Hidden geometry in two-dimensional scattering

SAGEX meeting 21/6/22

Patrick Dorey + Davide Polvara
Durham University

arXiv 2111.02210 + 2206.tomorrow + ...

Plan:

1. Perturbative integrability
2. Multiple scalar fields
3. Flipping (on-shell) diagrams
4. Affine Toda field theories
5. Higher-order poles (a.k.a. Landau singularities)

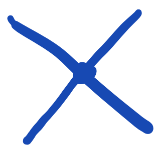
1. Perturbative Integrability

The simplest example... (can do party trick)

Consider $\lambda\phi^4$ theory in 1+1 dimensions

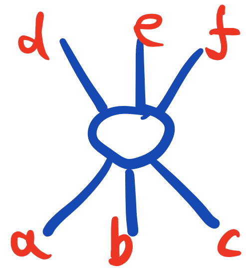
$$\mathcal{L}^{(4)} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

Feynman rules:  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$

 $= -i\lambda$

Task: check whether this theory is integrable by computing the connected $2 \to 4$ amplitude at tree level. (which should vanish if we're integrable)

Equivalently let's cross one "out" to "in" and compute a $3\to 3$ amplitude, with in particles a, b, c and out particles d, e, f.

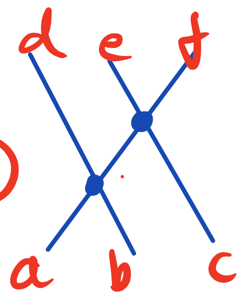


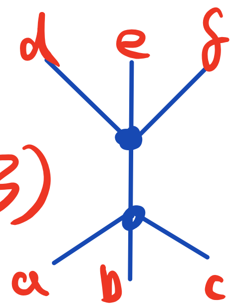
- Light cone momenta $(p, \bar{p}) = (p^0 + p^1, p^0 - p^1)$ are onshell when $p\bar{p} = m^2$ which we solve by writing in & out momenta as

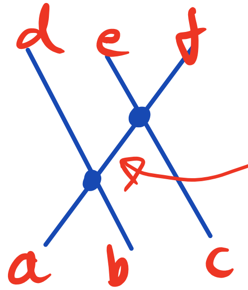
$$p_a = (ma, ma^{-1}) \quad , \quad p_b = (mb, mb^{-1})$$

etc, with $a, b \ldots \in \mathbb{R}^+$.

- Just two classes of diagrams:

(A) 

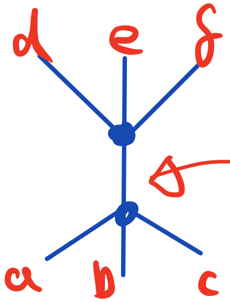
(B) 

(A) 

$$p = m(a+b-d,\ a^{-1}+b^{-1}-d^{-1})$$

$$\Rightarrow \frac{i}{p^2-m^2} = \frac{i}{m^2}\,\frac{1}{(a+b-d)(a^{-1}+b^{-1}-d^{-1})-1}$$

$$= \frac{i}{m^2}\,\frac{-abd}{(a+b)(a-d)(b-d)}$$

(B) 

$$p = m(a+b+c,\ a^{-1}+b^{-1}+c^{-1})$$

$$\Rightarrow \frac{i}{p^2-m^2} = \cdots = \frac{i}{m^2}\,\frac{abc}{(a+b)(b+c)(c+a)}$$

So $$\langle \text{out}|\text{in}\rangle_{\text{tree}} = -\frac{i\lambda^2}{m^2}\, A_{\text{legs}}\, H(a,b,c,d,e,f)$$

External leg factors (same for all diagrams)

internal propagator part, from

and $$H(a,b,c,d,e,f)$$

$$= \sum_{\substack{\text{cycl. }\{a,b,c\} \\ \text{cycl. }\{d,e,f\}}} \frac{-abc}{(a+b)(a-d)(b-d)} + \frac{abc}{(a+b)(b+c)(c+a)}$$

Looks complicated... but if

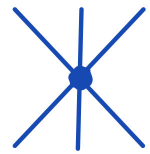
and
$$a+b+c = d+e+f$$
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1}{d}+\frac{1}{e}+\frac{1}{f}$$

overall conservation of left & right light-cone momenta

then $$\boxed{H(a,b,c,d,e,f) \equiv -1}$$

(exercise!)

<u>Conclude</u>: the production amplitude is <u>not</u> zero [so the theory isn't integrable], but it <u>is</u> unexpectedly simple.

Furthermore, $2 \to 4$ could be cancelled completely by adding a ϕ^6 term  to $\mathcal{L}^{(4)}$:

$$\mathcal{L}^{(4)} \to \mathcal{L}^{(6)} = \mathcal{L}^{(4)} - \frac{1}{6!}\frac{\lambda^2}{m^2}\phi^6$$

$$= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{1}{6!}\frac{\lambda^2}{m^2}\phi^6$$

$$= \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{\beta^2}\left[\frac{1}{2}\beta^2\phi^2 + \frac{1}{4!}\beta^4\phi^4 + \frac{1}{6!}\beta^6\phi^6\right]$$

where $\beta^2 = \lambda/m^2$

Now continue!

$$\mathcal{L}^{(6)} \xrightarrow{\quad} \mathcal{L}^{(8)} \xrightarrow{\quad} \mathcal{L}^{(10)} \xrightarrow{\quad} \cdots$$

$2 \to 6$ required to vanish

$2 \to 8$ required to vanish

Each step, if it works, determines the next term uniquely.

Infinitely-many diagrams later:

$$\mathcal{L}^{(\infty)} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{\beta^2}\left[\cosh\beta\phi - 1\right]$$

and we have rediscovered the sinh-Gordon theory.

[For a systematic recursive approach using a neat multi-Regge limit, see Gabai et al, arXiv 1803.0357]

Message: From the point of view of perturbation theory, a lot is happening "under the surface" to give the simple properties such as no particle production that we see in integrable QFT, even at tree level.

Our hope is to understand the mechanisms behind this 'perturbative integrability' in greater depth...

Further remark: The procedure just described gave us no choice at each step - whatever we end up with starting from $\lambda\phi^4$ is the <u>only</u>* such theory of a single massive scalar with no particle production.

[* small loophole: we could have started with $\lambda\phi^3$ - this gives the so-called Bullough-Dodd model, which is indeed the only other option]

This suggests a possible classification programme...

2. Multiple scalar fields

Natural next step: suppose we have r different scalar particles with

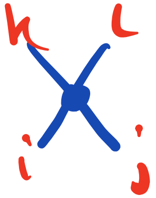
- masses $m_1, \ldots, m_r$
- non-zero 3-point couplings C^{ijk}

(& maybe higher-point couplings too)

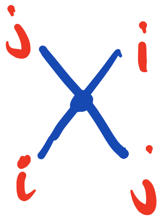
Two tasks:

- Understand the general constraints on the space of such theories implied by considerations like those just seen;
- For cases we know/suspect are integrable, understand what's going on 'under the hood' of perturbation theory.

With more particle types we can require 2→2 scattering to be diagonal:

S_{ij}^{kl} = 

$= 0$ unless $l = i$, $k = j$ [certainly needed by integrability if all masses differ]

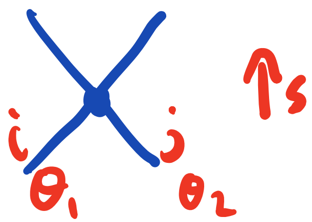
and write 

$= S_{ij}(\theta)$ where on shell momenta are

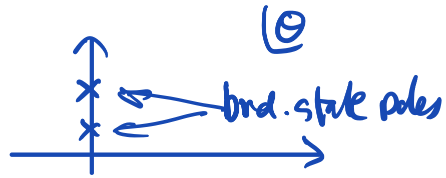
now parametrised by rapidity θ: $(p_i, \bar{p}_i) = (m_i e^{\theta}, m_i e^{-\theta})$

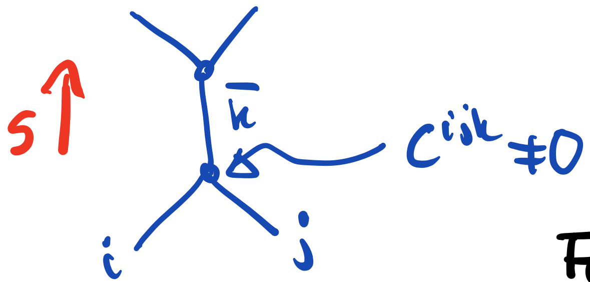
Note $s = (p_i + p_j)^2 = (m_i e^{\theta_1} + m_j e^{\theta_2})(m_i e^{\theta_1} + m_j e^{\theta_2})$

$$= m_i^2 + m_j^2 + 2 m_i m_j \cosh(\theta_1 - \theta_2)$$



If all particles are stable, bound state poles must be at imaginary rapidities (below threshold):

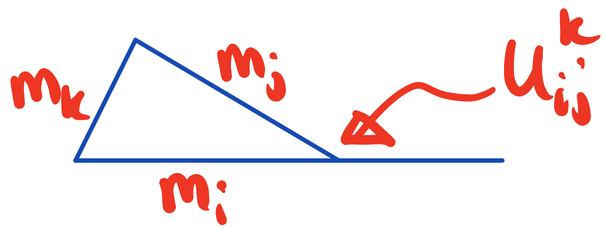




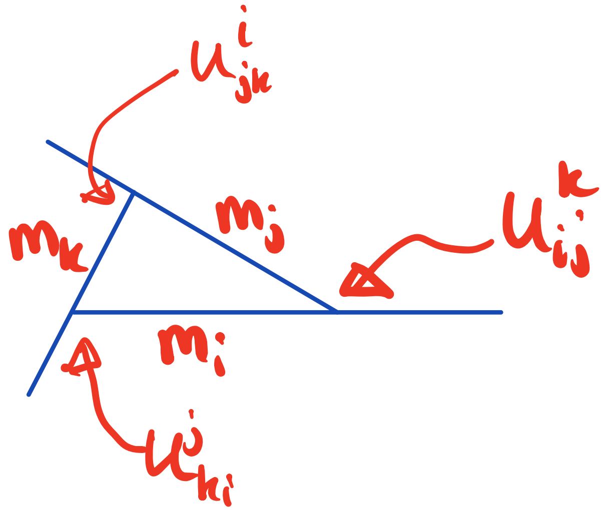
Forward channel pole when $\theta_1 - \theta_2 = i u_{ij}^k$, say, where the <u>fusing angle</u> u_{ij}^k is fixed by noting that at the pole we have

$$s = m_k^2 = m_i^2 + m_j^2 + 2 m_i m_j \cos u_{ij}^k$$

Geometrically it is an outside angle of a "mass triangle":

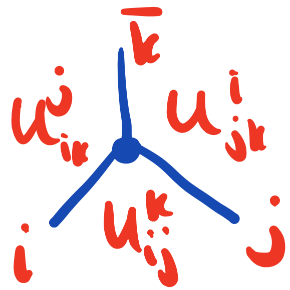


Likewise for u^i_{jk} and u^j_{ki} :

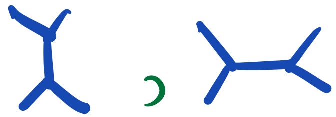
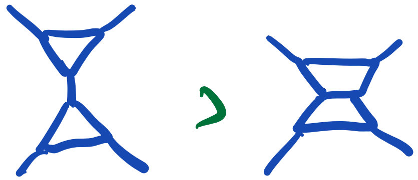


So the 3 closing angles satisfy $u^k_{ij} + u^i_{jk} + u^j_{ki} = 2\pi$

and the on-shell momenta at the vertex can be drawn in the Euclidean plane as



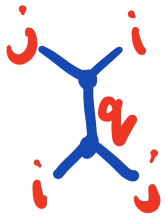
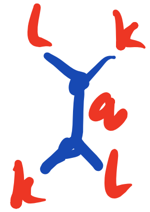
(relative angles fixed rigidly if on-shell & momentum conserving)

The geometry of these vertices, and how they can be plugged together, will determine the singularity structure of the S-matrix both at tree level () and at loop level () where they will signal loop diagrams with <u>Landau singularities</u> (multiple internal propagators on-shell for some particular external momenta)

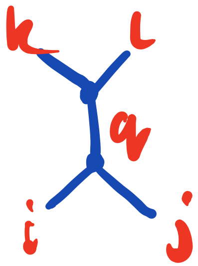
An elementary tree-level observation shows that they are highly constrained by integrability...

3. Flipping diagrams

Suppose $c^{ijq} \neq 0$ and $c^{klq} \neq 0$.

Then we'd expect to see poles in S_{ij}  and in S_{kl} 

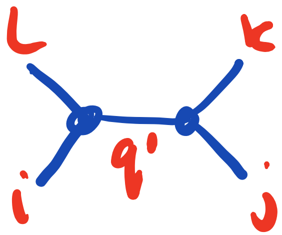
both from q propagating as a bound state.

But perturbation theory then suggests there should also be a pole in S_{ij}^{kl}  which is an amplitude we wanted to vanish.

How can this be?

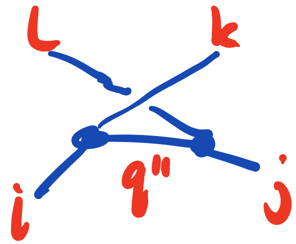
The only answer in perturbation theory is that there must be a competing process to cancel the amplitude, either in the u or t channels:

"u":



or

"t":



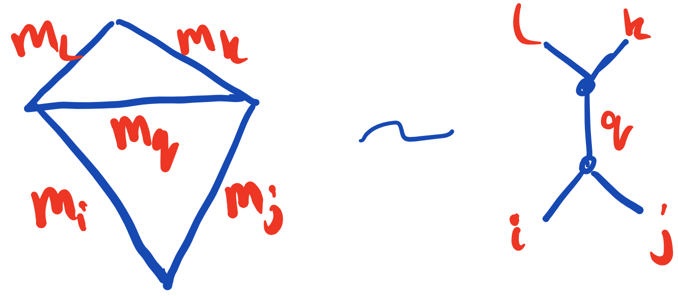
(or both)

Both have opposite residues for their poles, so there's a chance of cancellation.

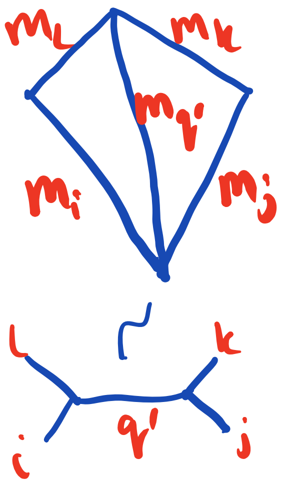
To be onshell at the right point, all external angles must agree, so the fusing angles for q' and q'' must be just right.

In terms of the mass triangles this has a simple 'dual' geometrical interpretation...

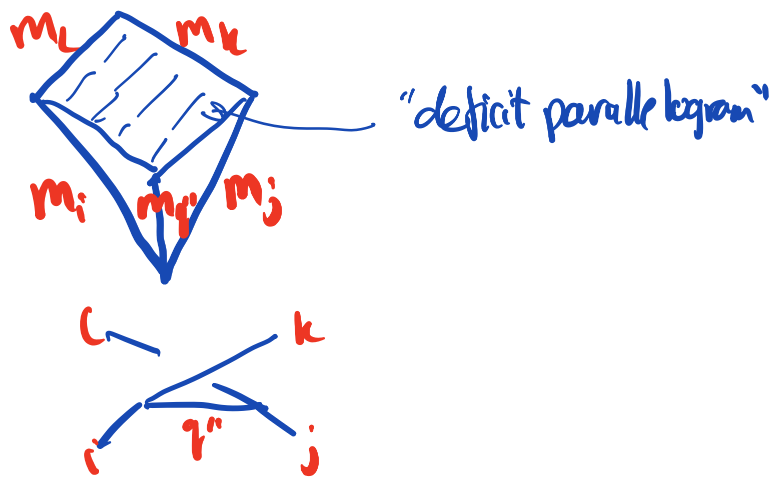
- Whenever a quadrilateral for an unwanted process $ij \to kl$ can be tiled with mass triangles in one way:

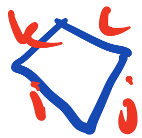


There must exist non-zero couplings and masses in the theory such that it can be tiled in another way, either as



or



We call this the __flipping rule__: the sets of masses $\{m_a\}$ and non-zero 3-pt couplings $\{c^{ijk}\}$ must be such that a whenever a disallowed quadrilateral 

can be tiled with mass triangles in one way, it can also be tiled in at least one other way.

It is not obvious that coherent sets of masses & couplings exist at all!

[while we don't have a complete classification, we were able to prove that, in theories where cancellations are always pairwise, the non-zero couplings must obey the __area rule__: $|c^{ijk}| \propto \triangle_{abc}$, the area of the mass triangle]

4. Affine Toda Field Theories

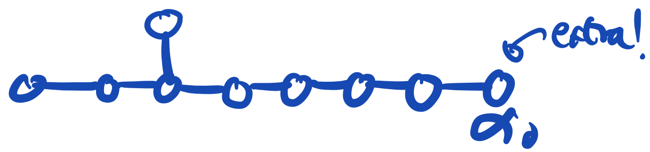
While we lack a complete answer to the classification problem, we do have a (possibly exhaustive) set of examples which solve the constraints in beautiful ways - the Affine Toda Field theories.

- Take an r component scalar field $\phi \in \mathbb{R}^r$ in 2d, & set

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{\beta^2}\sum_{a=0}^{r} n_a e^{\beta\alpha_a\cdot\phi} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

where $\alpha_0 \ldots \alpha_r$ are $r+1$ vectors in $\mathbb{R}^r$, with mutual inner products given by an affine Dynkin diagram (simply-laced) & $\sum_0^r n_a\alpha_a = 0$.

Eg for E_8:



Tactic: expand $V(\phi)$ in β to find mass^2 matrix, 3-pt couplings, etc.

Results
- masses form an eigenvector of the corresponding non-affine Cartan matrix [eg for E_8: $C_{ij} m_j = (2 - 2\cos\frac{\pi}{30}) m_i$]
- At one loop the mass ratios don't renormalise
- 3pt couplings obey the area rule, & their non-vanishing is encoded in a fusing rule based on the geometry of the corresponding root system

This data combined with non-perturbative constraints such as the bootstrap equations is enough to conjecture exact S-matrices for all cases - but these still need to be checked against perturbation theory!

5 Higher order poles

(one set of checks; independently interesting)

The conjectured S-matrices have higher-order poles, up to 12th order in the case of the E_8 theory, with surprisingly simple ('universal') coefficients.

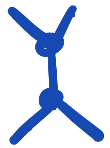
These should all have "prosaic" (in the language of Coleman & Thun) explanations in perturbation theory as anomalous threshold poles, ie the Landau singularities mentioned earlier (in $d>2$ these are branch points; but in 2d they are poles).

General ($d=2$) result:

A graph in which P internal propagators go simultaneously on-shell at some point, with L loops, will have a pole at that point of order $P-2L$.

dual pictures

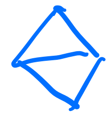
Examples:

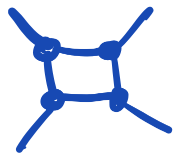


$P=1, L=0$

$P-2L=1$

$\rightarrow$ 1st order

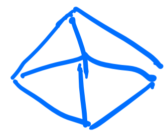


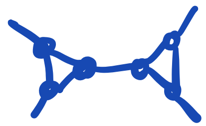


$P=4, L=1$

$P-2L=2$

$\rightarrow$ 2nd order

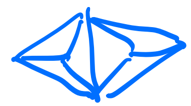


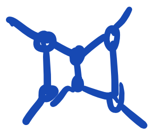


$P=7, L=2$

$P-2L=3$

$\rightarrow$ 3rd order

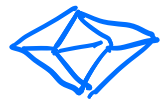




$P=7, L=2$

$P-2L=3$

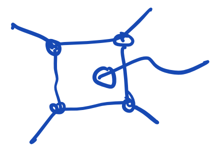
$\rightarrow$ also 3rd order



At each vertex of such an on-shell diagram, all angles are fixed to the fusing angles. So finding them is a "lego" problem of gluing together rigid vertices, or, in the dual picture of mass triangles, a geometrical <u>tiling problem</u>.

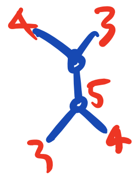
Example: (E_8) 8 particles masses $m_1 < \dots < m_8$

- $S_{11}(\theta)$ can't have higher poles by elementary geometry



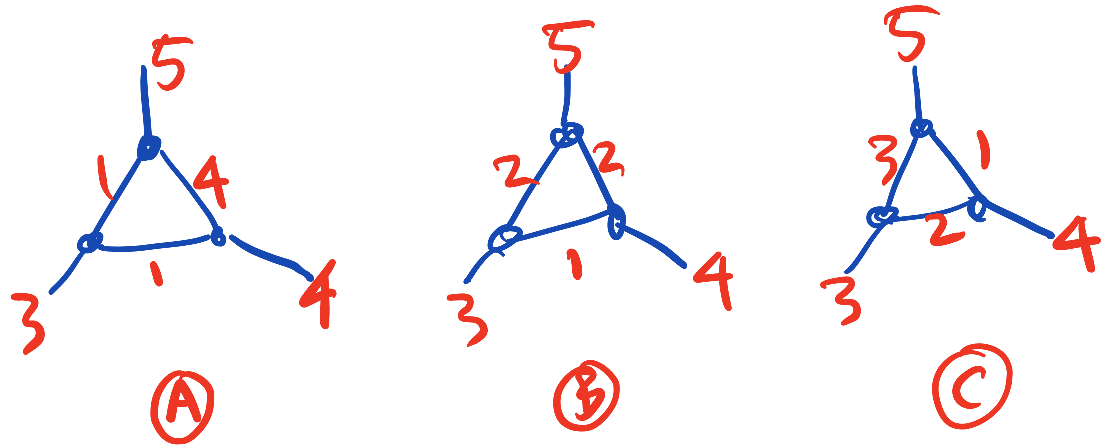
Internal lines must go outside here - can't be done

- But $S_{34}(\theta)$ has a 3^{rd} order pole at $\theta = \frac{8}{15}\pi i$, where naively a simple pole from $34 \to 5$

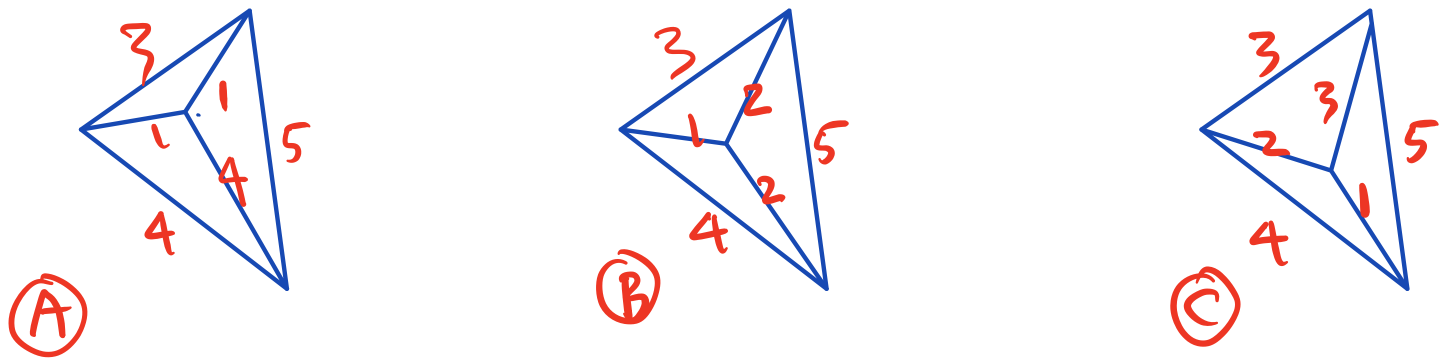


would be expected.

- Look more closely to find that the vertex corrections for C^{345} include 3 graphs with Landau singularities:

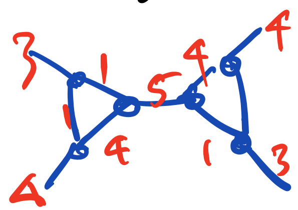


In the dual picture this corresponds to 3 different tilings of the 345 mass triangle:

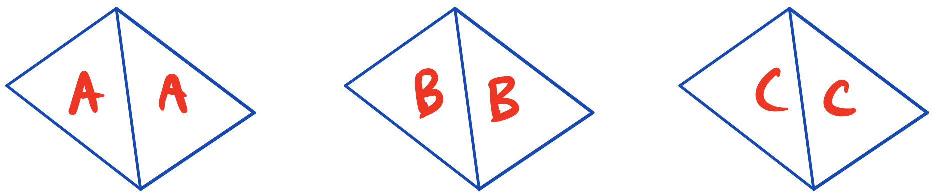


(not to scale!)

The geometrical constraint that the vertices and momenta in the on-shell diagrams fit together is equivalent to the 3 different sets of mass triangles all tiling the big triangle perfectly.

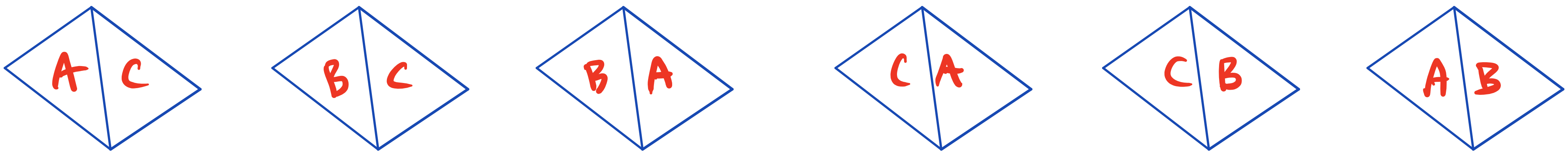
Joining these together gives 9 diagrams all contributing to the 3rd order pole,  etc.

We have



"diagonal"

and also

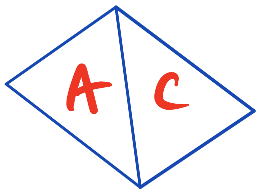


"off diagonal"

... adding these together gives a residue with the same absolute value as for the exact S-matrix (to leading β^6 order) but the <u>wrong sign</u>.

Something is missing, which can be found via the flipping idea.

Consider



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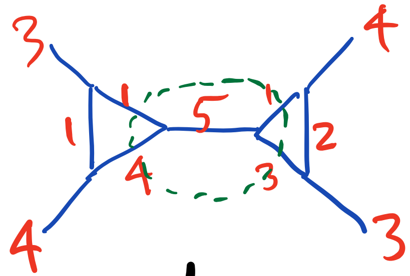
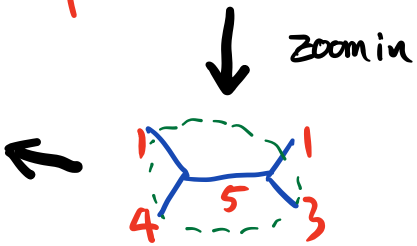
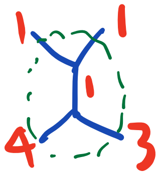
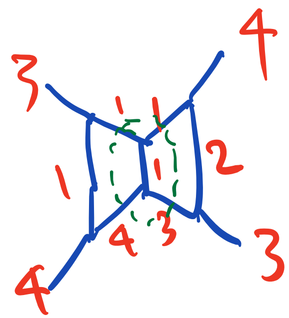




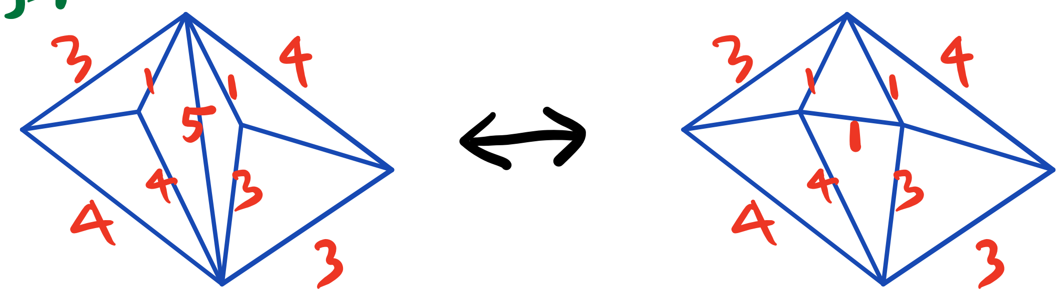
This is a disallowed tree-level process so it must have a flipped "friend", which turns out to be



So we can do a "surgery" on the initial diagram to find a new onshell diagram which must also contribute:



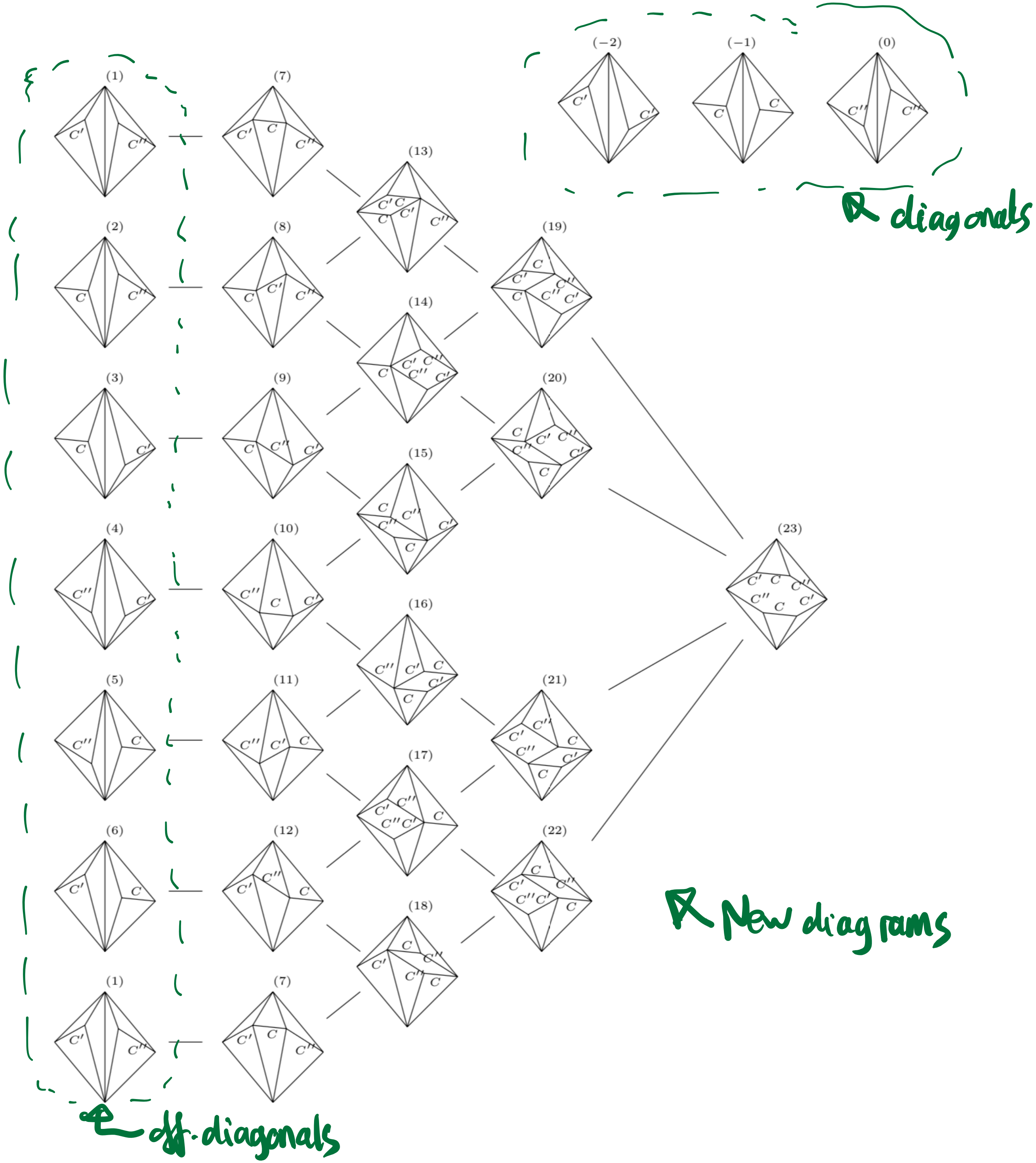
In terms of tilings:



Now repeat!

This generates 17 new diagrams which sum to exactly (-2)x the wrong-sign total of the first set

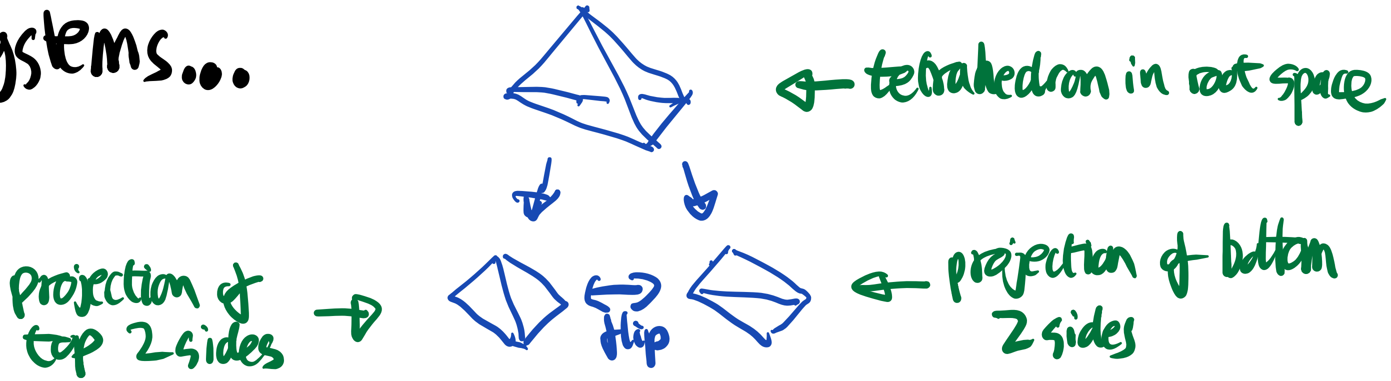
– so all is well.



This turns out to be generic for all ATFT 3rd order poles, and can be better understood via a cutting procedure on the diagrams (to appear; and for discussions of 2nd order poles, see arXiv.2022.tomorrow)

At higher order there are many (millions) more tilings but the final answers should still be simple (work in progress)

... and behind all this is the higher-dimensional geometry of root systems...



Thank you!