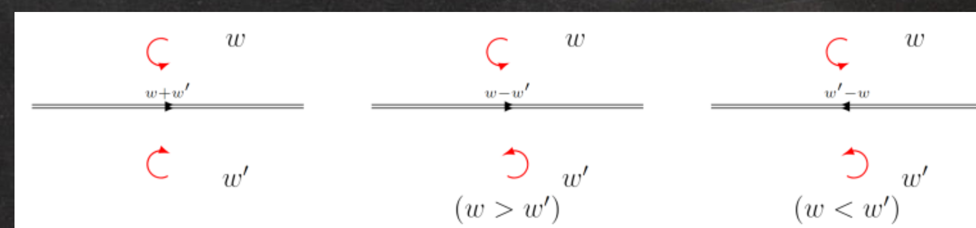
$$\lim_{y\to 0} y\,\Omega = dx\left(\frac{1}{x} - \frac{1}{x+1}\right) + 2dx\left(\frac{1}{x-1} - \frac{1}{x}\right) = \underbrace{\Omega([-1,0])}_{\text{External boundary}} + 2\underbrace{\Omega([0,1])}_{\text{Internal boundary}}$$

- Agrees with the formula

Suggested further generalisation:

Weighted Positive Geometry (WPG) assign constant weight for every region

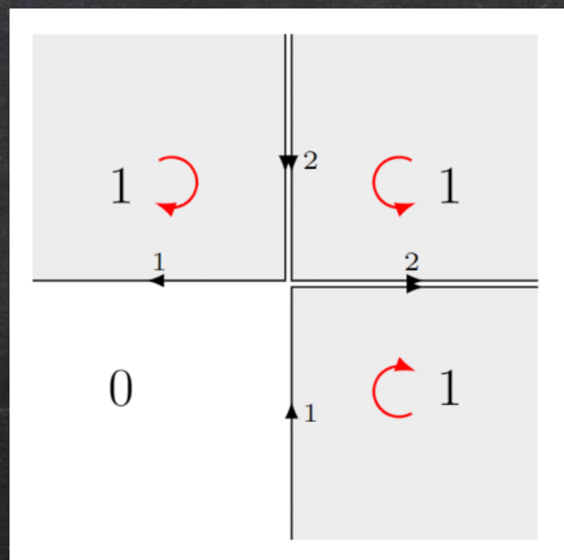
- Define geometry by a piecewise constant $\mathbb{Z}$-valued weight function w (and orientation form O)
- $(w, O) \sim (\mathrm{sign}(\lambda) w, \lambda O') \qquad \lambda \neq 0$
- Additive structure
- $(w_1, O_1) \oplus (w_2, O_2) = (w_1 + \mathrm{sign}(\lambda) w_2, O_1)$
- where $O_1 = \lambda O_2$
- Projection operator onto boundary:
  $\Pi_{\mathcal{C}}(w, O) = (w^+|_{\mathcal{C}}, O^+|_{\mathcal{C}}) \oplus (w^-|_{\mathcal{C}}, O^-|_{\mathcal{C}})$



Residue of canonical form is canonical form of the projection:

$$\mathrm{Res}_{\mathcal{C}}\Omega(w,O) = \Omega(\Pi_{\mathcal{C}}(w,O))$$

eg

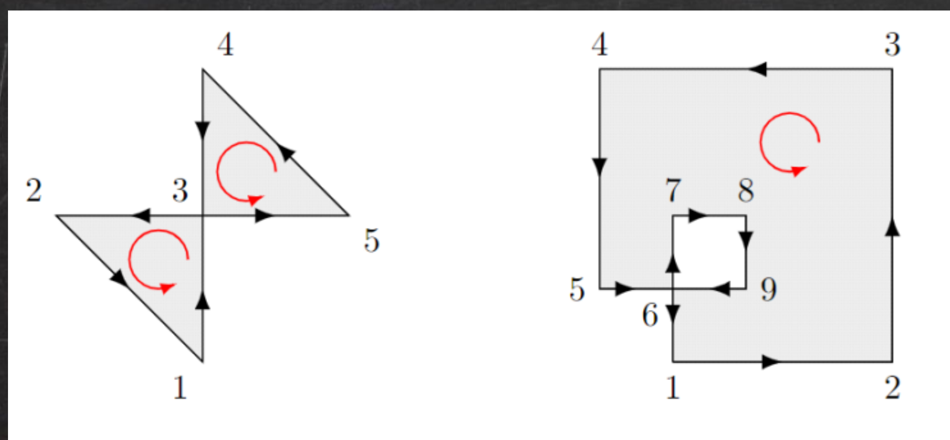


$$\mathrm{Res}_{y=0,x=0}\Omega = -\mathrm{Res}_{x=0,y=0}\Omega = 3$$

Anything that tesselates in WPGs is a WPG

- not true for positive geometries

Eg



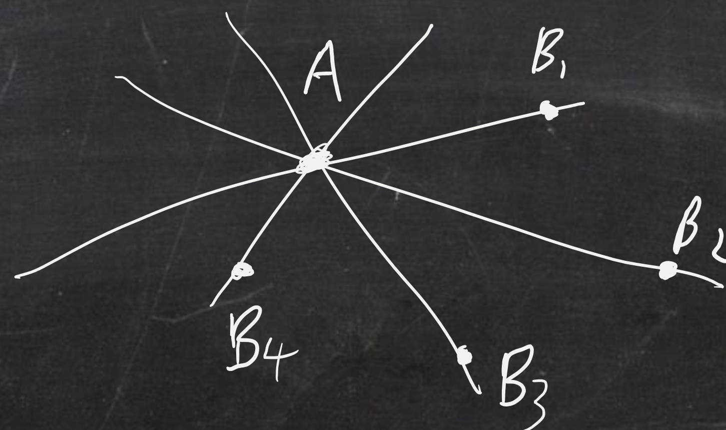
Deepest cuts

- All loop proposal for a multiple (2L-3)-dim residue of loop amplitudes
[Arkani-Hamed, Langer, Srikant, Trnka]

- Geometrical configuration: All loop lines intersect in a common point

$$A_i = A$$

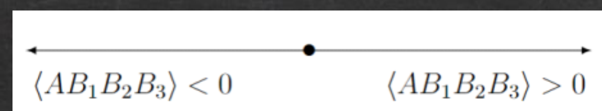
$$L_i = (A_i B_i)$$



- Canonical form of this configuration can be written down at arbitrary loop order

But:

- multiple residues to reach this configuration don't give this result!
- Instead one obtains two regions with opposite orientation separated by an internal boundary



- If this boundary were absent we would get the deepest cut result

At higher loops the deepest cut is also not unique

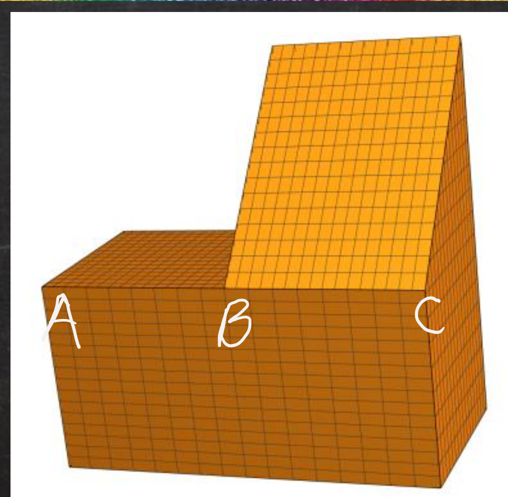
- NB, in general multiple residues depend on the order you take single residues
- Analogous statement true for boundaries - overlooked feature?

(Boundary component of)^k geometry

$\neq$

Codimension k boundary component of geometry

Simple example



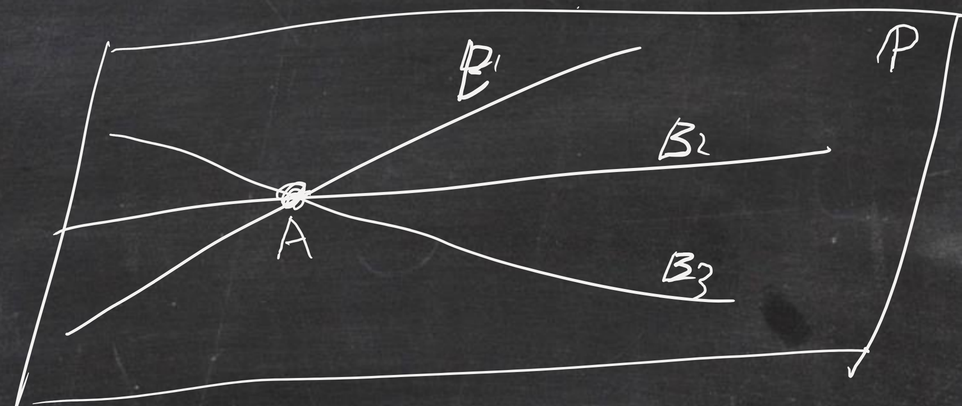
Codimension 2 boundary = [A,C]
Boundary of sloping roof boundary = [B,C]
Boundary of flat roof boundary = [A,B]

Nevertheless

- One *can* analyse the boundary of boundary .... Corresponding to multiple residues to very high loop order
- No explicit all loop formula but very quick algorithm!
- Obtain huge amounts of info about the amplitude
- Determine higher 4-point amplitude/correlator to even higher loop order?
- Construct the relevant f-graphs (rather than using a basis)

All loop order restored

All loops restored: All in one point AND all in one plane configuration



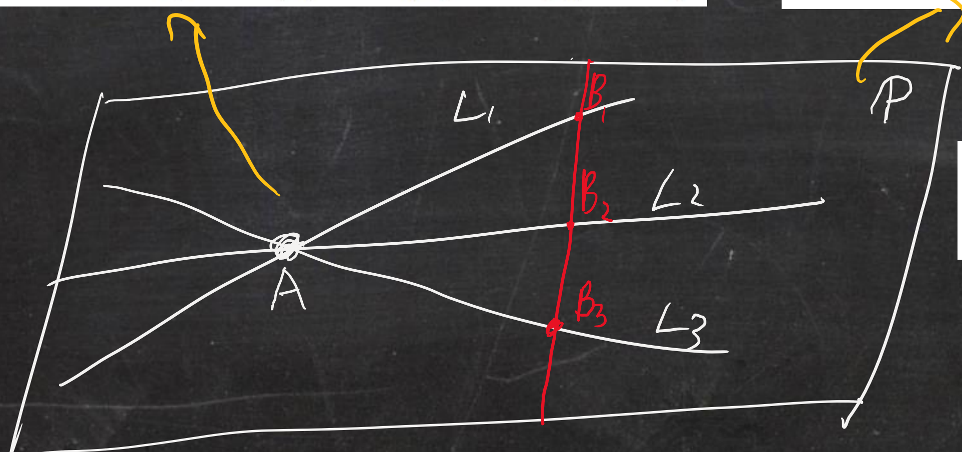
CLAIM: ANY way you reach this configuration gives the same answer (up to a numerical factor = number of internal boundaries crossed)

All loop Canonical form:

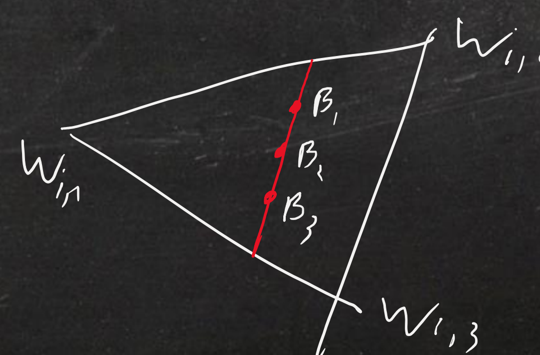
$$\Omega(A,P,B) = \sum_{i=1}^{4} \omega(A) \sum_{p=1}^{3} \lambda_{i,p}(P) \prod_{l=1}^{L} \kappa_{i,p}(B_l)$$

$$\omega(A) = -\frac{\langle 1234\rangle\langle A\mathrm{d}^3 A\rangle}{\langle A123\rangle\langle A234\rangle\langle A341\rangle\langle A412\rangle}$$

$$\lambda_i(P) = \frac{\langle AP\mathrm{d}P_1\rangle\langle AP\mathrm{d}P_2\rangle\langle AW_{i,1}W_{i,2}W_{i,3}\rangle}{\langle APW_{i,1}\rangle\langle APW_{i,2}\rangle\langle APW_{i,3}\rangle}$$



$$\kappa_{i,p}(B) = -\frac{\langle AW_{i,p+1}B\mathrm{d}B\rangle\langle AW_{i,p+1}W_pW_{i,p+2}\rangle}{\langle AW_{i,p+1}BW_{i,p}\rangle\langle AW_{i,p+1}BW_{i,p+2}\rangle}$$



Conclusions

- Amplituhedron-like=products of amplitudes (max nilp)
- Sum of amplituhedron-like = squared amplituhedron = limit of correlahedron
- squared amplitude contains non unit max residues
- So does loop amplitude, not a positive geometry!
- Internal boundaries (weighted positive geometries)
- Deepest cut investigated and fixed

Future:

- Non max nilpotent amplituhedron-like
- Correlahedron
- Use deepest cut to determine amplitude / correlator
- Applications of weighted positive geometry - cosmological polytope?

Further generalizations (future work)

- For maximum $k=n-4$ are there more general geometries?
- eg consider amplituhedron with most general sign choices
- Many possibilities? But many are equivalent under the map $Z_i \to -Z_i$ for one or more than one i.
- Only consider inequivalent geometries (can also relax even this eg Octagons of HLTZ)
- Impose manifest cyclicity (i.e. cyclicity up to equivalence)
- All such geometries are either cyclic or twisted cyclic.
- Different sign choices for $\langle Y i\, i{+}1\, i{+}2\, i{+}3 \rangle$, $\langle Y i\, i{+}1\, i{+}3\, i{+}4 \rangle$ etc.? A^3?

Further generalizations?

- What about the non-max k case?
- Naive generalization: $\boxed{H_{n,k,l}^{(k',l')} = A_{n,k',l'} * A_{n,k-k',l-l'}}$

  Not true! e.g. $k=0$

  (at least for positive z's)

$$H_{n,0,l}^{(0;\,\overbrace{0,..,0}^{l'},\,\overbrace{2,..,2}^{l-l'})} = \overline{A}_{n,0,l'}\, A_{n,0,l-l'}$$

only makes sense for MHV.

- But more possibilities for z's when $k<n-4$.

  - flipping number for z's?

- implications for correlahedron (non max case)?

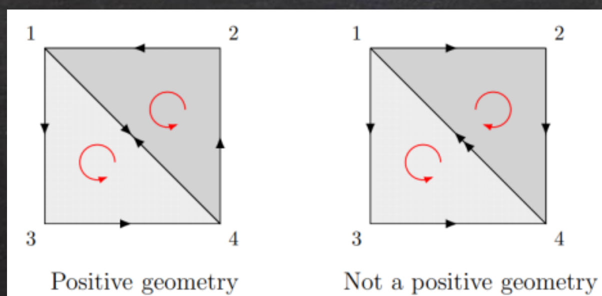
Canonical form.

- Defined only for "positive geometries" which are themselves defined recursively such that their boundaries are positive geometries.
- Positive geometry comes with it's own orientation.

eg.1. Disjoint union of positive geometry = positive geometry (any choice of orientation)

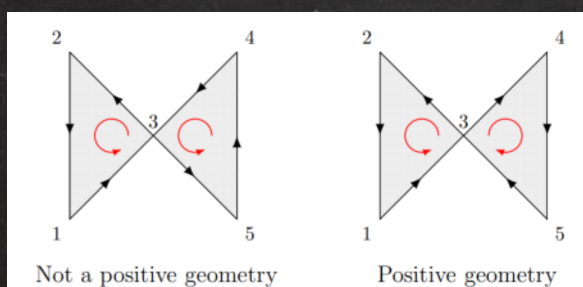
(similar recent discussion by DFLM)

eg.2.



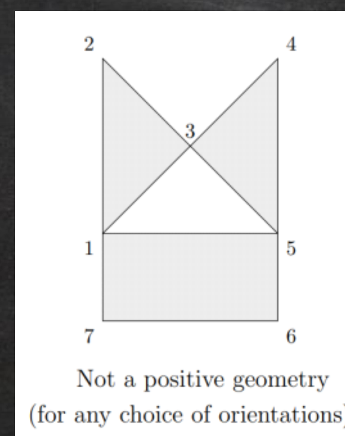
Positive geometry   Not a positive geometry

eg.3.



Not a positive geometry   Positive geometry

eg.4.



Not a positive geometry
(for any choice of orientations)

- Often no global orientation on $Gr(k, k+4)$.

Idea: oriented canonical form.

1.) Define geometries on oriented Grassmanian instead of Grassmanian

$$\boxed{\widetilde{Gr}(k,k+4) = GL_+(k) \backslash M_{k\times(k+4)}}$$

$\langle Y\, i\, i{+}1\, j\, j{+}1 \rangle > 0$ etc.

Note: geometry is defined by inequalities, initially on $\widetilde{Gr}$.

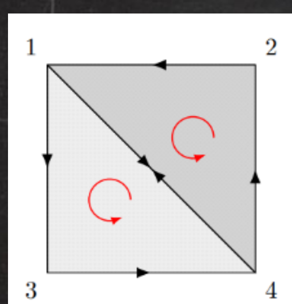
Normally then projected to Gr. Relax this last step.

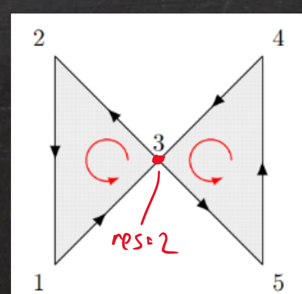
2.) Define "oriented canonical form" on any union of positive geometries, with orientation inherited from $\widetilde{Gr}$

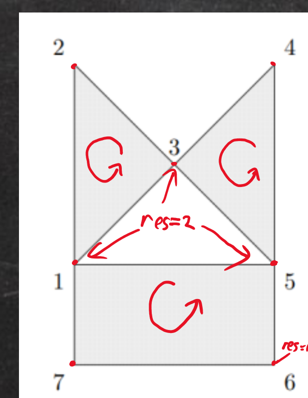
(generalizes to general algebraic variety (double cover)

- Any geometry defined by linear inequalities should have a unique oriented canonical form
- Computed by triangulation
- Cylindrical decomposition provides such a triangulation algorithmically (just procedure for converting integral over a shape in $\mathbb{R}^d$ into a sum of multiple integrals)

eg.







All have an oriented canonical form.

$$\mathscr{H}_{n,n-4,l} = \bigcup_{f,l'} \mathscr{H}^{(f,l')}_{n,n-4,l}$$

almost disjoint (touch on lower dimension)

oriented canonical form

$$H_{n,n-4,l} = \sum_{f,l'} H^{(f,l')}_{n,n-4,l} = \sum_{f,l'} A_{n,f,l'} * A_{n,n-4-f,l-l'}$$

$$= \left(A_n^{*2}\right)_{n-4,l}$$

recall

eg.

$$H^{(0;\overbrace{0,..,0}^{l'},\overbrace{2,..,2}^{l-l'})}_{n,0,l} = \overline{A}_{n,0,l'} A_{n,0,l-l'}$$

oriented + if canonical form   − if canonical form [AT?]

$$H_{4,0,1} = A_{4,0,0} A_{4,0,1} + A_{4,0,1} A_{4,0,0}$$

$$H_{5,0,1} = \bar{A}_{5,0,0} A_{5,0,1} + \bar{A}_{5,0,1} A_{5,0,0} = A_{5,1} + \bar{A}_{5,1}$$

"Proof" of claim (tree-level)

$$\boxed{H^{(f)}_{n,n-4} = A_{n,f} * A_{n,n-f-4}\,.}$$

- ⚠ Proofs modulo standard amplituhedron conjectures

- 1.) alternative description of amplituhedron-like geometry
  (m=4. generalizes to arbitrary m)

$$\mathscr{H}^{(f);\mathrm{alt}}_{n,n-m} := \left\{ Y = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cdot Z \mid C_1 \in Gr_{>}(f,n) \;\wedge\; C_2 \in \mathrm{alt}(Gr_{>})(n-m-f,n) \right\}$$

- analagous to original amplituhedron definition $\boxed{[Y = C \cdot Z,\ C \in Gr_{>}(k,n)]}$

$\mathrm{alt}(Gr_{>}) :=$ $Gr_{>}$ with odd colums flipped sigs

$$\mathscr{H}^{(f);\mathrm{alt}}_{n,n-m} \subseteq \mathscr{H}^{(f)}_{n,n-m}$$

Proof: take $Y \in \mathcal{H}^{\mathrm{alt}}$

$$\boxed{\begin{array}{ccc} Y = & Y_1 & Y_2 \\ & \parallel & \parallel \\ & C_1 \cdot Z & C_2 \cdot Z \end{array}}$$

$C_1 \in Gr_>$
$C_2 \in \mathrm{alt}(Gr_>)$

$\boxed{\text{project onto } Y_1^\perp}$:

Projected Z's:

$$\langle Z_J \rangle_{Y_1} := \langle Y_1 Z_J \rangle = \Delta_{\bar{J}}(C_1)\ \langle Z_{\bar{J}} Z_J \rangle \quad \text{re-order } \bar{J}J$$
$$= \Delta_{\bar{J}}(C_1)\ \langle 1 \cdots n \rangle\ (-1)^{\#_{\mathrm{odd}}(J)} (-1)^{g_{n,f}}$$

$$g_{n,f} := \lfloor \tfrac{n-f}{2} \rfloor + (n-f)n$$

$\tilde{Z}_i := (-1)^i Z_i \Rightarrow \boxed{\langle \tilde{Z}_J \rangle_{Y_1} > 0}$ (assume $g_{n,f} > 0$ for simplicity)

Further: $\boxed{Y_2 = C_2 \cdot Z = \tilde{C}_2 \cdot \tilde{Z}}$ $\quad \tilde{C}_2 \in Gr_>(n-f, n)$

flip odd columns

$\therefore$ $Y_1 \tilde{Z}$'s projected onto $Y_1^\perp$ equivalent to amplituhedron $\boxed{A_{n,n-m-f}(Y_2, \tilde{Z})}$

(Proof of $\mathscr{H}^{(f);\mathrm{alt}}_{n,n-m} \subseteq \mathscr{H}^{(f)}_{n,n-m}$ continued..)

$\therefore$ Y_2 satisfies sign flip defn of amplituhedron

$\langle Y_2 \tilde{Z}_i \tilde{Z}_{i+1} \tilde{Z}_j \tilde{Z}_{j+1} \rangle_{Y_1} > 0, \quad (-1)^{n-f} \langle Y_2 \tilde{Z}_i \tilde{Z}_{i+1} \tilde{Z}_1 \tilde{Z}_n \rangle_{Y_1} > 0, \quad \{\langle Y_2 \tilde{Z}_1 \tilde{Z}_2 \tilde{Z}_3 \tilde{Z}_i \rangle_{Y_1}\}$ has $n-4-f$ sign flips

$$\Downarrow \quad \tilde{Z}_i \to Z_i = \tilde{Z}_i (-1)^i$$

$\langle Y Z_i Z_{i+1} Z_j Z_{j+1} \rangle > 0, \quad (-1)^f \langle Y Z_i Z_{i+1} Z_1 Z_n \rangle > 0, \quad \{\langle Y Z_1 Z_2 Z_3 Z_i \rangle\}$ has f sign flips

$\parallel$ $\boxed{\text{amplituhedron-like}}$

$= \mathcal{H}^{(f)}_{n,n-4}$

(here $m=4$ for definiteness)

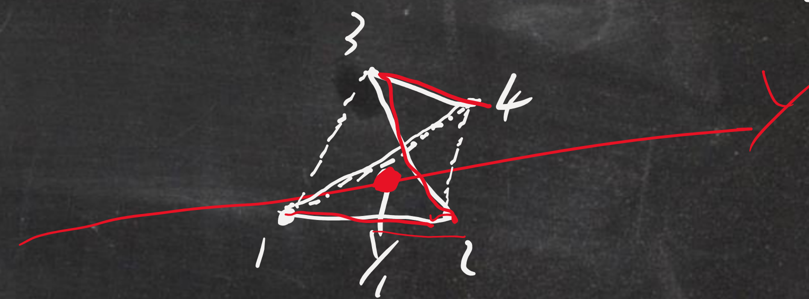
Converse?

$\mathcal{H}^{(f)}_{n,n-m} \subseteq \mathcal{H}^{(f;\mathrm{alt})}_{n,n-m}$

- enough to show that for any $\boxed{Y \in H^{(f)}_{n,n-m} \text{ (sign flip def}^{n}\text{:)}}$

$\exists$ $\boxed{C_1 \in Gr_{>}}$ s.t. $\boxed{Y = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cdot Z}$ rest follows automatically.

[eg $m=2$, $k=2$, $f=1$, $n=4$



Y always goes through tetrahedron]

- Not proven in general...

from now on assume $\boxed{H^{(f)} = H^{(f;\mathrm{alt})}}$

2.) On-shell diagrams

- On-shell diagrams provide triangulation of amplituhedron
- Products of on-shell diagrams provide triangulation of amplituhedron-like !
- Skipping details! to each (affine) permutation $\boxed{\sigma}$ [A-H, B, C, G, P, T]

$$\downarrow$$

$\boxed{C_\sigma(\alpha)}$ co-ords s.t. $\alpha > 0 \Rightarrow C_\sigma(\alpha) \in \boxed{Gr_{\geq}(k,n)}$

$\therefore$ corresponding geometry: $\Pi_\sigma^{>} = \{C_\sigma(\alpha) : \alpha_i > 0\}$ in $\mathrm{Gr}_{\geq 0}(k,n)$ "positroid cell"

corresponding object in super twistor space:

$$f_\sigma^{(k)} = \int \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{4k}}{\alpha_{4k}} \delta^{(4|4)\times k}(C_\sigma(\alpha) \cdot \mathcal{Z})$$

corresponding object in amplituhedron space (canonical form)

$$\Omega(\Pi_{\sigma_i}) = \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{4k}}{\alpha_{4k}}$$

where $\boxed{Y = C_\sigma(\alpha) \cdot Z}$

Product of on-shell diagrams.

- 1st in supertwistor space. If $\boxed{f_\sigma^{(k_1)}}$, $\boxed{f_\tau^{(k_2)}}$ two diagrams

Product:

$$\boxed{f_\sigma^{(k_1)} f_\tau^{(k_2)}} = \int \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{4k_1}}{\alpha_{4k_1}} \frac{d\beta_1}{\beta_1} \cdots \frac{d\beta_{4k_2}}{\beta_{4k_2}} \delta^{(4|4\times k)}\left(\boxed{\begin{pmatrix} C_\sigma(\alpha) \\ C_\tau(\beta) \end{pmatrix}} \cdot \mathcal{Z}\right)$$

- Corresponding geometry?

$$\Pi^{>}_{\sigma,\tau} := \{C_{\sigma,\tau} = \begin{pmatrix} C_\sigma(\alpha) \\ \mathrm{alt}(C_\tau)(\beta) \end{pmatrix} \quad \text{for} \quad \alpha_i, \beta_i > 0\}$$

- This clearly lies inside $\mathcal{H}^{(k_1)\,\mathrm{alt}}_{n,\,k=k_1+k_2}$

- Canonical form:

$$\boxed{\Omega(Z(\Pi^{>}_{\sigma,\tau})) = \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{4k_1}}{\alpha_{4k_1}} \frac{d\beta_1}{\beta_1} \cdots \frac{d\beta_{4k_2}}{\beta_{4k_2}}}$$

for $Y = C_{\sigma,\tau} \cdot Z$

But covariantizing:

$$\Omega(Z(\Pi^{>}_{\sigma,\tau})) = \Omega(Z(\Pi^{>}_{\sigma})) * \Omega(Z(\Pi^{>}_{\tau}))$$

Proof of main conjecture

$$\boxed{H^{(f)}_{n,n-4} = A_{n,f} * A_{n,n-f-4}\,.}$$

Consider two sets of on-shell diagrams $\{f^{(k_1)}_{\sigma_i}\}, \{f^{(k_2)}_{\tau_j}\}$

s.t.

$$\mathcal{A}_{n,k_1} = \sum_i f^{(k_1)}_{\sigma_i}, \qquad \mathcal{A}_{n,k_2} = \sum_j f^{(k_2)}_{\tau_j}$$

with

$k = k_1 + k_2 = n - m$

want to prove $Z(\Pi_{\sigma_i,\tau_j})$ is a triangulation of $\mathcal{H}^{(k_1);\text{alt}}_{n,k}$ amplituhedron_like.

i.e. prove that $\forall\, Y \in \mathcal{H}^{(k);\text{alt}}_{n,k}$, Y belongs to a unique $Z(\Pi_{\sigma_i\tau_j})$.

Take $Y \in H_{n,k}^{(k_1);\mathrm{alt}}$. Then $Y = Y_1 Y_2$; $Y_1 = C_1 \cdot Z$, $Y_2 = C_2 \cdot Z$

$C_1 \in Gr_>(k_1,n)$, $C_2 \in \mathrm{alt}(Gr_>)(k_2,n)$

- As before, project onto $Y_1^\perp$, find $\boxed{A_{n,k_2}((Y_2)_{Y_1},(\tilde{Z})_{Y_1})}$
- $\{f_{\tau_j}\}$ provides a triangulation of this $\Rightarrow \exists j_*$ s.t. $\boxed{(Y_2)_{Y_1} = C_{\tau_{j^*}}(\beta)\cdot(\tilde{Z})_{Y_1}}$ for some $\beta > 0$.
- Key-point: $\boxed{\text{Now we can project back away from } Y_1^\perp.}$ define $\boxed{\hat{Y}_2 = C_{\tau_{j^*}}(\beta)\cdot\hat{Z} = \mathrm{alt}(C_{\tau_{j^*}}(\beta))\cdot Z}$ with $\boxed{Y = Y_1\hat{Y}_2 = Y_1 Y_2.}$
- Similarly, project onto $\hat{Y}_2^\perp$. $\boxed{\text{Find } (Y_1)_{\hat{Y}_2} \in A_{n,k_1}((Y_1)_{\hat{Y}_2},(Z)_{\hat{Y}_2})}$

$$\Rightarrow \exists i^* \text{ s.t. } (Y_1)_{\hat{Y}_2} = C_{\sigma_{i^*}}(\alpha)\cdot(Z)_{\hat{Y}_2} \quad \text{some } \alpha_i > 0$$

project back: $\boxed{\hat{Y}_1 = C_{\sigma_{i^*}}(\alpha)\cdot Z}$ with $\boxed{Y = \hat{Y}_1\hat{Y}_2}$

$\therefore$ Any $Y \in \mathcal{H}_{n,k}^{k_1; alt}$ belongs to one region $\boxed{z(\Pi_{\sigma_{i^*}\tau_{j^*}})}$

$$\therefore \boxed{\mathcal{H}_{n,k}^{k_1; alt} = \bigcup_{i,j} z(\Pi_{\sigma_i \tau_j})}$$

(disjoint union.)

(N.B. some of these will not have maximal dimension $\Leftrightarrow$ contribute 0 eg if $f_{\sigma_i} = f_{\tau_j}$)

Finally this means corresponding canonical form!

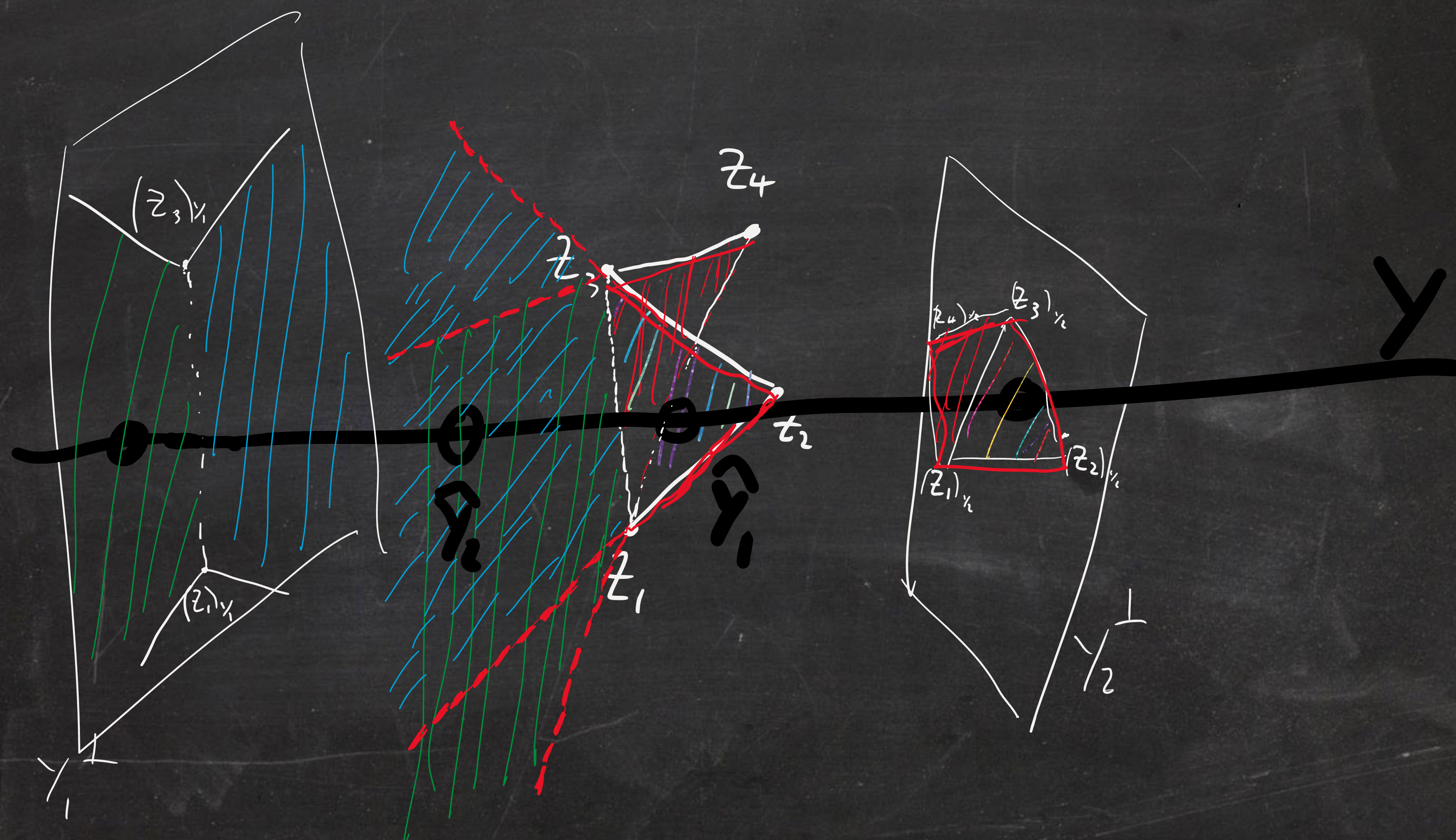
$$H_{n,n-m}^{(k_1)} = \sum_{i,j} \Omega\left(z(\Pi^{?}_{\sigma_i \tau_j})\right) = \sum_{i} \Omega\left(z(\Pi^{?}_{\sigma_i})\right) + \sum_{j} \Omega\left(z(\Pi^{?}_{\tau_j})\right)$$

$$= A_{n,k_1} + A_{n,k_2}$$

$\square$

Example picture

$n=4,\ k=2,\ f=1$



Loop level proof?

- Alternative definition for loop amplituhedron-like

$$(AB)_i = D_i \cdot Z$$

$$Y = C \cdot Z$$

$$Y \in H^{(f)}_{n,n-4}$$

$$\mathcal{H}_{n,n-4,l}^{(f;\overbrace{f+2,\dots,f+2}^{l'},\overbrace{f,\dots,f}^{l-l'})} := \begin{cases} \begin{pmatrix} C_1 \\ D_i \end{pmatrix} \in Gr_>(f+2,n) & \forall \quad i \le l' \\ \begin{pmatrix} C_2 \\ D_i \end{pmatrix} \in \mathrm{alt}(Gr_>)(n-f-2,n) & \forall \quad i > l' \\ \det \begin{pmatrix} C \\ D_i \\ D_j \end{pmatrix} > 0 & \forall \quad i \neq j \end{cases}$$

- Map on-shell diagrams + follow tree-level proof??

  Future work...

- $f = k = n-4$ direct proof, due to manifest factorization

Loop-level proof for $f=k=n-4$

Prove:

$$H_{n,n-4,l}^{(n-4;\overbrace{n-2,..,n-2}^{l'},\overbrace{n-4,..,n-4}^{l-l'})} = A_{n,n-4,l'}\, A_{n,0,l-l'}$$

$= MHV_{l-l'}$

$\boxed{\text{anti-MHV}} = \text{anti-MHV}_{\text{tree}} \times \boxed{\overline{MHV}_{l'}}$

$\therefore$ Prove:

$$H_{n,n-4,l}^{(n-4;\overbrace{n-2,..,n-2}^{l'},\overbrace{n-4,..,n-4}^{l-l'})} = A_{n,n-4,0}\, \overline{A_{n,0,l'}}\, A_{n,0,l-l'}$$

- Factorization can be seen geometrically

1.) $$\mathscr{H}_{n,n-4,l}^{(n-4;\overbrace{n-2,..,n-2}^{l'},\overbrace{n-4,..,n-4}^{l-l'})}(Y,(AB)_i;Z) = \mathscr{A}_{n,n-4}(Y;Z) \times \boxed{\mathscr{H}_{n,0,l}^{(0;\overbrace{0,..,0}^{l'},\overbrace{2,..,2}^{l-l'})}(-(AB)_i;\tilde{Z})}$$

$\tilde{Z}_i = Z_i(-1)^i$

(by examining defining inequalities – no mixing between Y, AB.)

Key-point: $\langle ijkl\rangle_Y = \langle Yijkl\rangle$ max flipping number $\Rightarrow \langle \tilde{i}\tilde{j}\tilde{k}\tilde{l}\rangle > 0$.

2.)

$$\mathscr{H}_{n,0,l}^{(0;\overbrace{0,..,0}^{l'},\overbrace{2,..,2}^{l-l'})} = \mathscr{H}_{n,0,l'}^{(0,0)}\,\mathscr{A}_{n,0,l-l'}$$

- again check defining inequalities
- Only $\langle (AB)_i (AB)_j \rangle > 0$ can prevent factorization but this is automatic when $(AB)_i$ has $f_i = 0$, $(AB)_j$ has $f_j = 2$

(1-loop amplituhedron $\Rightarrow (AB)_j = \sum c_{lm} Z_l Z_m$, $c_{lm} > 0$)

$\Rightarrow \langle (AB)_i (AB)_j \rangle = \sum c_{lm} \langle (AB)_i Z_l Z_m \rangle$

> 0 ← 0-flipping #

$$H_{n,n-4,l}^{(n-4;\overbrace{n-2,..,n-2}^{l'},\overbrace{n-4,..,n-4}^{l-l'})} = A_{n,n-4}\,A_{n,0,l-l'}\,H_{n,0,l'}^{(0,0)}$$

- ⋮
- Finally $H_{n,0,l'}^{(0,0)}$ = amplituhedron MHV, loop with minimal loop flipping # $= \boxed{\overline{A_{n,0,l'}}}$

Proof? Put $\boxed{l'=l}$

Implication, non max k amplituhedron-like

· This then implies the following interpretation for MHV any n, arbitrary winding # l-l′ :

$$H_{n,0,l}^{(0;\overbrace{0,..,0}^{l'},\overbrace{2,..,2}^{l-l'})} = \overline{A}_{n,0,l'}\, A_{n,0,l-l'}$$

[A-M, T, T]

· This is a non-maximal k case! Difficult to generalize, $\bar{A}$?

(arises from maximal k case.)

Stress tensor correlators in N=4 SYM [See SAGEX review!!]

stress tensor multiplet: (analytic superspace)

$O_2 = \mathrm{Tr}(\varphi^2)$ L = chiral Lagrangian

$$\boxed{G_{n;k} = \langle \underbrace{O_2 \cdots O_2}_{n-k} \underbrace{L \cdots L}_{k} \rangle + \ldots} \quad 0 \leq k \leq n-4$$

New: All half BPS correlators generated from these! [Caron-Huot, Coronado]

All half BPs correlators described by correlahedron!

compare with amplitudes in momentum superspace

$$A_{n;k} \qquad 0 \leq k \leq n-4$$

$\uparrow$

$N^k MHV$

Loop level = tree level

(l-loop correlator)

$$\mathcal{G}_{n;k,l} \in \mathcal{G}_{n+l;k+l}$$

Loop integrand = Lagrangian insertion

$\Rightarrow$

Loop-level correlators = higher point tree-level correlators

- Only two parameter-family $\boxed{\mathcal{G}_{n,k}}$

- versus 3 parameter family $\boxed{A_{n,k,l}}$

Correlator /amplitude duality

Light-like limit of correlators = Wilson loop (adjoint rep) [Alday, Eden, Korchemsky, Maldacena, Sokatchev; Adamo, Bullimore, Mason, Skinner]

Wilson loop (fundamental) = amplitude (planar)

[Alday, Maldacena; Drummond, Henn, Korchemsky, Sokatchev, Brandhuber, Travaglini, PH; Mason Skinner, Caron-Huot, ..]

Light-like limit of correlators = $(\text{amplitude})^2$

[Eden, Korchemsky, Sokatchev, PH]

2, parameters

3, parameters

$$G_{n+l,k+l} \to G_{n,k,l} \to \left(A^2\right)_{n,k,l} = \sum_{k',l'} A_{n,k',l'} A_{n,k-k',l-l'}$$

- so one tree correlator contains many combinations of amplitudes
- by taking different light-like limits.

(4-pt) correlator

$$\boxed{G_{n;k=n-4} = G^{(n-4)}_{4;0}\,.}$$

1 parameter family.

max nilpotent = 4-point $l=n-4$ loop.

$k=n-4$

(cf $\overline{\mathrm{MHV}}$)

- Superspace dependence Factorises (like $\overline{\mathrm{MHV}}$)

- Hidden Sn permutation symmetry (mixes loops and points)
- Known explicitly to n=14 (4-pnt ten loops)
- All amplitudes can be extracted even from this simplest correlator? [Tran, PH]

correlahedron

squared amplituhedron $= \bigcup$ amplituhedron-like

Loops too!!

loop amplituhedron

$$\mathscr{A}_{n,k,l} := \left\{ Y, (AB)_1, .., (AB)_l \;\middle|\; \begin{array}{ll} Y \in \mathscr{A}_{n,k} & \\ \langle Y (AB)_j ii+1 \rangle > 0, & \forall\, j,\ \forall i = 1, .., n-1 \\ \langle Y (AB)_j 1n \rangle\, (-1)^{k+1} > 0 & \forall\, j \\ \{\langle Y (AB)_j 1i \rangle\} & \text{has } k{+}2 \text{ flips as } i = 2, .., n, \forall j \\ \langle (AB)_i (AB)_j \rangle > 0 & \forall i \neq j \end{array} \right\}$$

for $Z \in Gr_{>}(k+4, n)$

loop amplituhedron-like

$$\mathscr{H}^{(f;f_1,..,f_l)}_{n,k,l} := \left\{ Y, (AB)_1, .., (AB)_l \;\middle|\; \begin{array}{ll} Y \in \mathscr{H}^{(f)}_{n,k} & \\ \langle Y (AB)_j ii{+}1 \rangle > 0, & \forall\, j,\ \forall i = 1, .., n-1 \\ \langle Y (AB)_j 1n \rangle\, (-1)^{f_j} > 0 & \forall\, j \\ \{\langle Y (AB)_j 1i \rangle\} & \text{has } f_j \text{ flips as } i = 2, .., n, \forall j \\ \langle Y (AB)_i (AB)_j \rangle > 0 & \forall i \neq j \end{array} \right\}$$

for $Z \in Gr_{>}(k+4, n)$

$f_j = f$ or $f{+}2$ only!

flipping # f for each loop.

$$\mathscr{H}^{(f;l')}_{n,n-4,l} := \bigcup_{\sigma \in S_l/(S_{l'} \times S_{l-l'})} \mathscr{H}^{(f;\sigma(\overbrace{f+2,..,f+2}^{l'},\overbrace{f,..,f}^{l-l'}))}_{n,n-4,l}$$

integrand, symmetrise over integration variables.

Loop-level claim:

$$H_{n,n-4,l}^{(f;\overbrace{f+2,..,f+2}^{l'},\overbrace{f,..,f}^{l-l'})} = A_{n,f,l'}(AB_1,\cdots,AB_{l'}) * A_{n,n-f-4,l-l'}(AB_{l'+1},\cdots,AB_l)$$

or equivalently, more compactly!

$$\boxed{H_{n,n-4,l}^{(k',l')} = \binom{l}{l'} A_{n,k',l'} * A_{n,n-k'-4,l-l'}\,.}$$

(individual terms in squared amplitude)

N. B. sum over all distributions of loop momenta with weight one.

Canonical form

- Amplitude=canonical Form of amplituhedron.
- Canonical form (unique) = Volume form with log divergences on amplituhedron boundaries (and no divergences elsewhere)
- Canonical Form defined recursively [Arkani-Hamed, Lam] via it's residues.

boundary on f=0

$\Downarrow$

Residue at f=0

$$\operatorname{Res}_{f=0}\Omega = \lim_{f\to 0} f\Omega = df \wedge \omega$$

canonical form of geometry on f=0