Construction of a third order approximation for heavy flavour production in deep inelastic scattering

Master Thesis in Physics at the University of Rome "La Sapienza"

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- In the parton model the hadronic structure functions for deep inelastic scattering (DIS) are computed as

$$F_a(x, Q^2) = x \sum_{i=q,\bar{q},g} \int_x^1 dz\, f_i(z, \mu_F^2) C_{a,i}\left(\frac{x}{z}, Q^2, \mu_R^2, \mu_F^2, \alpha_s(\mu_R^2)\right)$$

  with $a = 2, L$
- f_i are the parton distribution functions (PDF).
- $C_{a,i}$ are the coefficient functions (partonic structure functions for DIS) and are computed as an expansion in α_s:

$$C_{a,i} = C_{a,i}^{(0)} + \alpha_s C_{a,i}^{(1)} + \alpha_s^2 C_{a,i}^{(2)} + \alpha_s^3 C_{a,i}^{(3)} + \dots$$

Introduction

- Massless coefficient functions: known exactly up to $\mathcal{O}(\alpha_s^3)$

$$C_{a,i}^{\text{light}}(z) = C_{a,i}^{\text{light}\,(0)}(z)+\alpha_s C_{a,i}^{\text{light}\,(1)}(z)+\alpha_s^2 C_{a,i}^{\text{light}\,(2)}(z)+\alpha_s^3 C_{a,i}^{\text{light}\,(3)}(z)+\dots$$

$a = 2, L$ and $i = q, \bar{q}, g$

- Massive coefficient functions: known exactly up to $\mathcal{O}(\alpha_s^2)$

$$C_{a,i}(z) = \alpha_s C_{a,i}^{(1)}(z) + \alpha_s^2 C_{a,i}^{(2)}(z) + \alpha_s^3 C_{a,i}^{(3)}(z) + \dots$$

$C_{a,i}^{(3)}(z)$ is not fully known yet.

The subject of the thesis has been the construction of an approximation for the gluon coefficient function in heavy quark production for F_2 at $\mathcal{O}(\alpha_s^3)$.

- $C_{a,i}^{(3)}(z, Q^2, \mu^2)$ can be decomposed as (where $\mu \equiv \mu_F = \mu_R$)

$$C_{a,i}^{(3)}(z, Q^2, \mu^2) = C_{a,i}^{(3,0)}(z, Q^2) + C_{a,i}^{(3,1)}(z, Q^2) \log \frac{\mu^2}{m^2} + C_{a,i}^{(3,2)}(z, Q^2) \log^2 \frac{\mu^2}{m^2}$$

- The μ-dependent part is exactly known
- The only unknown part is the μ-independent one

Therefore, we constructed an approximation for the μ-independent part and then we reinserted the exact μ-dependent one:

$$C_{a,i}^{\mathrm{approx}\,(3)}(z, Q^2, \mu^2) = C_{a,i}^{\mathrm{approx}\,(3,0)}(z, Q^2) + C_{a,i}^{(3,1)}(z, Q^2) \log \frac{\mu^2}{m^2} + C_{a,i}^{(3,2)}(z, Q^2) \log^2 \frac{\mu^2}{m^2}$$

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Strategy

- We started from the results of [Kawamura, Lo Presti, Moch, Vogt: arXiv:1205.5727]
- It uses the fact that even if $C_{2,g}^{(3,0)}$ is not known, its limits in three kinematic regions are known. Such regions are:

  - High-scale limit ($Q^2 \gg m^2$)
  - High-energy limit ($s \to \infty$, where s is the partonic center-of-mass energy and is given by $s = Q^2(1/z - 1)$)
  - Threshold limit ($s \to s_{\min} = 4m^2$)

- Combining these functions we can obtain a function that approaches the exact coefficient function in these regions, and that is an interpolation of the three functions in the intermediate regions.

High-scale limit

- $Q^2 \gg m^2$ limit.
- Neglects power terms $\left(\frac{m^2}{Q^2}\right)^k$ but keeps logarithmic terms $\log^h\left(\frac{m^2}{Q^2}\right)$.
- Known exactly up to $\mathcal{O}(\alpha_s^2)$, while at $\mathcal{O}(\alpha_s^3)$ some terms are known only in approximate form.
- Doesn't approximate the exact function for $Q^2 \sim m^2$.
- Cannot describe the large-z region for any Q^2 since in this region the mass effects are not negligible.

High-scale limit

$$C_{2,g}\Big|_{Q^2 \gg m^2} = \sum_{j=q,\bar{q},g,c,\bar{c}} C_{2,j}^{\text{light}} \otimes K_{jg}$$

High-energy limit

- Limit for $s \to \infty$ with $s = Q^2(1/z - 1)$.
- Therefore, it is the limit for $z \to 0$ with Q^2 fixed.
- It is constructed as an expansion for small-z:

High-energy limit

$$
\begin{aligned}
zC^{(1)}_{2,g} &= 0 \\
zC^{(2)}_{2,g} &= a_2 \log^0 z \\
zC^{(3)}_{2,g} &= a_3 \log z + b_3 \log^0 z \\
zC^{(4)}_{2,g} &= a_4 \log^2 z + b_4 \log z + c_4 \log^0 z
\end{aligned}
$$

LL *NLL* *NNLL*

- Known exactly up to leading logarithm (LL), while at next-to-leading logarithm (NLL) is known only in an approximate form.

Threshold limit

- Limit for $s \to 4m^2$, i.e. for $z \to z_{\max} = \frac{1}{1+4m^2/Q^2}$
- It gives a good approximation of the exact coefficient function in the threshold region, for all the values of Q^2.
- Known exactly up to $\mathcal{O}(\alpha_s^2)$, while at $\mathcal{O}(\alpha_s^3)$ some terms are known only in approximate form.
- It is constructed in such a way that it goes to zero for $z \to 0$.

Missing terms at $\mathcal{O}(\alpha_s^3)$

- We have said that the three limits are known exactly up to $\mathcal{O}(\alpha_s^2)$
- In fact, some terms needed for the construction of these functions at $\mathcal{O}(\alpha_s^3)$ are known only in an approximate form:
  - $C_{2,g}\big|_{Q^2 \gg m^2} = \sum_{j=q,\bar{q},g,c,\bar{c}} C_{2,j}^{\text{light}} \otimes K_{jg}$. Where $K_{Qg}^{(3)} = k_{Qg}^{(3)} + (\mu\text{-dependent part})$
  - $zC_{2,g}^{(3,0)}\big|_{s\to\infty} = a_3 \log z + b_3 \log^0 z$
  - $C_{2,g}^{(3,0)}\big|_{s\to 4m^2} = C_{2,g}^{(3,0)}\big|_{s\to 4m^2}^{\text{const}} + (\beta\text{-dependent part})$
- The approximate expressions of $k_{Qg}^{(3)}$ and $C_{2,g}^{(3,0)}\big|_{s\to 4m^2}^{\text{const}}$ have been taken from [Alekhin, Blümlein, Moch, Placakyte: arXiv:1701.05838] and [Kawamura, Lo Presti, Moch, Vogt: arXiv:1205.5727]
- The approximation for b_3 has been computed from small-x resummation.

Combination of the limits

- In [Kawamura, Lo Presti, Moch, Vogt: arXiv:1205.5727] the three limits are combined in the following way:

Approximation from [Kawamura, Lo Presti, Moch, Vogt: arXiv:1205.5727]

$$C_{2,g}^{(3,0)\mathrm{approx}} =$$
$$C_{2,g}^{(3,0)}\Big|_{s\to 4m^2} f_1\left(z, \frac{Q^2}{m^2}\right) + C_{2,g}^{(3,0)}\Big|_{s\to\infty} f_2\left(z, \frac{Q^2}{m^2}\right) + C_{2,g}^{(3,0)}\Big|_{Q^2\gg m^2} f_3\left(z, \frac{Q^2}{m^2}\right)$$

- The three limits are treated as independent contributions, each one with its own damping function.
- The form of the damping functions has been extracted applying the approximation to the (known) NNLO and comparing the approximate and the exact results.

Combination of the limits

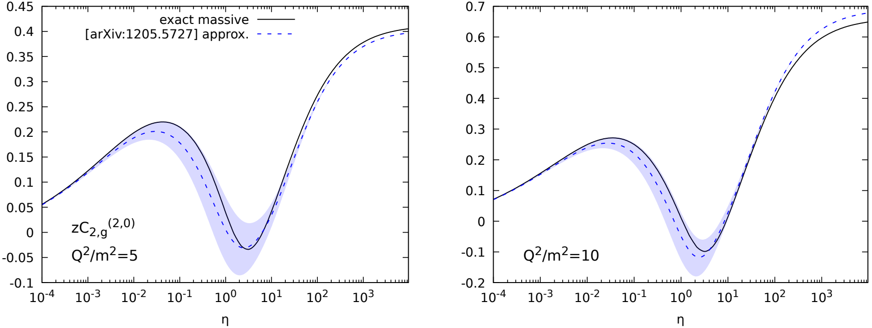
- The approximation proposed in [Kawamura, Lo Presti, Moch, Vogt: arXiv:1205.5727] is the central value of the band given by the two extremes:

$$
\begin{aligned}
C_{2,g}^{(3,0)\mathrm{approx,A}} &= C_{2,g}^{(3,0)}\Big|'_{s\to 4m^2} + \left(1 - f(Q^2/m^2)\right)\beta\, C_{2,g}^{(3,0),\mathrm{A}}\Big|_{Q^2\gg m^2} \\
&\quad + f(Q^2/m^2)\beta^3\left[-C_{2,g}^{(3,0)\mathrm{LL}}\frac{\log\eta}{\log z} + C_{2,g}^{(3,0)\mathrm{NLL,A}}\frac{\eta^\gamma}{C+\eta^\gamma}\right] \\
C_{2,g}^{(3,0)\mathrm{approx,B}} &= C_{2,g}^{(3,0)}\Big|'_{s\to 4m^2} - 2f(Q^2/m^2)\, C_{2,g}^{(3,0)}\Big|^{\mathrm{const}}_{s\to 4m^2} \\
&\quad + \left(1 - f(Q^2/m^2)\right)\beta^3\, C_{2,g}^{(3,0),\mathrm{B}}\Big|_{Q^2\gg m^2} \\
&\quad + f(Q^2/m^2)\beta^3\left[-C_{2,g}^{(3,0)\mathrm{LL}}\frac{\log\eta}{\log z} + C_{2,g}^{(3,0)\mathrm{NLL,B}}\frac{\eta^\delta}{D+\eta^\delta}\right]
\end{aligned}
$$

Combination of the limits

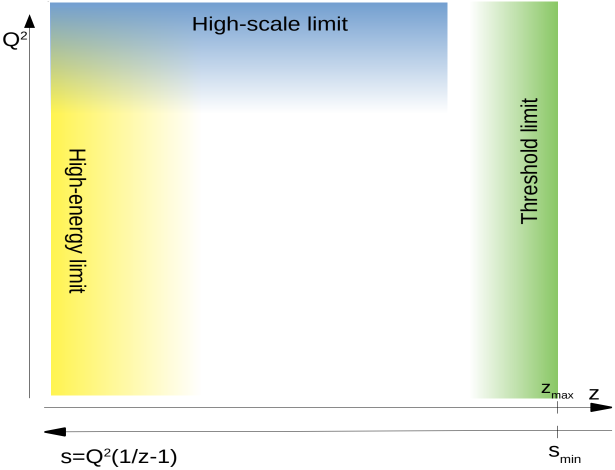
- $C_{2,g}^{(3,0)}\Big|_{s\to 4m^2}^{\mathrm{const}}$ is the approximation of the constant term of the threshold limit at $\mathrm{N^3LO}$.
- $C_{2,g}^{(3,0)}\Big|_{s\to 4m^2}^{\prime}$ is the threshold limit defined without $C_{2,g}^{(3,0)}\Big|_{s\to 4m^2}^{\mathrm{const}}$.
- $C_{2,g}^{(3,0),\mathrm{A}}\Big|_{Q^2\gg m^2}$ and $C_{2,g}^{(3,0),\mathrm{B}}\Big|_{Q^2\gg m^2}$ are the two extremes of the uncertainty band of the $\mathrm{N^3LO}$ high-scale limit.
- $C_{2,g}^{(3,0)\mathrm{LL}}$ is the known LL expansion for small-z, while $C_{2,g}^{(3,0)\mathrm{NLL,A}}$ and $C_{2,g}^{(3,0)\mathrm{NLL,B}}$ are the two extremes of the uncertainty band of an approximation of the $\mathrm{N^3LO}$ NLL expansion for small-z (that is different from our approximation of the NLL term).
- $f\left(\frac{Q^2}{m^2}\right) = \frac{1}{1+\exp\left(2(Q^2/m^2-4)\right)}$ and β is the velocity of the final heavy quarks.
- $\gamma = 1.0,\ C = 20.0,\ \delta = 0.8$ and $D = 10.7$.

Combination of the limits: Results at NNLO



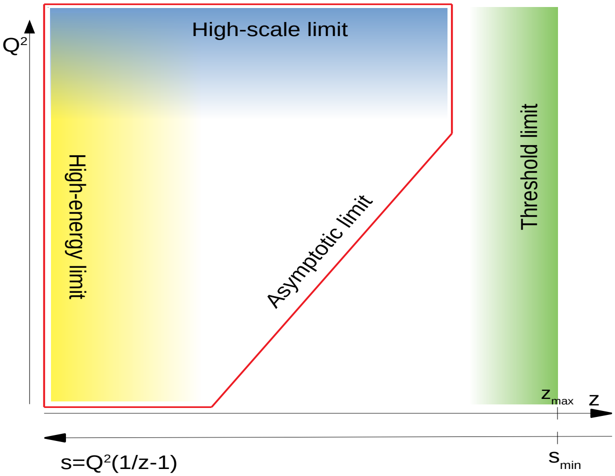
with $\eta = \frac{s}{4m^2} - 1 = \frac{Q^2}{4m^2}\left(\frac{1}{z} - 1\right) - 1$

- However, the high-scale and the high-energy cannot be treated separately since they overlap in a certain region.



Combination of the limits

- For this reason, we slightly modified the way in which the limits are combined, defining the asymptotic limit.



Asymptotic limit

Combining the high-scale and the high-energy limits we constructed the asymptotic limit.

- It is constructed reinserting in the high-scale limit the $z \to 0$ limit of the power terms.

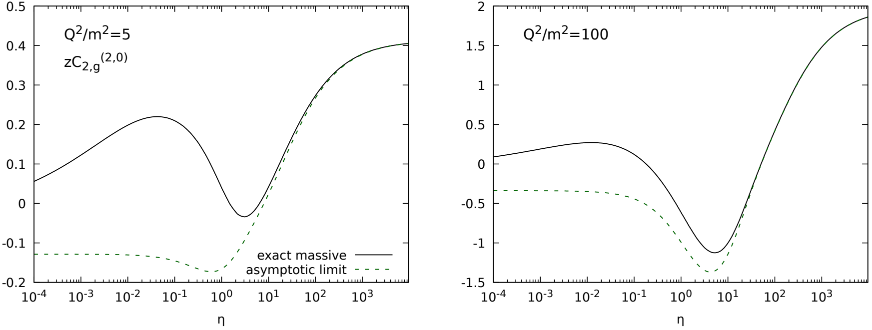
Power terms for $z \to 0$

$$\Delta C_{2,g}^{(3,0)}\Big|_{s\to\infty} = C_{2,g}^{(3,0)}\Big|_{s\to\infty} - C_{2,g}^{(3,0)}\Big|_{\substack{s\to\infty\\ Q^2\gg m^2}}$$

Asymptotic limit

$$C_{2,g}^{(3,0)}\Big|_{\rm asymp} = C_{2,g}^{(3,0)}\Big|_{Q^2\gg m^2} + \Delta C_{2,g}^{(3,0)}\Big|_{s\to\infty}$$

Asymptotic limit: NNLO



- It approaches the exact function for $\eta \to \infty$ for any Q^2. Therefore, we have corrected the high-η behavior for all the values of Q^2.
- For $Q^2 \gg m^2$ it approaches the exact function in the whole range of η, with the exception of the threshold region.

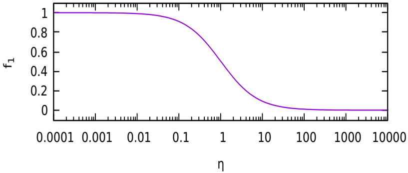
Combination of Asymptotic and Threshold

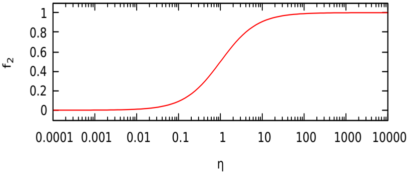
Approximation that we propose

$$C_{2,g}^{(3,0)\mathrm{approx}} = C_{2,g}^{(3,0)}\Big|_{s\to 4m^2} f_1(\eta) + C_{2,g}^{(3,0)}\Big|_{\mathrm{asymp}} f_2(\eta)$$

with

$$f_1(\eta) \xrightarrow{\eta\to 0} 1, \quad f_1(\eta) \xrightarrow{\eta\to\infty} 0, \qquad f_2(\eta) \xrightarrow{\eta\to 0} 0, \quad f_2(\eta) \xrightarrow{\eta\to\infty} 1$$





- In this way our approximation approaches the exact function in the threshold and asymptotic regions.
- For intermediate values of η the approximation is an interpolation between asymptotic and threshold limits. Its accuracy in this zone will depend on the form of the damping functions.

Damping functions

- We have to choose the functional forms of the functions f_1 and f_2 in order to have the best accuracy in the interpolation zone.

Damping functions

$$f_1(\eta) = \frac{1}{1 + \left(\frac{\eta}{h}\right)^k}$$

$$f_2(\eta) = 1 - f_1(\eta)$$

where $h = h(\frac{Q^2}{m^2})$, $k = k(\frac{Q^2}{m^2})$ and $\eta = \frac{s}{4m^2} - 1 = \frac{Q^2}{4m^2}\left(\frac{1-z}{z}\right) - 1$

- h and k must be functions of Q^2/m^2 because:
  - The exchange between asymptotic and threshold limit becomes more strict as Q^2/m^2 increases $\implies k(\frac{Q^2}{m^2})$ must increase as Q^2/m^2 increases.
  - The center of the interpolation zone moves leftwards as Q^2/m^2 increases $\implies h(\frac{Q^2}{m^2})$ must decrease as Q^2/m^2 increases.

Damping functions

Forms of $h\left(\frac{Q^2}{m^2}\right)$ and $k\left(\frac{Q^2}{m^2}\right)$

$$h\left(\frac{Q^2}{m^2}\right) = A + \frac{B - A}{1 + \exp\left(a(\log\frac{Q^2}{m^2} - b)\right)}$$

$$k\left(\frac{Q^2}{m^2}\right) = C + \frac{D - C}{1 + \exp\left(c(\log\frac{Q^2}{m^2} - d)\right)}$$

In order to extract the parameters and to check the accuracy of the approximation we applied it to the NNLO.

Damping functions

Parameters at NNLO

$$A = 1.7, \qquad B = 2.5,$$
$$C = 2.5, \qquad D = 1.2,$$
$$a = c = 2.5, \qquad b = d = 5$$

- These parameters have been extracted studying the agreement between the approximate and the exact NNLO curves.

Parameters at NLO

$A = 0.2, \quad B, C, D, a, b, c, d$ are unchanged

Parameters at N^3LO

$A = 0.3, \quad B, C, D, a, b, c, d$ are unchanged

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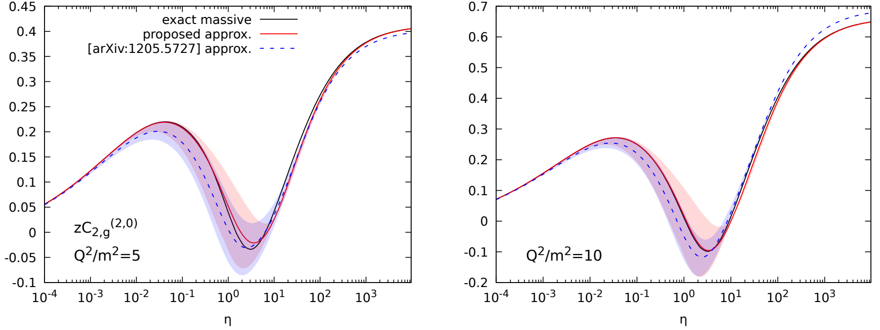
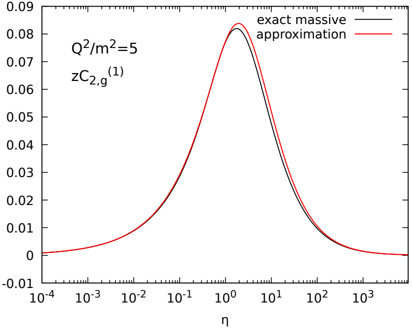
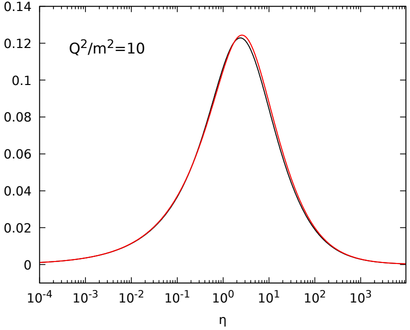


- The uncertainty band has been constructed varying the parameters of the damping functions and taking the envelope of the curves obtained in this way.

Approximation: NLO

- Before applying the approximation to the N^3LO, we applied it to the NLO.
- This was done in order to verify that our construction wasn't too specific for the NNLO (since the other orders are in principle different functions with respect to the NNLO).

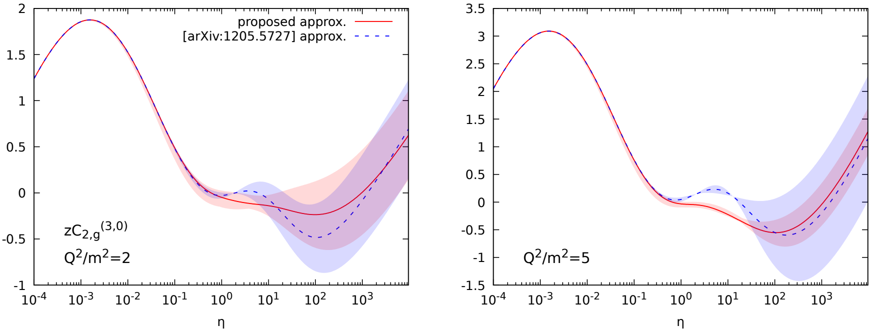




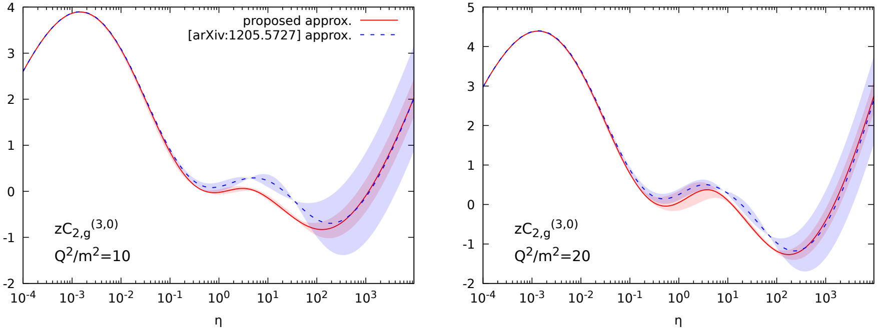
Approximation: N^3LO

Now we can apply our construction to the N^3LO.

- We chose the same functions for $h(Q^2/m^2)$ and $k(Q^2/m^2)$ that we have tuned from the NNLO.
- For all the approximate contributions that we have at N^3LO we have used the center of the uncertainty band.



Approximation: N^{3}LO



Conclusions

- Using the three known kinematics limits of the N^3LO gluon coefficient function we constructed an approximation that is valid in the whole range of z.
- It has been validated on the NNLO, that is known, and it has been compared with other approximations available in the literature.
- We can apply the same approximation procedure on the N^3LO quark coefficient function.
- Such result is an important ingredient for a N^3LO PDF fit, that is fundamental in the study of precision physics at LHC.

Thank you for your attention!

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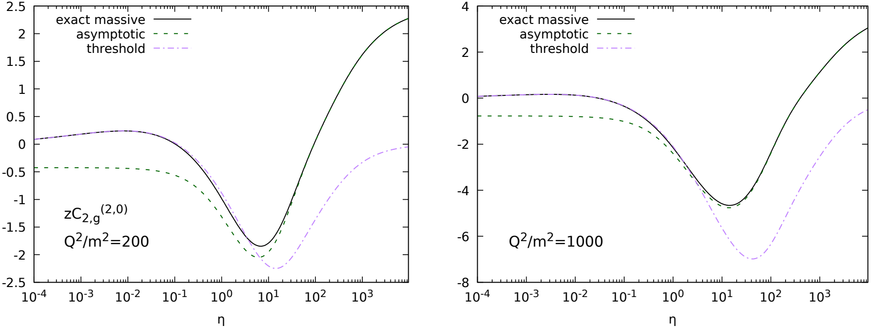
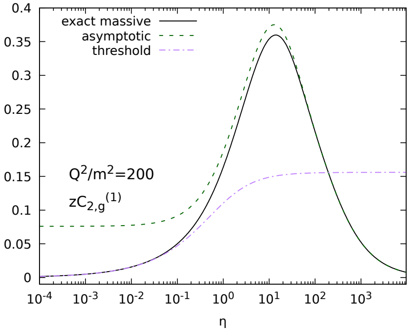
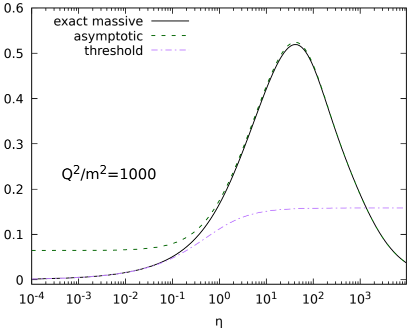
Extraction of the parameter A: NNLO

- Studying the asymptotic and the threshold limits for large Q^2 we extracted the parameter A



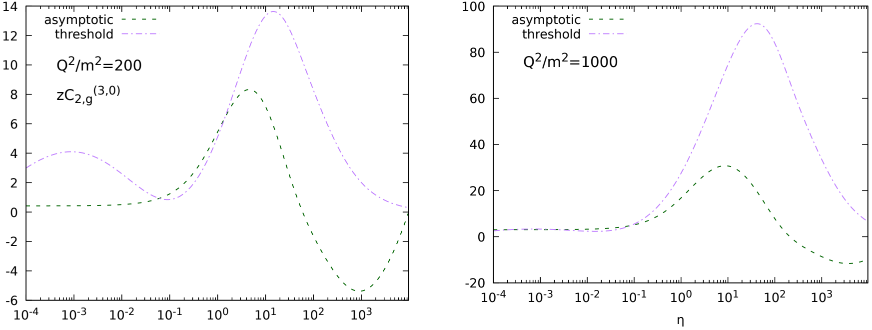
- We did the same for the NLO





Extraction of the parameter A: N^{3}LO

- And for the N^{3}LO



- Obviously, in this case we don't have the exact function that helps us to choose the value of A. This will give a bigger uncertainty in the final result.

Computation of the approximate NLL expansion of $C_{2,g}^{(3)}$

- In Mellin space we have that

$$F_{2,N}(\xi) = \hat{F}_{2,N}(\xi) f_{g,N}(\mu^2)$$

with

$$\hat{F}_{2,N}(\xi) = K_{2,N}(\xi,\gamma) h(\gamma) \left(\frac{m^2}{\mu^2}\right)^{\gamma}$$

- Expanding in γ we find

$$\hat{F}_{2,N}(\xi) = \hat{F}_{2,N}^{(0)} + \gamma \hat{F}_{2,N}^{(1)} + \gamma^2 \hat{F}_{2,N}^{(2)} + \mathcal{O}(\gamma^3)$$

- Then we make the substitution

$$\begin{aligned}
\gamma^0 &\to \left[\gamma^0\right] = 1 \\
\gamma^1 &\to \left[\gamma^1\right] = \gamma = \alpha_s \gamma_0 + \alpha_s^2 \gamma_1 + \mathcal{O}(\alpha_s^3) \\
\gamma^2 &\to \left[\gamma^2\right] = \gamma(\gamma - \alpha_s \beta_0) = \alpha_s^2(\gamma_0^2 - \gamma_0 \beta_0) + \mathcal{O}(\alpha_s^3)
\end{aligned}$$

- Therefore we find

$$\hat{F}_{2,N}(\xi) = \hat{F}_{2,N}^{(0)} + \alpha_s \hat{F}_{2,N}^{(1)}\gamma_0 + \alpha_s^2\Big(\hat{F}_{2,N}^{(1)}\gamma_1 + \hat{F}_{2,N}^{(2)}(\gamma_0^2 - \gamma_0\beta_0)\Big)$$

- In order to find our approximation we used

$$\gamma_0^{\mathrm{NLL}} = \frac{a_{11}}{N} + \frac{a_{10}}{N+1} \qquad \gamma_1^{\mathrm{NLL}} = \frac{a_{21}}{N} - \frac{2a_{21}}{N+1}$$

where the coefficients a_{11}, a_{10} and a_{21} are given by

$$a_{11} = \frac{C_A}{\pi}$$

$$a_{10} = -\frac{11C_A + 2n_f(1 - 2C_F/C_A)}{12\pi}$$

$$a_{21} = n_f\frac{26C_F - 23C_A}{36\pi^2}$$

- Going back from Mellin to x space we found our approximate result