Waveforms from Amplitudes

David A. Kosower
Institut de Physique Théorique, CEA–Saclay

Work with **Andrea Cristofoli** (Edinburgh), **Riccardo Gonzo** (Trinity), & Donal O'Connell [2107.10193]

Also based on earlier work with **Ben Maybee** (Imperial) & Donal O'Connell (Edinburgh) [1811.10950]

SAGEX Closing Meeting
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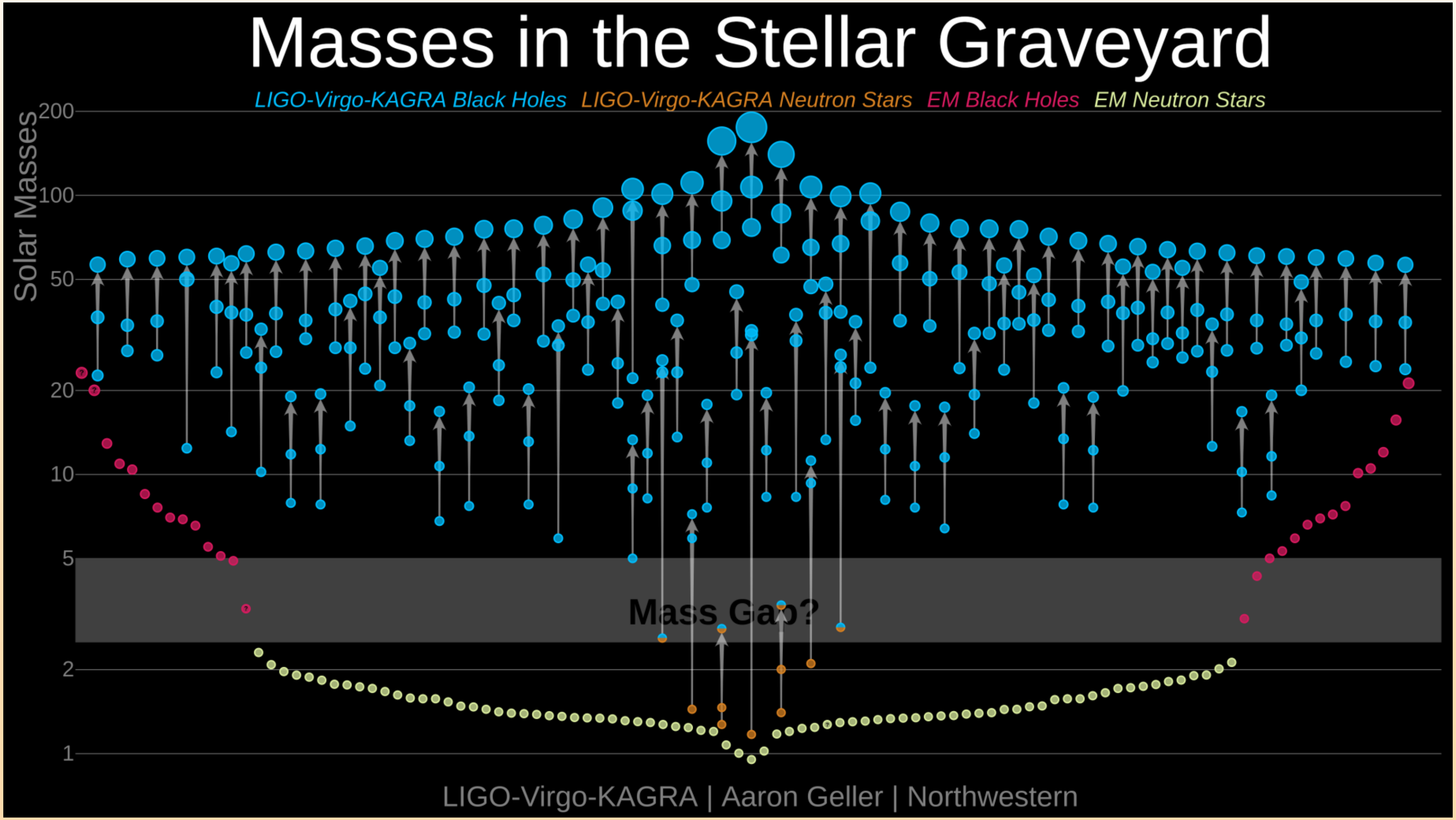
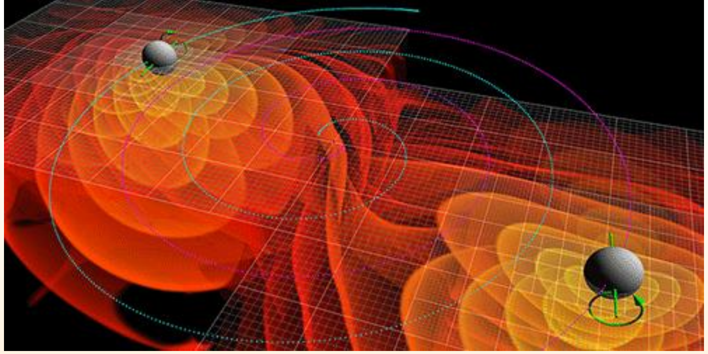
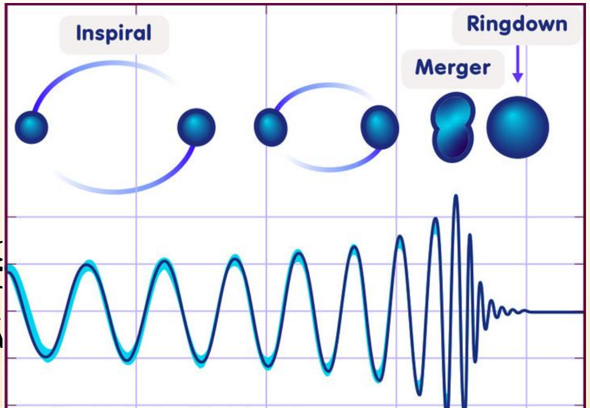
Scattering Amplitudes: from Geometry to Experiment


Masses in the Stellar Graveyard
LIGO-Virgo-KAGRA Black Holes
LIGO-Virgo-KAGRA Neutron Stars
EM Black Holes
EM Neutron Stars
Solar Masses
200
100
50
20
10
5
2
1
Mass Gap?
LIGO-Virgo-KAGRA | Aaron Geller | Northwestern



Goals



LIGO/Daniela Leitner/Einstein-Online

- Enhance detection and analysis of sig
and future gravitational-wave observ

- Compute waveforms for gravitational waves from binary inspirals (black holes, neutron stars, white dwarves)

Bound states

Not today

Today

- Waveforms in unbound scattering
  - Also possibly of observational interest
  - Black-hole clusters
  - Scattering events suck energy out of binary systems & accelerate decay

March of the Bs

- Related talks by

Brandhuber

Britto

Bern

Bjerrum-Bohr

to break the pattern

Plefka

Observables-Based Formalism

- Pick well-defined observables in the quantum theory that are also relevant classically

- Express them in terms of scattering amplitudes in the quantum theory
  - Amplitudes are our friends
  - But they are not directly observable

- Understand how to take the classical limit efficiently

Observables-Based Formalism

- Connection to LVK analysis pipeline is further off
- No arbitrary divisions between conservative and radiation-reaction contributions
  - No need to worry about tails, or tails of tails, or hair on tails

Classical Physics

- Classical limit requires $\hbar \to 0$: restore $\hbar$ via dimensional analysis (keep everything relativistic, $c = 1$)

$$[M] \neq [L]^{-1}$$

$$[|p\rangle] = [M]^{-1}$$

$$[\mathrm{Ampl}_n] = [M]^{4-n}$$

- Set up initial state
- Two sources of $\hbar$
  - Couplings: $e \to e/\sqrt{\hbar}$; $\kappa \to \kappa/\sqrt{\hbar}$
  - Messenger wavenumbers: $\bar{\boldsymbol{p}} = \boldsymbol{p}/\hbar$
- Turn the crank $\left\langle\!\!\left\langle \cdots \right\rangle\!\!\right\rangle$
  - Laurent-expand in $\hbar$ where needed
  - In physical observables, singular terms in $\hbar$ will cancel
  - Integrate over phase space

Classical Limit

- If we just consider $\hbar$s from the first source
  - Net: n-point, L-loop amplitude in scalar QED scales as

$$\hbar^{1-\frac{n}{2}-L}$$

  - We would get confused
  - But it's not the whole story, of course

Observables

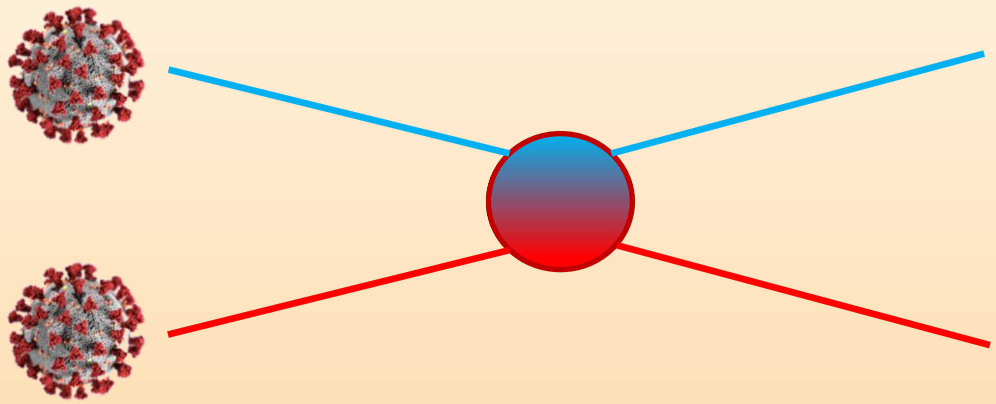
- Change in momentum ('impulse') $\langle \Delta p \rangle$ of a scattered particle
- Radiated momentum $\langle K \rangle$
- Waveform

- Plain perturbative expansion, just in G: relativistic

- Conservative & dissipative (radiation-reaction)
  - Potentials focus on the first
  - Can do both together

Set-up

- Scatter two 'things'



- If they're both massive, look at point particles

Wave Packets

- Point particles: localized positions and momenta
- Wavefunction $\phi(p)$
- Initial state: integral over on-shell phase space

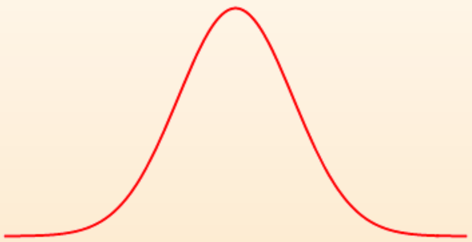
$$|\psi\rangle_{\text{in}} = \int \hat{d}^4 p_1 \hat{d}^4 p_2 \, \hat{\delta}^{(+)}(p_1^2 - m_1^2)\hat{\delta}^{(+)}(p_2^2 - m_2^2) \, \phi(p_1)\phi(p_2) \times e^{ib\cdot p_1/\hbar} \, |p_1 p_2\rangle_{\text{in}}$$

$$= \int d\Phi(p_1) d\Phi(p_2) \, \phi(p_1)\phi(p_2) \; e^{ib\cdot p_1/\hbar} \, |p_1 p_2\rangle_{\text{in}}$$

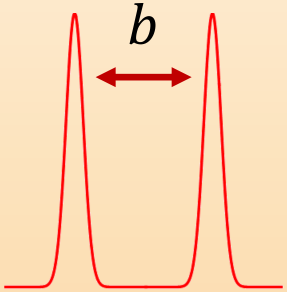
Notation tidies up 2πs

Simple example: $\phi(p) = \exp(-p \cdot \frac{u}{m\xi})$

Classical Limit, part 2



- Three scales
  - ℓ_c: Compton wavelength
  - ℓ_w: wavefunction spread
  - ℓ_s: scattering length ~ impact parameter b
- Particles localized: $\ell_c \ll \ell_w$
- Well-separated wave packets: $\ell_w \ll b$



More careful analysis confirms this ‘Goldilocks’ condition

$$\ell_c \ll \ell_w \ll b$$

Massless Scatterers

- What about massless particles, like photons or gravitons?
- Compton wavelength is infinite: can’t localize them
- But plane waves are still not appropriate
- Solution is to use coherent states

Coherent States

- Introduce the coherent-state operator

$$\mathbb{C}_{\alpha,(\eta)} \equiv \mathcal{N}_\alpha \exp\left[\int d\Phi(k)\, \alpha(k)\, a^\dagger_{(\eta)}(k)\right]$$

Creates a state of indefinite messenger number $|\alpha^\eta\rangle$

Waveshape

- Eigenstate of creation operator

$$a_{(+)}(k)|\alpha^+\rangle = \alpha(k)|\alpha^+\rangle\,,$$

$$a_{(-)}(k)|\alpha^+\rangle = 0\,,$$

$$\langle\alpha^+|a^\dagger_{(+)}(k) = \langle\alpha^+|\alpha^*(k)\,,$$

$$\langle\alpha^+|a^\dagger_{(-)}(k) = 0\,,$$

Connection to Classical Field

- Look at the electromagnetic field operator

$$\mathbb{A}_\mu(x) = \frac{1}{\sqrt{\hbar}} \sum_\eta \int d\Phi(k) \left[a_{(\eta)}(k)\varepsilon_\mu^{(\eta)*}(k)\, e^{-ik\cdot x/\hbar} + a^\dagger_{(\eta)}(k)\varepsilon_\mu^{(\eta)}(k)\, e^{+ik\cdot x/\hbar}\right]$$

- Compute its expectation in the state $|\alpha^+\rangle$

$$\langle\alpha^+|\mathbb{A}_\mu(x)|\alpha^+\rangle = \frac{1}{\sqrt{\hbar}} \int d\Phi(k) \left[\alpha(k)\varepsilon_\mu^{(+)*}(k)e^{-ik\cdot x/\hbar} + \alpha^*(k)\varepsilon_\mu^{(+)}(k)e^{+ik\cdot x/\hbar}\right]$$

$$= \int d\Phi(\bar{k}) \left[\bar{\alpha}(\bar{k})\varepsilon_\mu^{(+)*}(\bar{k})e^{-i\bar{k}\cdot x} + \bar{\alpha}^*(\bar{k})\varepsilon_\mu^{(+)}(\bar{k})e^{+i\bar{k}\cdot x}\right] \equiv A_{\mathrm{cl}\,\mu}(x)\,,$$

Fourier coefficients

- So long as we set $\bar{\alpha}(\bar{k}) = \hbar^{3/2}\alpha(k)$

Occupation Number

- Number of photons

$$N_\gamma = \langle \alpha^+ | \sum_\eta \int d\Phi(k)\, a^\dagger_{(\eta)}(k) a_{(\eta)}(k) | \alpha^+ \rangle$$
$$= \frac{1}{\hbar} \int d\Phi(\bar{k}) |\bar{\alpha}(\bar{k})|^2$$

- Large as required when $\hbar \to 0$ so long as $\bar{\alpha}$ is not parametrically small

- Waveshape $\bar{\alpha}(\bar{k})$ chosen to give form of classical wave

Light Deflection

- Initial state for massive–massless scattering (point particle–classical wave)

$$|\psi_w\rangle_{\text{in}} = \int d\Phi(p_1)\, \phi_1(p_1)\, e^{ib\cdot p_1/\hbar}|p_1\, \alpha_2^\eta\rangle_{\text{in}}$$

- Compute impulse

$$\langle \Delta p_1^\mu \rangle = \langle \psi_w|\, i[\mathbb{P}_1^\mu, T]\, |\psi_w\rangle + \langle \psi_w|\, T^\dagger [\mathbb{P}_1^\mu, T]\, |\psi_w\rangle$$

- At lowest order, need just the first term

$$\int d\Phi(p_1) d\Phi(p_1')\, e^{-ib\cdot(p_1'-p_1)/\hbar} \phi_1(p_1)\phi_1^*(p_1')\, i(p_1'-p_1)^\mu \langle p_1'\, \alpha_2^\eta|T|p_1\, \alpha_2^\eta\rangle$$

Evaluation

- Matrix elements of coherent states are not of definite order in perturbation theory
- Would ordinarily introduce complete sets of states of definite particle number on each side of T
- Sum is complicated because we need to sum over disconnected pieces for most messengers
- Instead introduce representation (Weinberg) of T matrix in terms of creation and annihilation operators

Evaluation

$$T = \sum_{\tilde\eta,\tilde\eta'} \int d\Phi(\tilde r_1, \tilde r_1', \tilde k_2, \tilde k_2')\, \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \rangle\; a^\dagger_{(\tilde\eta')}(\tilde k_2') a^\dagger(\tilde r_1')\, a(\tilde r_1) a_{(\tilde\eta)}(\tilde k_2) + \cdots$$

- Required matrix element is then

$$\begin{aligned}\langle p_1'\, \alpha_2^\eta | T | p_1\, \alpha_2^\eta \rangle &= \hat\delta_\Phi(\tilde r_1 - p_1)\, \hat\delta_\Phi(\tilde r_1' - p_1')\, \delta_{\tilde\eta,\eta} \delta_{\tilde\eta',\eta} \alpha_2(\tilde k_2) \alpha_2^*(\tilde k_2')\, \langle \tilde r_1' \tilde k_2'^{\tilde\eta'} | T | \tilde r_1 \tilde k_2^{\tilde\eta} \rangle \\ &= \hat\delta_\Phi(\tilde r_1 - p_1)\, \hat\delta_\Phi(\tilde r_1' - p_1')\, \delta_{\tilde\eta,\eta} \delta_{\tilde\eta',\eta} \alpha_2(\tilde k_2) \alpha_2^*(\tilde k_2') \\ &\quad \times \mathcal{A}(\tilde r_1\, \tilde k_2^{\tilde\eta} \to \tilde r_1'\, k_2'^{\tilde\eta'})\, \hat\delta^4(\tilde r_1 + \tilde k_2 - \tilde r_1' - \tilde k_2')\end{aligned}$$

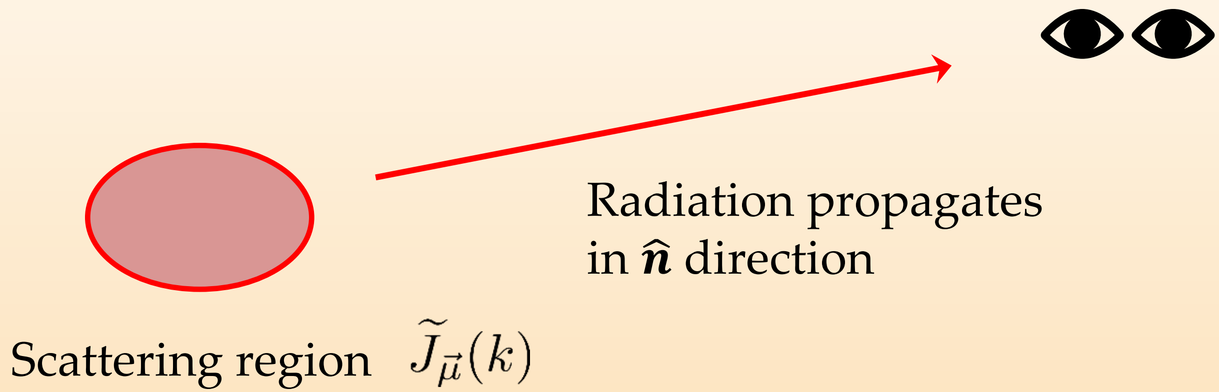
- Gravitational scattering of photon off neutral massive scalar
- Within classical regime, choose geometric optics with collimated beam

$$\ell_c \ll \lambda \ll \ell_\perp \ll \ell_s \sim b\,.$$

- Reproduce well-known value

$$\theta = \frac{4\, G_N m}{|\boldsymbol{b}|}$$

Point-Like Observables



- Local radiation observable

$$R_{\vec{\mu}}(x) = i \int d\Phi(\bar{k}) \left[\widetilde{J}_{\vec{\mu}}(\bar{k}) e^{-i\bar{k}\cdot x} - \widetilde{J}^{*}_{\vec{\mu}}(-\bar{k}) e^{+i\bar{k}\cdot x}\right]$$

- Waveform is leading large-distance behavior

$$R_{\vec{\mu}}(x) = \frac{1}{|\boldsymbol{x}|} W_{\vec{\mu}}(t, \hat{\boldsymbol{n}}; x)$$

Waveforms

- Measure electromagnetic field in massive–massive scattering

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle \equiv {}_{\text{out}}\langle\psi|\mathbb{F}_{\mu\nu}(x)|\psi\rangle_{\text{out}}$$

- Rewrite it in terms of the incoming state

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle = {}_{\text{in}}\langle\psi|S^\dagger\mathbb{F}_{\mu\nu}(x)S|\psi\rangle_{\text{in}}$$

- Fits into our general form with current

$$\widetilde{J}_{\mu\nu}(\bar{k}) = -2\hbar^{3/2}\sum_\eta\langle\psi|S^\dagger a_{(\eta)}(k)S|\psi\rangle\,\bar{k}_{[\mu}\varepsilon^{(\eta)*}_{\nu]}(\bar{k})$$

- In spinorial form, as a Newman–Penrose scalar

$$\Phi^0_2(t,\hat{\mathbf{n}}) = -\frac{\hbar^{3/2}}{4\pi}\int\hat{d}\omega\,\Theta(\omega)\,\omega\big[e^{-i\omega t}\langle\psi|S^\dagger a_{(-)}(\bar{k})S|\psi\rangle + e^{+i\omega t}\langle\psi|S^\dagger a^\dagger_{(+)}(-\bar{k})S|\psi\rangle\big]\Big|_{\bar{k}=(\omega,\omega\hat{\mathbf{n}})}$$

Example: LO EM Waveform

- Rewrite *S* matrix

$$\langle F^{\text{out}}_{\mu\nu}(x)\rangle = 2\,\text{Re}\, i\langle\psi|\mathbb{F}_{\mu\nu}(x)T|\psi\rangle + \langle\psi|T^\dagger\mathbb{F}_{\mu\nu}(x)T|\psi\rangle\,.$$

- At LO, only the first term contributes

$$\frac{4}{\hbar^{3/2}}\,\text{Re}\sum_\eta\int d\Phi(p_1)d\Phi(p_2)d\Phi(p_1')d\Phi(p_2')d\Phi(k)\; e^{-ib\cdot(p_1'-p_1)/\hbar} \times \phi(p_1)\phi^*(p_1')\phi(p_2)\phi^*(p_2')k^{[\mu}\varepsilon^{(\eta)\nu]*}e^{-ik\cdot x/\hbar}\langle p_1'\,p_2'|a_{(\eta)}(k)\,T|p_1\,p_2\rangle$$

- The matrix element is a five-point amplitude

$$\begin{aligned}\langle p_1'\,p_2'|a_{(\eta)}(k)\,T|p_1\,p_2\rangle &= \langle p_1'\,p_2'\,k^\eta|T|p_1\,p_2\rangle \\ &= \mathcal{A}(p_1,p_2\to p_1',p_2',k^\eta)\hat{\delta}^4(p_1+p_2-p_1'-p_2'-k)\end{aligned}$$

EM Waveform

- We find the same radiation kernel as in the total radiated momentum

$$\mathcal{R}^{(0)}(\bar{k}^\eta; b) \equiv \hbar^2 \prod_{i=1,2} \int \hat{d}^4\bar{q}_i\ \hat{\delta}(p_i \cdot \bar{q}_i)\ e^{-ib\cdot\bar{q}_1}\hat{\delta}^4(\bar{q}_1 + \bar{q}_2 + \bar{k})$$

$$\times \bar{\mathcal{A}}(p_1, p_2 \to p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2, \hbar\bar{k}^\eta)$$

- The usual classical limit + long-distance expansion gives

$$\langle F^{\mu\nu}(x)\rangle_{1,\text{cl}} = g^3 \left\langle\!\!\left\langle \text{Re} \sum_\eta \int d\Phi(\bar{k}) \bar{k}^{[\mu} \varepsilon^{(\eta)\nu]*} e^{-i\bar{k}\cdot x} \mathcal{R}^{(0)}(\bar{k}^\eta; b) \right\rangle\!\!\right\rangle$$

- Or directly for the spectral Newman–Penrose scalar

$$\tilde{\Phi}_2^0(\omega, \hat{\mathbf{n}}) = -\frac{ig^3\omega}{16\pi} \left\langle\!\!\left\langle \Theta(\omega)\mathcal{R}^{(0)}(\omega(1,\hat{\mathbf{n}})^-; b) + \Theta(-\omega)\mathcal{R}^{(0)*}(-\omega(1,\hat{\mathbf{n}})^+; b) \right\rangle\!\!\right\rangle.$$

Building blocks

- Bessel functions in frequency domain

$$I_1 = -\frac{1}{2\pi\sqrt{\gamma^2-1}} K_0\big(\sqrt{-b^2}\, u_2\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\,,$$

$$I_{4'} = \frac{1}{2\pi\,(\gamma^2-1)} K_1\big(\sqrt{-b^2}\, u_1\cdot\bar{k}/\sqrt{\gamma^2-1}\big)\,.$$

- Functions like

$$\Xi^{\zeta}_{ia}(t,\hat{\mathbf{n}};\mathbf{v}) = \frac{\sqrt{\gamma^2-1}}{\rho_1(t)} - \zeta\frac{(\gamma^2-1)(t+\mathbf{v}\cdot\hat{\mathbf{n}})}{\rho_1^{3/2}(t)}\,\mathrm{arcsinh}\left(\frac{\sqrt{\gamma^2-1}}{\sqrt{-b^2}u_{1,\hat{\mathbf{n}}}}(t+\mathbf{v}\cdot\hat{\mathbf{n}})\right)$$
$$- \frac{i\pi}{2}\frac{(\gamma^2-1)(t+\mathbf{v}\cdot\hat{\mathbf{n}})}{\rho_1^{3/2}(t)}$$

in time domain

Start of a new tower of ‘interesting functions’

- In terms of these building blocks,

$$\Phi_2^0(t,\hat{\mathbf{n}}) =$$

$$-\frac{ig^3 Q_1^2 Q_2}{(4\pi)^3\sqrt{2}m_1\,u_{1,\hat{\mathbf{n}}}}\Big[\langle\hat{n}|\,u_2\,u_1\,|\hat{n}\rangle\;\Xi_{1a}^{+}(t,\hat{\mathbf{n}};\mathbf{b}) - [\hat{n}|\,u_2\,u_1\,|\hat{n}]\;\Xi_{1a}^{-}(t,\hat{\mathbf{n}};\mathbf{b})$$

$$+\,i\big(\langle\hat{n}|\,b\,u_1\,|\hat{n}\rangle - [\hat{n}|\,b\,u_1\,|\hat{n}]\big)\,\Xi_{1b}(t,\hat{\mathbf{n}};\mathbf{b})\Big]$$

$$-\frac{ig^3 Q_1 Q_2^2}{(4\pi)^3\sqrt{2}m_2\,u_{2,\hat{\mathbf{n}}}}\Big[\langle\hat{n}|\,u_1\,u_2\,|\hat{n}\rangle\;\Xi_{2a}^{+}(t,\hat{\mathbf{n}};\mathbf{0}) - [\hat{n}|\,u_1\,u_2\,|\hat{n}]\;\Xi_{2a}^{-}(t,\hat{\mathbf{n}};\mathbf{0})$$

$$+\,i\big(\langle\hat{n}|\,b\,u_2\,|\hat{n}\rangle - [\hat{n}|\,b\,u_2\,|\hat{n}]\big)\,\Xi_{2b}(t,\hat{\mathbf{n}};\mathbf{0})\Big]$$

Summary

- Observables-based formalism for computing classical physics via scattering amplitudes
  - Observables valid in both quantum and classical theories
  - Simple limit
  - $\hbar$s from dimensional analysis
  - Momenta for massive particles, wavenumbers for massless

- Classical waves correspond to coherent states of massless particles

- Waveform for radiation *is* the five-point amplitude