Two-loop QCD corrections to five-particle amplitudes with one massive leg

Simone Zoia

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European Research Council
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DI TORINO

Outline

Special function basis for planar 2-loop 5-pt amplitudes with 1 external massive leg

2102.02516 with **Simon Badger**, **Heribertus Bayu Hartanto**

2110.10111 with **Dmitry Chicherin**, **Vasily Sotnikov**

Two-loop leading colour amplitudes:

2102.02516 ($Wb\bar{b}$)

2107.14733 ($Hb\bar{b}$)

2XXX.XXXXX ($W\gamma j$)

Simon Badger, Heribertus Bayu Hartanto

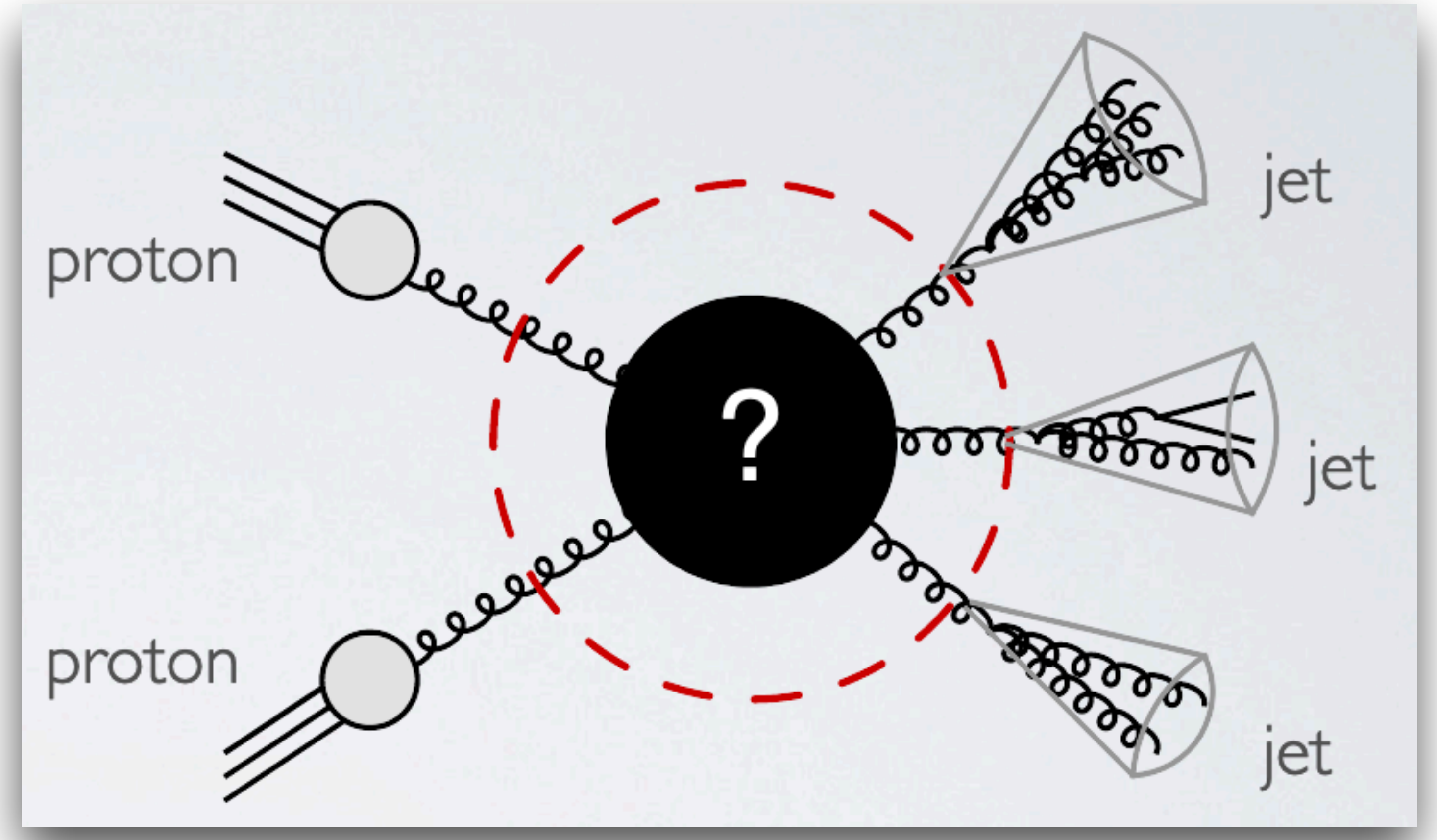
Jakub Kryś

Urgent demand for NNLO QCD predictions

NNLO QCD for $2 \to 1{,}2$ processes under control

Great interest in $2 \to 3$ scattering

$(2 \to 3)/(2 \to 2)$ ratios $\Rightarrow$ high-precision observables



[courtesy of S. Badger]

$$pp \to 3j\,,\, 3\gamma\,,\, \gamma\gamma + j\,,\, H + 2j\,,\, H + b\bar{b}\,,\, V + 2j\,,\, V + b\bar{b}\,,\, VV' + j\,, \ldots$$

[from Les Houches 2019 “Precision wish-list”]

Dramatic progress for massless 5-particle scattering

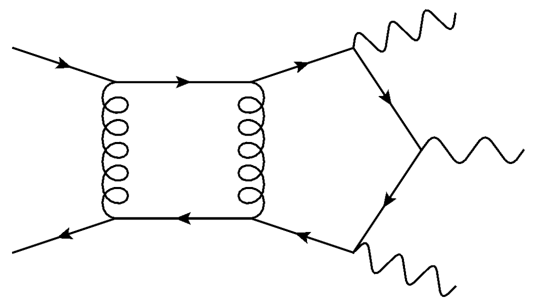
Analytic results for all Feynman integrals

[Gehrmann, Henn, Lo Presti 2015;
Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser 2018;
Abreu, Page, Zeng 2018; Chicherin, Henn, Mitev 2018;
Abreu, Dixon, Herrmann, Page, Zeng 2018;
Chicherin, Gehrmann, Henn, Wasser, Zhang, **SZ** 2018]

Special function basis [Chicherin, Sotnikov 2020]

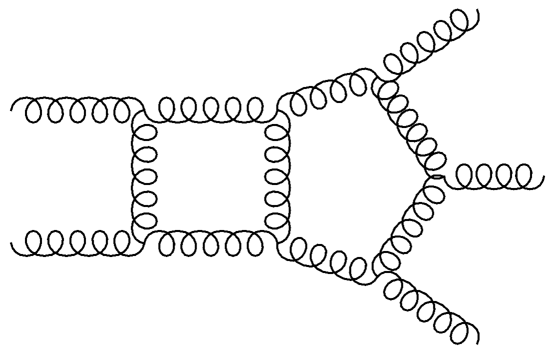
Analytic results for scattering amplitudes

3γ planar



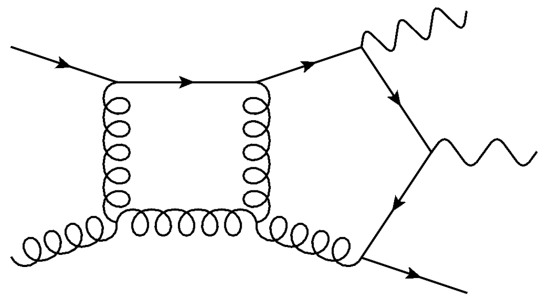
[Abreu, Page, Pascual, Sotnikov 2020; Chawdhry, Czakon, Mitov, Poncelet 2021]

$3j$ planar

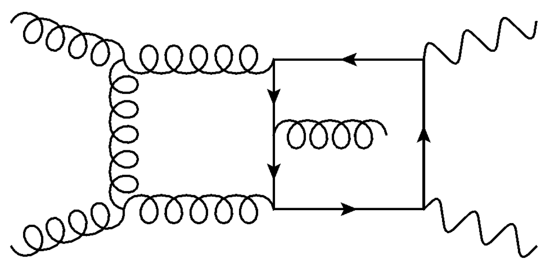


[Abreu, Febres-Cordero, Ita, Page, Sotnikov 2021; Badger, Brönnum-Hansen, Bayu Hartanto, Peraro, Moodie, **SZ** 202?]

$2\gamma + j$ full colour



[Agarwal, Buccioni, von Manteuffel, Tancredi 2021 x2; Chawdhry, Czakon, Mitov, Poncelet 2021]



[Badger, Brönnum-Hansen, Chicherin, Gehrmann, B. Hartanto, Henn, Marcoli, Moodie, Peraro, **SZ** 2021]

$d\sigma$ **@NNLO QCD:**

$pp \to 3\gamma$ [Kallweit, Sotnikov, Wiesemann; Chawdhry, Czakon, Mitov, Poncelet]

$pp \to 2\gamma + j$ [Chawdhry, Czakon, Mitov, Poncelet; Badger, Gehrmann, Marcoli, Moodie]

$pp \to 3j$ [Czakon, Mitov, Poncelet]

Five-particle scattering with one off-shell leg

$$pp \to 3j\,,\ 3\gamma\,,\ \gamma\gamma + j\,,\ H + 2j\,,\ H + b\bar{b}\,,\ V + 2j\,,\ V + b\bar{b}\,,\ VV' + j\,,\ \ldots$$

✔ ✔ ✔

[from Les Houches 2019 "Precision wish-list"]

Massless internal legs, focus on QCD corrections

Rich potential phenomenology

High algebraic and analytic complexity

massless		**1 external mass**
5 scalar variables	→	6 scalar variables
1 pseudo-scalar		1 pseudo-scalar
1 square root		4 square roots (planar)

Phenomenology is very demanding

$$\text{amplitude} = \sum \text{Feynman diagrams}$$

?

$$\text{amplitude} = \sum \text{rational coeffs} \times \text{special funcs}$$

Phenomenology is very demanding

amplitude $= \sum$ **Feynman diagrams**

?

amplitude $= \sum$ **rational coeffs** $\times$ **special funcs**

Simplification and cancellation of the poles in ϵ

Compact analytic expressions?

Fast/stable evaluation across physical phase space?

Phenomenology is very demanding

$$\textbf{amplitude} = \sum \textbf{Feynman diagrams}$$

?

$$\textbf{amplitude} = \sum \textbf{rational coeffs} \times \textbf{special funcs}$$

Finite field arithmetics +
functional reconstruction

Canonical DEs +
Chen’s iterated integrals

Amplitude workflow

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i \textbf{Feynman diagram}_i$$

FORM + Mathematica

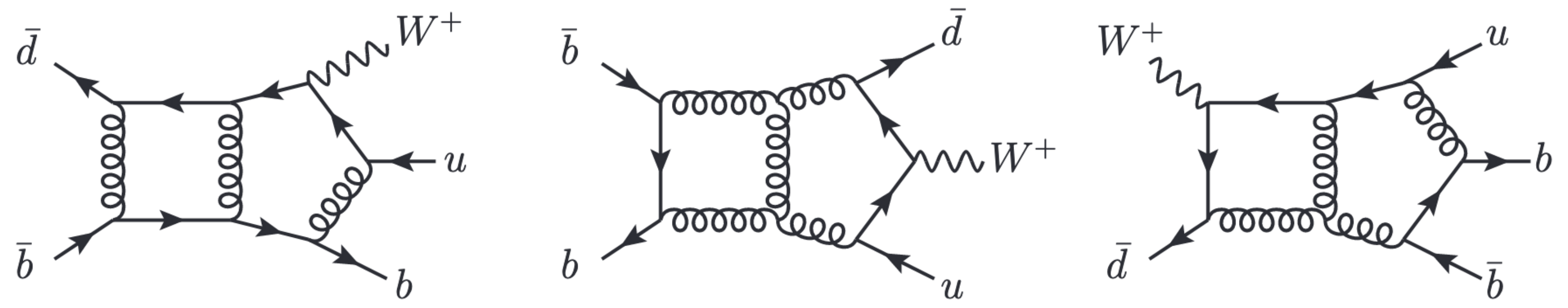
$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i c_i\left(\{p\},\epsilon\right)\, I_i\left(\{p\},\epsilon\right)$$

Integration-by-parts (IBPs)

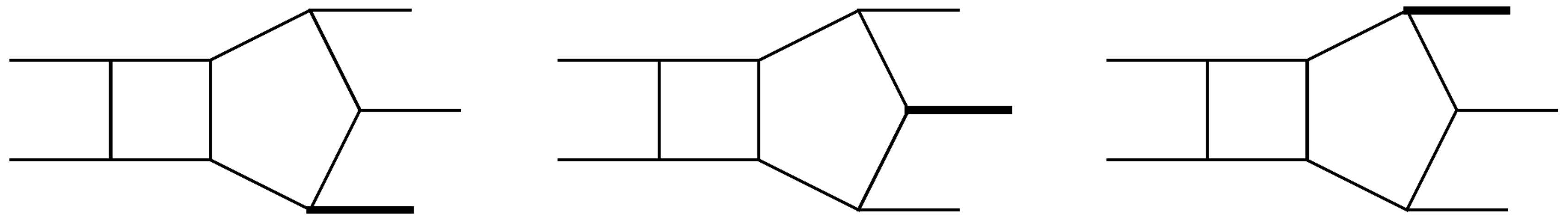
$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i d_i\left(\{p\},\epsilon\right)\, \textbf{MI}_i\left(\{p\},\epsilon\right)$$

Master integrals

QGRAF



Scalar Feynman integrals



Very large linear system

Complicated solution

Dramatic improvement from finite field arithmetic

[von Manteuffel, Schabinger 2015; Peraro 2016]

Evaluate rational functions at numerical points $\left(\{p\}, \epsilon\right)$ modulo prime number

Reconstruct only the final result from multiple numerical evaluations

Framework **FiniteFlow** [Peraro 2019]

Amplitude workflow

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FORM + Mathematica

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Numerical algorithm to evaluate the coefficients

IBPs generated with **LiteRed** [Lee 2012]

Solved numerically with the Laporta algorithm within **FiniteFlow** [Laporta 2001]

Amplitude workflow

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We don't want to stop here!

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IBPs generated with **LiteRed** [Lee 2012]

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Integrating by differentiating

$$\frac{d}{ds_{12}} \overrightarrow{\mathbf{MI}} \left(\{p\}, \epsilon\right) = A_{s_{12}} \left(\{p\}, \epsilon\right) \cdot \overrightarrow{\mathbf{MI}} \left(\{p\}, \epsilon\right)$$

[Kotikov '91; Bern, Dixon, Kosower '94; Remiddi '97; Gehrmann, Remiddi 2000]

Integrating by differentiating

DEs in the *canonical form*: $$d\,\overrightarrow{\mathbf{MI}}(\{p\},\epsilon) = \epsilon\, d\tilde{A}\left(\{p\}\right) \cdot \overrightarrow{\mathbf{MI}}\left(\{p\},\epsilon\right)$$ [Henn 2013]

Integrating by differentiating

DEs in the *canonical form*:

$$d\,\overrightarrow{\mathbf{MI}}(\{p\},\epsilon) = \epsilon\, d\tilde{A}\left(\{p\}\right) \cdot \overrightarrow{\mathbf{MI}}\left(\{p\},\epsilon\right)$$

[Henn 2013]

$$\tilde{A}\left(\{p\}\right) = \sum_i a_i \log w_i\left(\{p\}\right)$$

Constant matrices

Letters: algebraic functions of kinematics

Integrating by differentiating

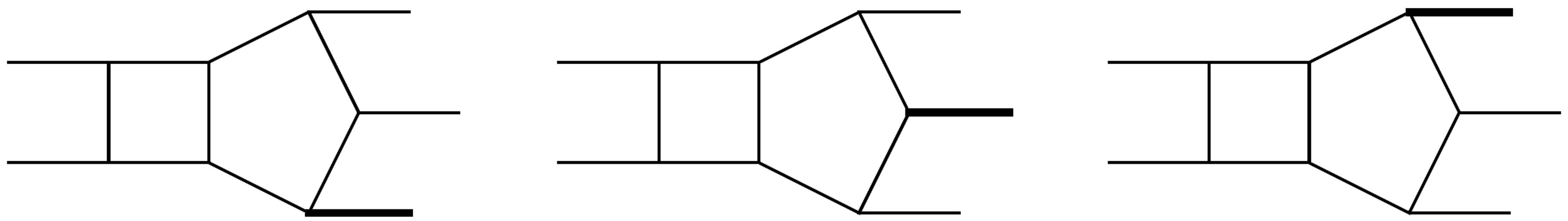
DEs in the *canonical form*: $$d\,\overrightarrow{\mathbf{MI}}(\{p\},\epsilon) = \epsilon\, d\tilde{A}\left(\{p\}\right)\cdot\overrightarrow{\mathbf{MI}}\left(\{p\},\epsilon\right)$$ [Henn 2013]

$$\tilde{A}\left(\{p\}\right) = \sum_i a_i \log w_i\left(\{p\}\right)$$

Letters: algebraic functions of kinematics

Constant matrices

Planar integral families:



[Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020]

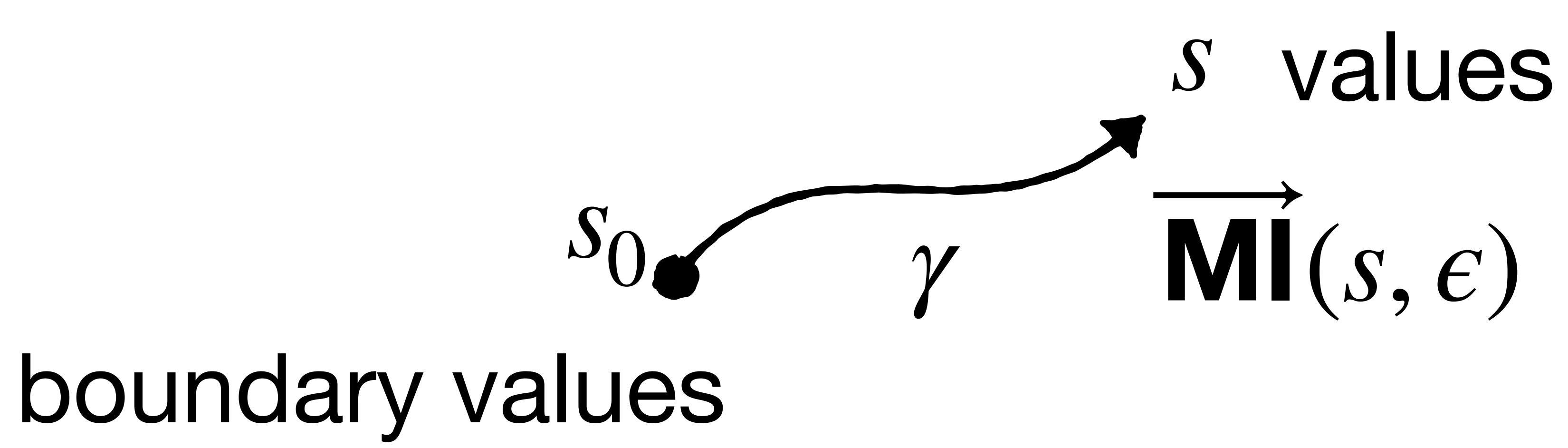
156 letters, 4 square roots

(Massless: 31 letters, 1 square root)

How to solve the DEs?

1. Integrate DEs numerically along path using **generalised series expansions**

[Moriello 2019; Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020]



😀 Very flexible and easy to set up

🙁 Forced to evaluate the MIs

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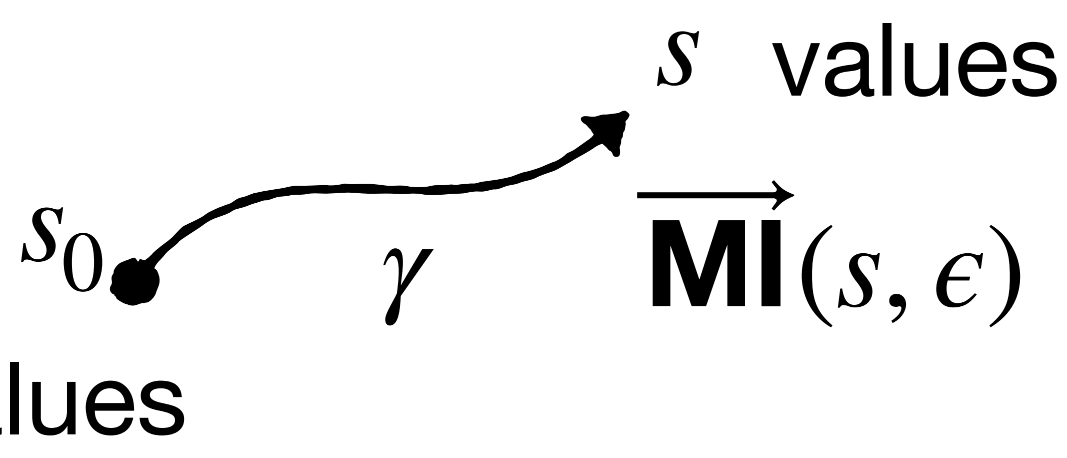

😀 Very flexible and easy to set up

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2. Express MIs in terms of **multiple polylogarithms**

[Canko, Papadopoulos, Syrrakos 2020; Syrrakos 2020]

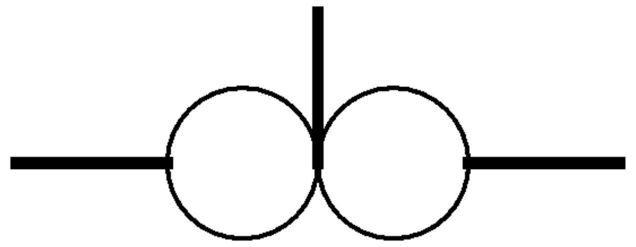
$$G\left(z_1,\ldots,z_n;x\right) = \int_0^x \frac{dt_1}{t_1 - z_1}\int_0^{t_1}\frac{dt_2}{t_2 - z_2}\cdots\int_0^{t_{n-1}}\frac{dt_n}{t_n - z_n}$$

😀 Well understood functions, libraries for numerical evaluation

🙁 Analytic continuation, functional relations, difficult to obtain

Our approach: special function basis

Construct a **basis** of algebraically independent special functions: $\vec{f} = \{f_i^{(w)}\}$

$\epsilon^2(1-2\epsilon)^2$  $= 1 - \epsilon\left(f_1^{(1)}+f_6^{(1)}\right) + \epsilon^2\left[\frac{1}{2}\left(f_1^{(1)}\right)^2 + f_1^{(1)}f_6^{(1)} + \frac{1}{2}\left(f_6^{(1)}\right)^2 - \frac{1}{6}f_1^{(2)}\right] + \mathcal{O}\left(\epsilon^3\right)$

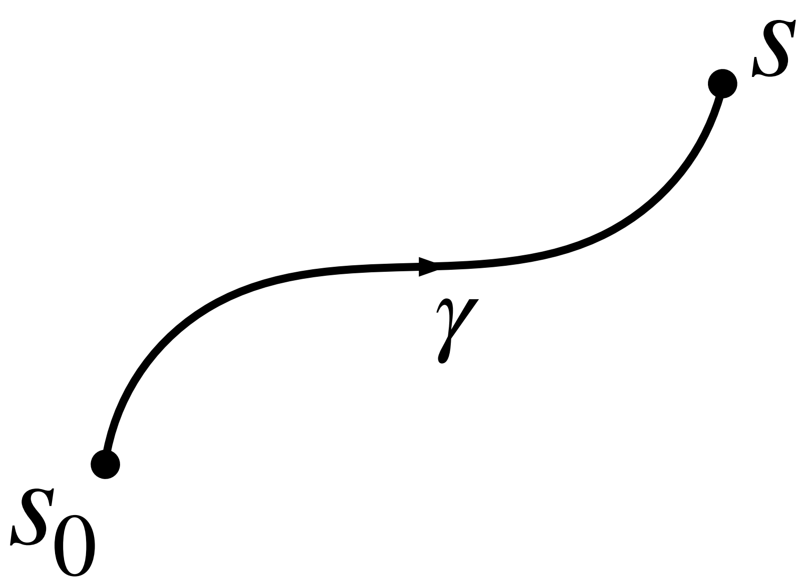
Analytic cancellations and simplifications

- Simpler reconstruction
- More compact expressions

More efficient evaluation

- Generalised series expansion
- Tailored representation and C++ library

Chen's iterated integrals



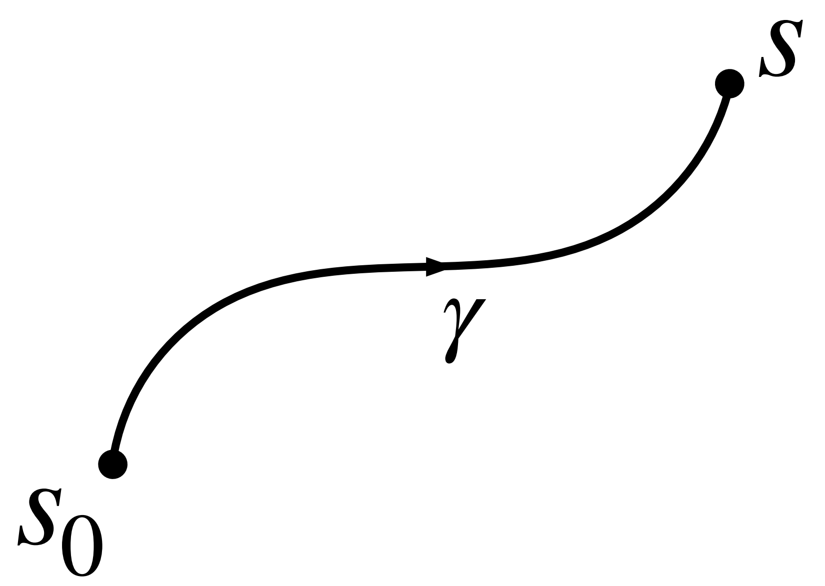
$$\left[w_{i_1},\ldots,w_{i_n}\right]_{s_0}(s) = \int_\gamma d\log w_{i_n}(s')\left[w_{i_1},\ldots,w_{i_{n-1}}\right]_{s_0}(s')$$

$n =$ transcendental weight

All functional relations become manifest in terms of iterated integrals

$$Li_2(z) + \frac{1}{2}\log^2(-z) + Li_2\left(\frac{1}{z}\right) + \frac{\pi^2}{6} = 0$$

Chen’s iterated integrals



$$\left[w_{i_1}, \ldots, w_{i_n}\right]_{s_0}(s) = \int_\gamma d\log w_{i_n}(s')\,\left[w_{i_1}, \ldots, w_{i_{n-1}}\right]_{s_0}(s')$$

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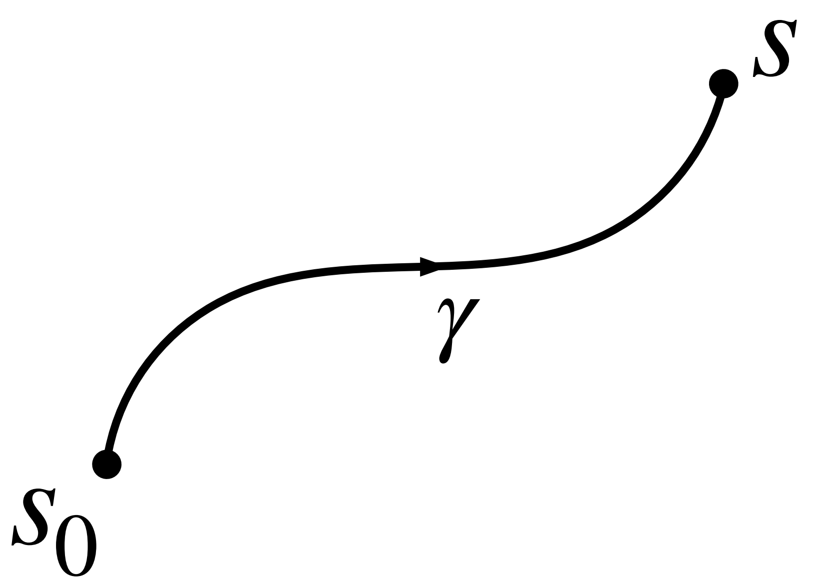
$$Li_2(z) = -\left[1-z, z\right]_{-1} - \log 2\,\left[z\right]_{-1} - \frac{\pi^2}{12}$$

$$Li_2\left(\frac{1}{z}\right) = \left[1-z, z\right]_{-1} - \left[z, z\right]_{-1} + \log 2\,\left[z\right]_{-1} - \frac{\pi^2}{12}$$

$$\frac{1}{2}\log^2(-z) = \left[z, z\right]_{-1}$$

Chen's iterated integrals



$$[w_{i_1}, \ldots, w_{i_n}]_{s_0}(s) = \int_\gamma d\log w_{i_n}(s')\,[w_{i_1}, \ldots, w_{i_{n-1}}]_{s_0}(s')$$

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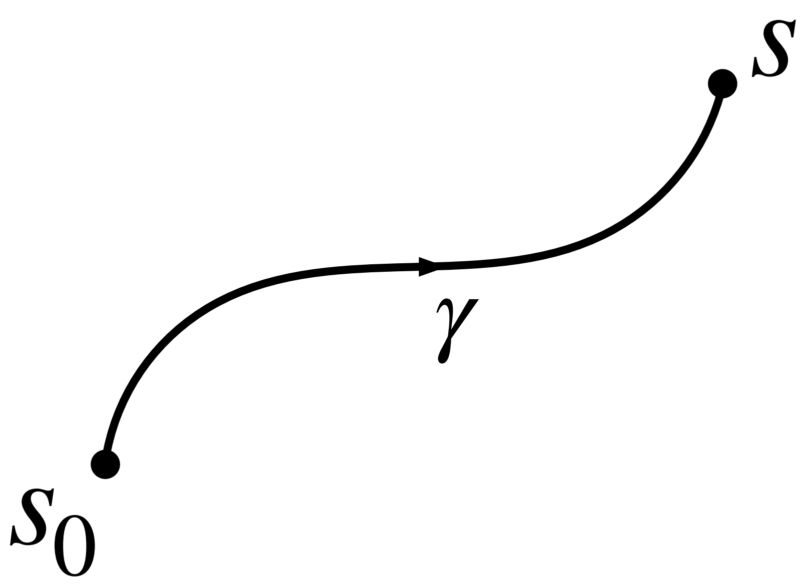
Chen's iterated integrals



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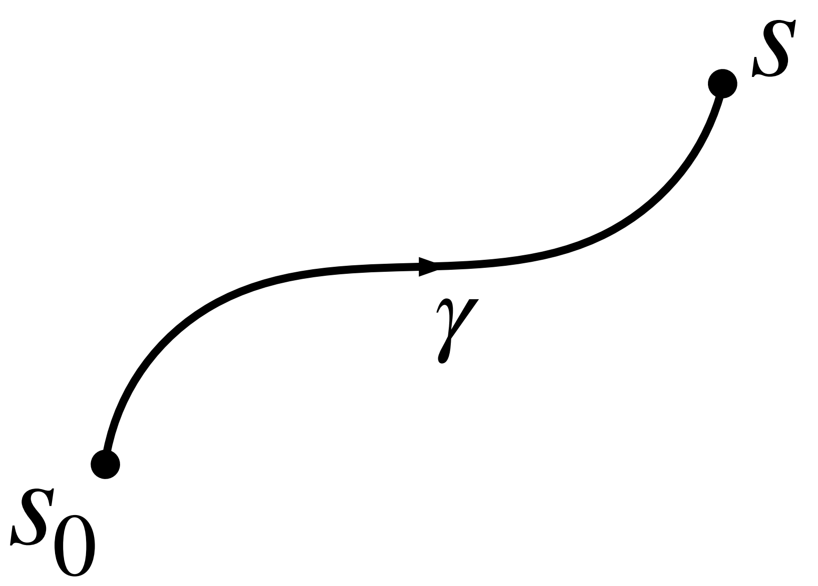

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$$\frac{1}{2}\log^2(-z) = \left[z, z\right]_{-1}$$

Constructing a basis of special functions becomes a linear algebra problem

1. Write the MIs in terms of iterated integrals $\overrightarrow{\mathbf{MI}}(s,\varepsilon) = \sum_{w\geq 0} \varepsilon^w \overrightarrow{\mathbf{MI}}^{(w)}(s)$

2. Extract basis from the terms of the ε-expansion of the MIs

$$\left\{\mathbf{MI}_i^{(w)}(s)\right\} \implies \left\{f_i^{(w)}(s)\right\}, \qquad w = 1,\ldots,4$$

Some work required to construct basis of transcendental constants:

high-precision evaluation + PSLQ algorithm

Numerical evaluation through generalised series expansion

[Simon Badger, Heribertus Bayu Hartanto, **SZ** 2021]

Apply generalised series expansion method directly to the special functions

$$\vec{f} = \begin{pmatrix} \epsilon^4 f_i^{(4)} \\ \epsilon^3 f_i^{(3)} \\ \epsilon^2 f_i^{(2)} \\ \epsilon^1 f_i^{(1)} \\ 1 \end{pmatrix}$$

$$d\vec{f} = \epsilon \, d\tilde{B} \cdot \vec{f}$$

Much simpler than the DEs for the master integrals

Generalised series expansion implemented in **DiffExp** [Hidding 2020]

Evaluation in any kinematic region

One-mass pentagon function library

[Chicherin, Sotnikov, **SZ** 2021]

Hand-crafted expressions in terms of logs and dilogarithms up to weight 2

$$\log\left(p_1^2\right) = \left[W_1\right]_{s_0}(s) \qquad Li_2\left(1 - \frac{s_{13}}{p_1^2}\right) = \left[\frac{W_1}{W_3}, \frac{W_{13}}{W_1}\right]_{s_0}(s) + \log 3 \left[\frac{W_1}{W_{13}}\right]_{s_0}(s) + Li_2(-2)$$

One-fold integral representations at weight 3 and 4 [Caron-Huot, Henn 2014]

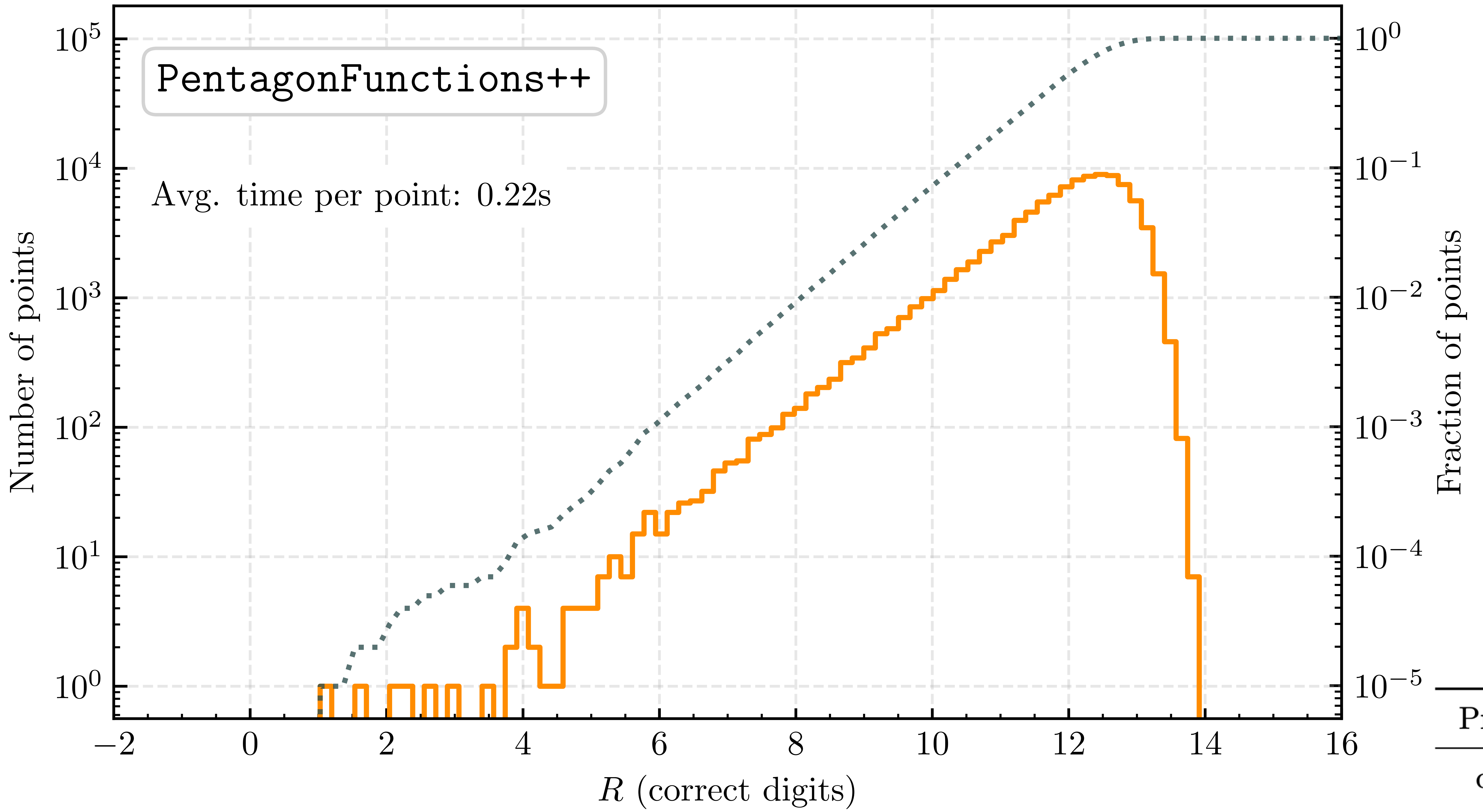
$$f_i^{(3)}(s) = \sum_{j,k} c_{ijk} \int_0^1 dt \left(\partial_t \log W_j(t)\right) h_k^{(2)}(t) + \tau_i^{(3)}$$

Implemented in C++ library **PentagonFunctions++**

Valid in the physical scattering regions

One-mass pentagon function library

[Chicherin, Sotnikov, **SZ** 2021]



Precision	Correct digits	Timing (s)
double	12	0.19
quadruple	28	159
octuple	60	1695

Ready for phenomenology!

Target the finite remainder

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i \text{Feynman diagram}_i$$

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i c_i\left(\{p\},\epsilon\right)\, I_i\left(\{p\},\epsilon\right)$$

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i d_i\left(\{p\},\epsilon\right)\, \text{MI}_i\left(\{p\},\epsilon\right)$$

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_{i=-4}^{0} \varepsilon^i \sum_j e_{i,j}\left(\{p\}\right)\, \text{mon}_j\left(\vec{f}\right)$$

$$F^{(2)}\left(\{p\},\epsilon\right) = \sum_i q_i\left(\{p\}\right)\, \text{mon}_i\left(\vec{f}\right)$$

Map onto special function basis + Laurent expansion

UV/IR subtraction

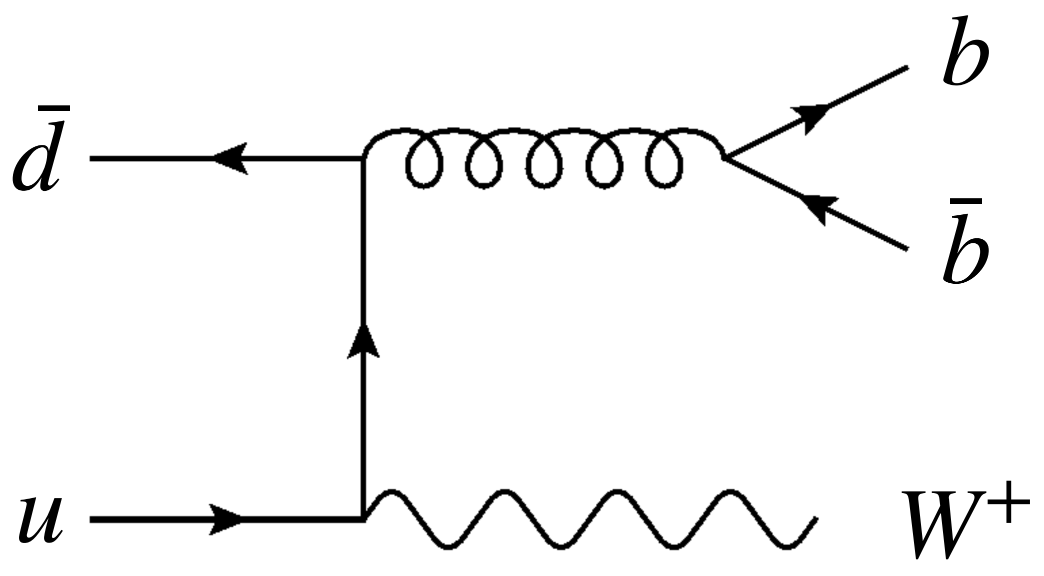
Reconstruct rational coefficients of the finite remainder

$pp \to Wb\bar{b}$

[Badger, B. Hartanto, **SZ** 2021]

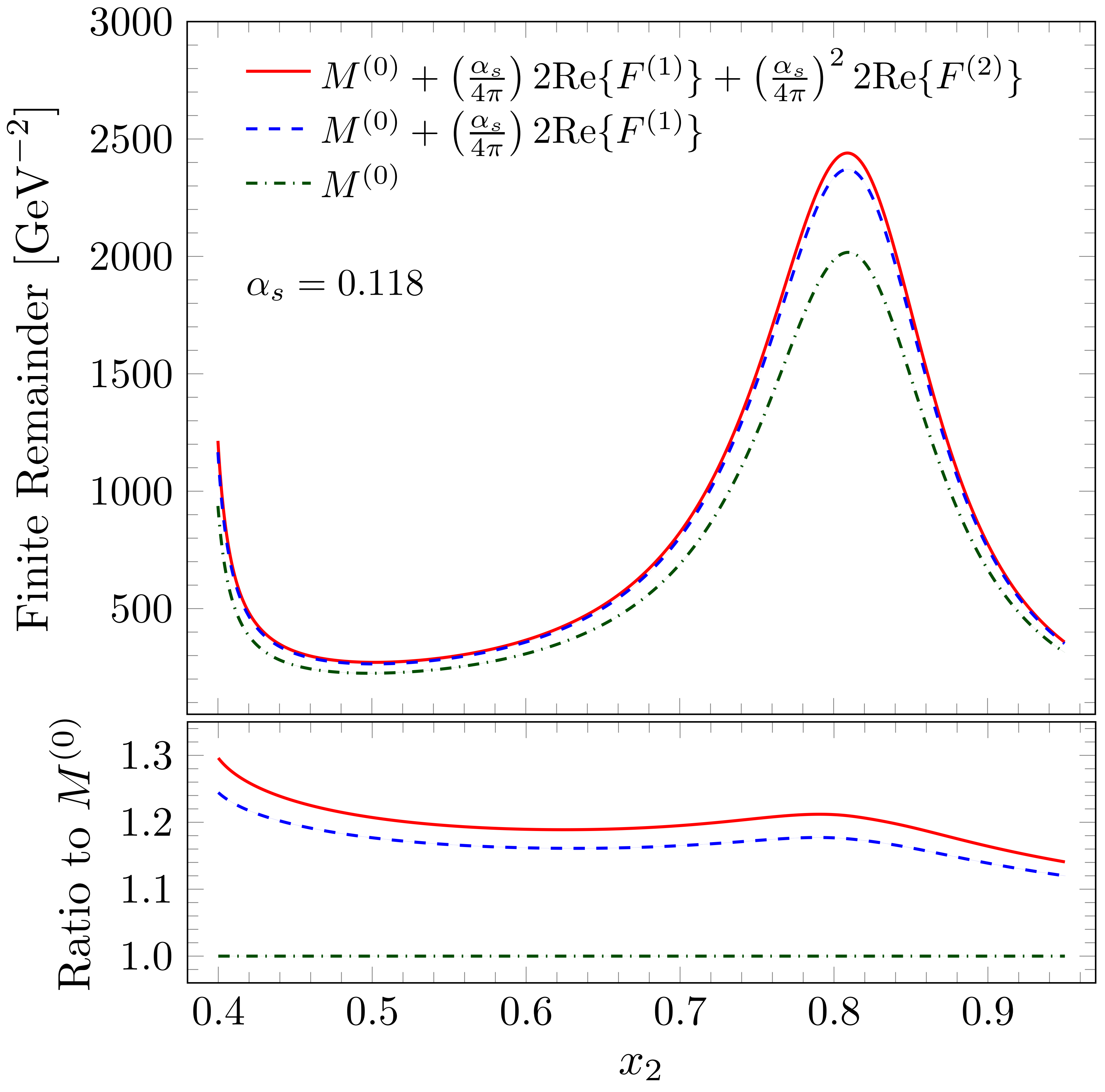
Background to $pp \to WH$

Squared matrix elements for $\bar{d}(p_1) + u(p_2) \to b(p_3) + \bar{b}(p_4) + W^+(p_5)$



Leading colour, massless b quark, on-shell W

Special functions evaluated with **DiffExp**
[Hidding 2020]

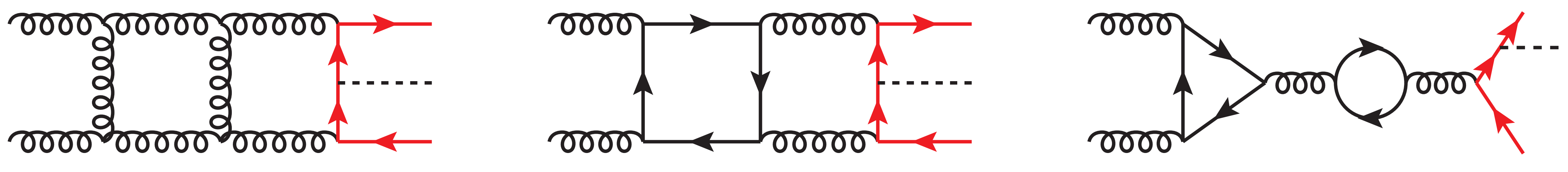


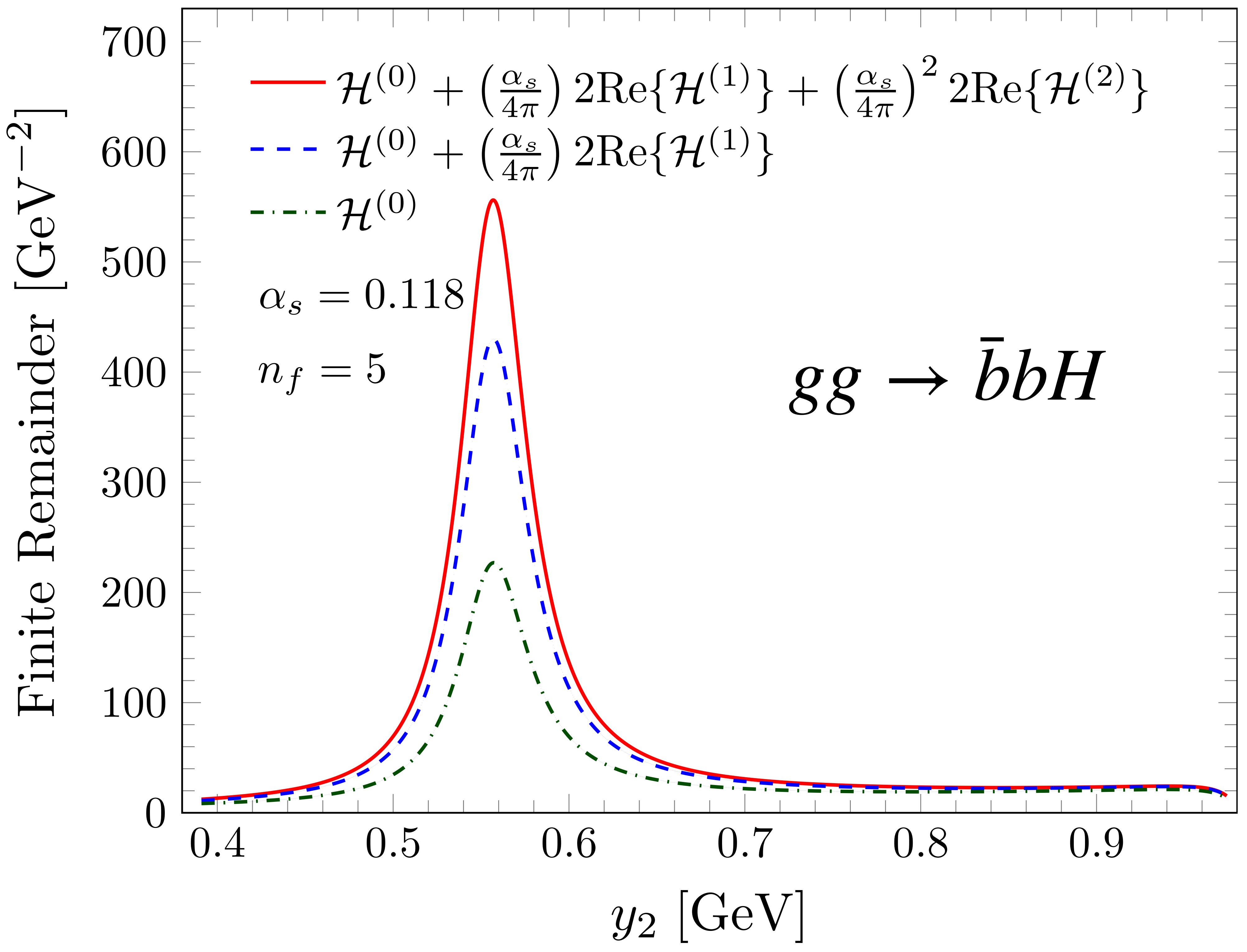
$pp \to Hb\bar{b}$ [Badger, B. Hartanto, Kryś, **SZ** 2021]

Complete set of helicity amplitudes

$$0 \to \bar{b}(p_1) + b(p_2) + g(p_3) + g(p_4) + H(p_5)$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{q}(p_3) + q(p_4) + H(p_5)$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{b}(p_3) + b(p_4) + H(p_5)$$

Leading colour, massless b quark

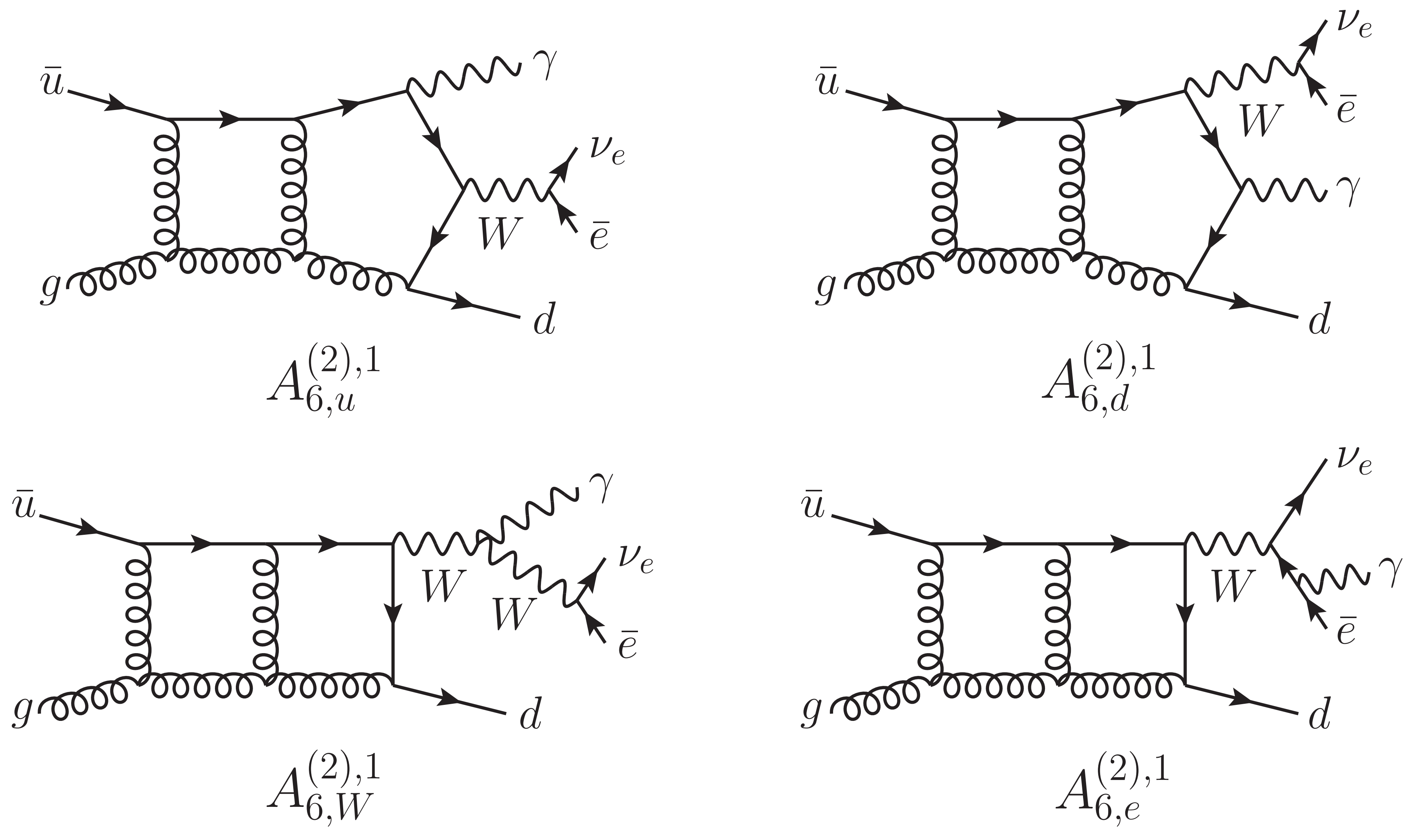


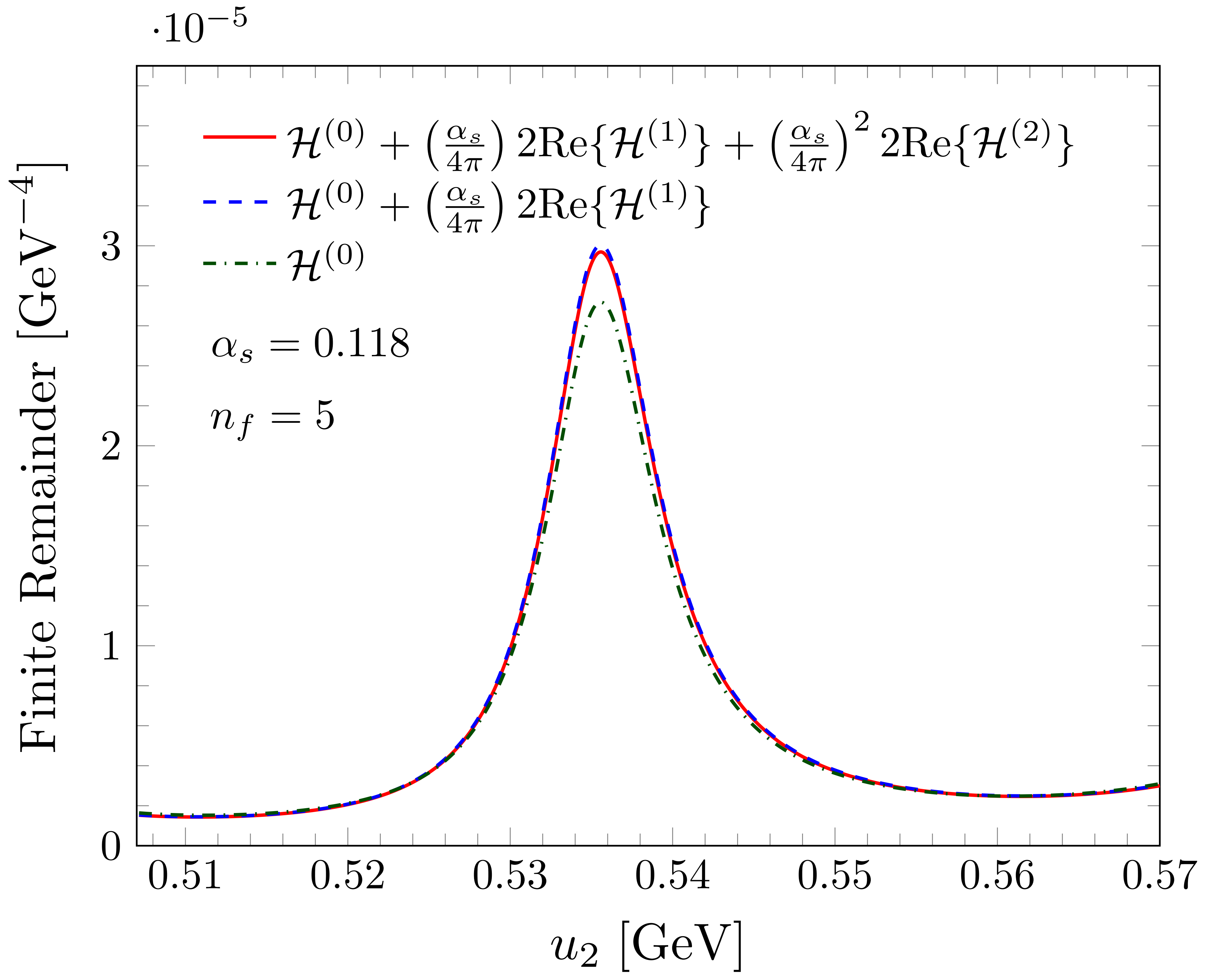


Special functions evaluated with **DiffExp**
[Hidding 2020]

$pp \to W\gamma j$ [Badger, B. Hartanto, Kryś, **SZ** 202?]

Leading colour helicity amplitudes: $0 \to \gamma(p_1) + \bar{u}(p_2) + g(p_3) + d(p_4) + \nu_e(p_5) + e^+(p_6)$





$u + g \to \gamma + d + \nu_e + e^+$

One-mass pentagon functions (**PentagonFunctions++**) [Chicherin, Sotnikov, **SZ** 2021]

Conclusions

Function basis for all planar 2-loop 5-particle amplitudes with 1 mass (& C++ library)

[Badger, B. Hartanto, **SZ** 2021;
Chicherin, Sotnikov, **SZ** 2021]

First analytic amplitudes (leading colour):

- $pp \to Wb\bar{b}\,,\ Hb\bar{b}\,,\ W\gamma j$ [Badger, B. Hartanto, **SZ** 2021; Badger, B. Hartanto, Kryś, **SZ** 2021 + 202?]
- $pp \to V + 2j$ [Abreu, F. Cordero, Ita, Klinkert, Page, Sotnikov 2021]

Progress on non-planar integrals [Papadopoulos, Wever 2019; Abreu, Ita, Page, Tschernow 2021]

Looking forward to phenomenology!

Back-up slides

Algebraic dependence on the kinematic variables

- 6 scalar invariants: $(s_{12}\,, s_{23}\,, s_{34}\,, s_{45}\,, s_{51}\,, p_5^2)$

- 1 pseudo-scalar invariant: $tr_5 = \mathbf{tr}\left(\gamma_5 p_1 p_2 p_3 p_4\right) = 4i\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$

$$tr_5^2 = \Delta_5 = s_{12}^2 s_{23}^2 + 2 s_{12} s_{23}^2 s_{34} + \ldots \quad \longrightarrow \quad tr_5 = \pm\sqrt{\Delta_5}$$

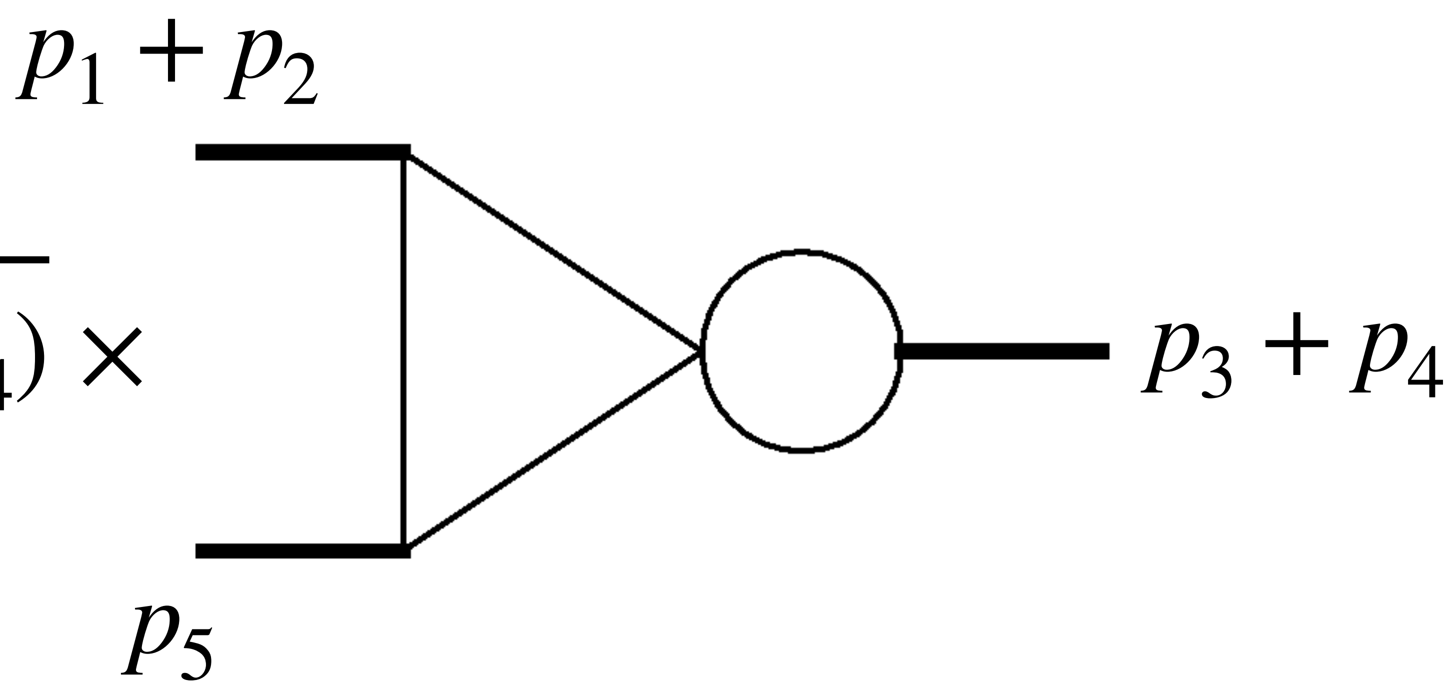
Parity label

Other 3 square roots from the Feynman integrals

No rational parameterisation!

Overall square roots in the definition of the canonical basis integrals

$$\epsilon^3(1-2\epsilon)\sqrt{(p_5^2)^2+(s_{12}-s_{34})^2-2p_5^2(s_{12}+s_{34})}\times$$

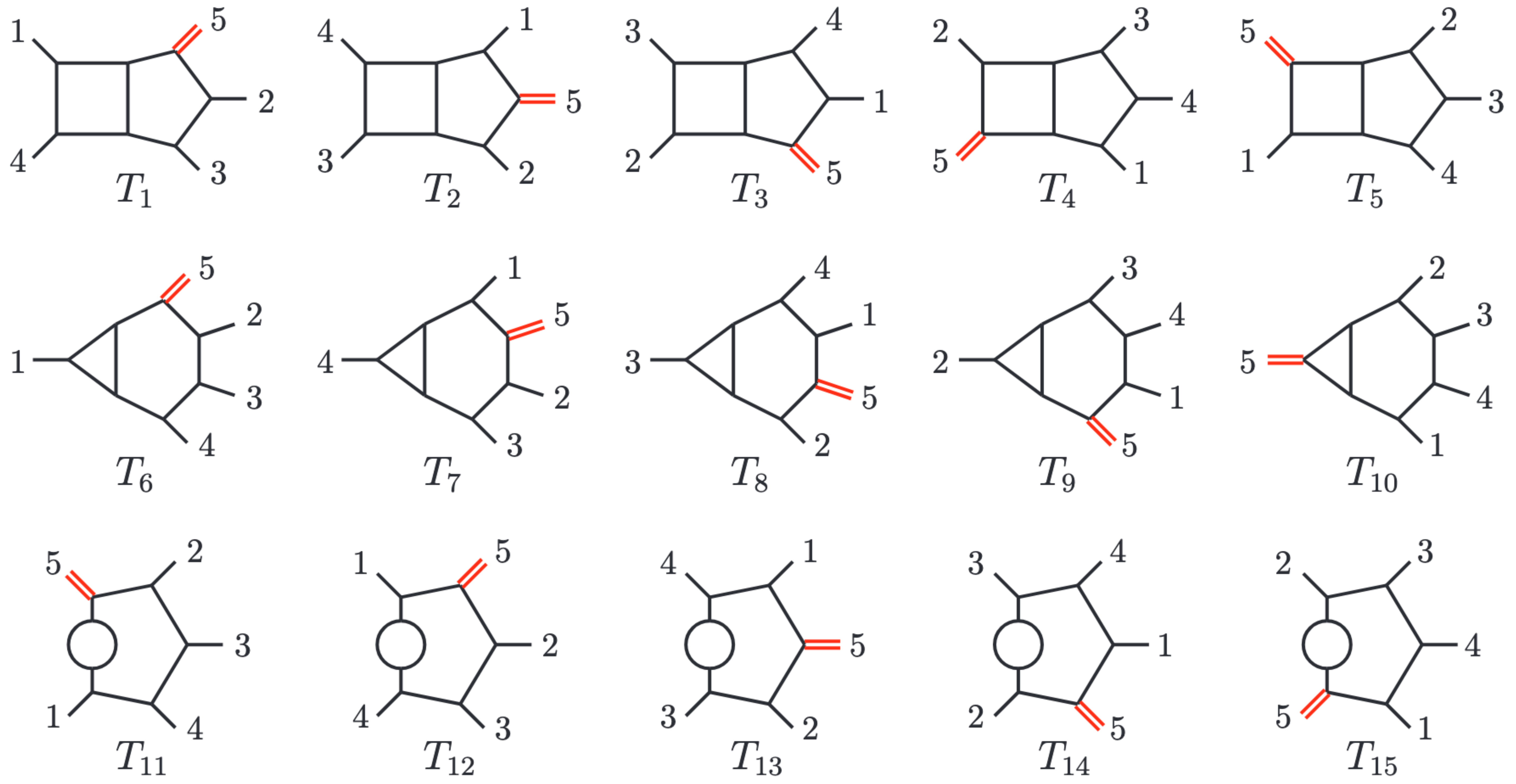


$$F^{(2)}(\{p\}) = \sum_i e_i\left(\{p\}\right)\ \textbf{mon}_i\left(\sqrt{\Delta_j}, f\right) + \mathcal{O}(\epsilon)$$

Rational functions

Families of scalar Feynman integrals

$$\mathcal{I}\left(a_1, a_2, \ldots, a_{11}\right) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{D_1^{a_1} D_2^{a_2} \ldots D_{11}^{a_{11}}}$$



$$D_{T_1,1} = k_1^2\,,$$
$$D_{T_1,2} = (k_1 + p_5)^2\,,$$
$$D_{T_1,3} = (k_1 + p_5 + p_2)^2\,,$$
$$D_{T_1,4} = (k_1 + p_5 + p_2 + p_3)^2\,,$$
$$D_{T_1,5} = k_2^2\,,$$
$$D_{T_1,6} = (k_2 + p_5 + p_2 + p_3)^2\,,$$
$$D_{T_1,7} = (k_2 - p_1)^2\,,$$
$$D_{T_1,8} = (k_1 - k_2)^2\,,$$

ISPs

$$D_{T_1,9} = (k_1 - p_1)^2\,,$$
$$D_{T_1,10} = (k_2 + p_5)^2\,,$$
$$D_{T_1,11} = (k_2 + p_5 + p_2)^2$$

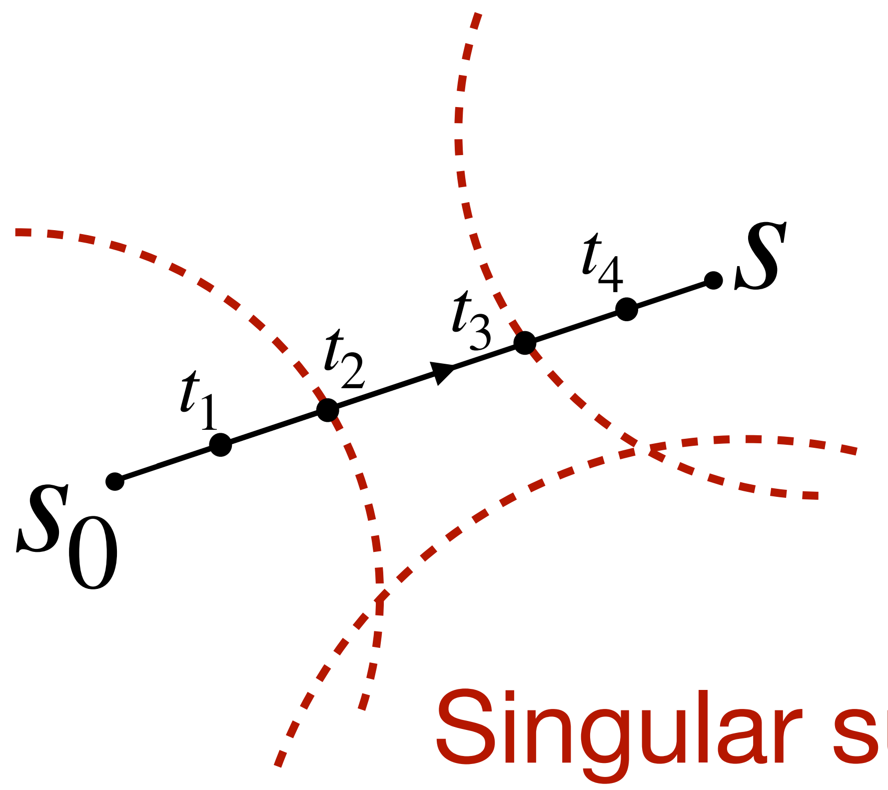
Series solution of the DEs

[Moriello 2019]

Integrate DEs along 1-dim. path γ

$$\overrightarrow{\mathbf{MI}}(t,\epsilon) := \overrightarrow{\mathbf{MI}}\left(s=\gamma(t),\epsilon\right)$$

$$\frac{d}{dt}\overrightarrow{\mathbf{MI}}(t,\epsilon) = \epsilon\, A(t)\cdot\overrightarrow{\mathbf{MI}}(t,\epsilon)$$



$$\gamma(t) = s_0 + t(s-s_0)$$

Generalised series solution around any point t_k

$$\overrightarrow{\mathbf{MI}}^{(w)}(t) = \sum_{j_1\geq 0}\sum_{j_2=0}^{w} \vec{c}_{j_1,j_2}(t-t_k)^{\frac{j_1}{2}}\log^{j_2}(t-t_k)$$

Compute solutions at various t_k and match them $\quad \overrightarrow{\mathbf{MI}}(0,\epsilon) \longrightarrow \overrightarrow{\mathbf{MI}}(1,\epsilon)$

Boundary values from the MPL expressions

[Canko, Papadopoulos, Syrrakos 2020; Syrrakos 2020]

3000-digit precision using `GiNaC` [Vollinga, Weinzierl 2004]

`PSLQ` algorithm to find basis of transcendental constants

$$G(0,1;1) = -1.644934067\ldots$$
$$G(3/2,2;1) = 0.4060916335\ldots \longrightarrow 3\,G(0,1;1) + 4\,G(3/2,1;1) - 2\,G(3/2,2;1) = 0$$
$$G(3/2,1;1) = 1.436746367\ldots$$

Solving the canonical DEs in terms of iterated integrals is trivial

$$\begin{cases} d[w_{i_1}, \ldots, w_{i_n}]_{s_0}(s) = d\log w_{i_n}(s)\, [w_{i_1}, \ldots, w_{i_{n-1}}]_{s_0}(s) \\ [w_{i_1}, \ldots, w_{i_n}]_{s_0}(s_0) = 0 \end{cases}$$

Chen's iterated integrals

Canonical DEs

$$\begin{cases} d\,\overrightarrow{\mathbf{MI}}^{(w)}(s) = \sum_i a_i\, d\log w_i(s)\, \overrightarrow{\mathbf{MI}}^{(w-1)}(s) \\ \overrightarrow{\mathbf{MI}}^{(w)}(s_0) = \overrightarrow{\mathbf{MI}}_0^{(w)} \end{cases}$$

Insensitive to square roots!

MPL-expressions + `PSLQ` algorithm

We also need cancellations between products

$$F^{(2)} = \lim_{\epsilon \to 0} \left[A^{(2)} - Z^{(2)} A^{(0)} - Z^{(1)} A^{(1)} \right]$$

Products of $f_i^{(w)}$

Shuffle product of iterated integrals:

$$[a] \times [b, c] = [a, b, c] + [b, a, c] + [b, c, a]$$

Remove the $f_i^{(w)}$ that can be rewritten in terms of products of lower-weight functions

$$[a, b] + [b, a] = [a] \times [b]$$

Express MIs in terms of monomials of $f_i^{(w)}$

Classical polylogarithms: $\frac{dLi_n(z)}{dz} = \frac{Li_{n-1}(z)}{z}\,, \qquad Li_1(z) = -\log(1-z)$

$$d\begin{pmatrix} \epsilon^2 Li_2(z) \\ \epsilon^2 Li_2(1-z) \\ \epsilon \log z \\ \epsilon \log(1-z) \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & 0 & 0 & -d\log z & 0 \\ 0 & 0 & -d\log(1-z) & 0 & 0 \\ 0 & 0 & 0 & 0 & d\log(z) \\ 0 & 0 & 0 & 0 & d\log(1-z) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \epsilon^2 Li_2(z) \\ \epsilon^2 Li_2(1-z) \\ \epsilon \log z \\ \epsilon \log(1-z) \\ 1 \end{pmatrix}$$

$$\vec{f}\left(\frac{1}{2}\right) = \begin{pmatrix} \epsilon^2\left(\frac{\pi^2}{12} - \frac{1}{2}\log^2 2\right) \\ \epsilon^2\left(\frac{\pi^2}{12} - \frac{1}{2}\log^2 2\right) \\ -\epsilon\log 2 \\ -\epsilon \log 2 \\ 1 \end{pmatrix}$$

$$Li_2(z) = -\left[1-z, z\right]_{1/2} + \log(2)\left[z\right]_{1/2} + \frac{\pi^2}{12} - \frac{1}{2}\log^2 2$$

$$\log(1-z) = \left[1-z\right]_{1/2} - \log(2)$$

b-quark mass effects

Neglecting bottom-quark mass effects overestimates the NLO total cross-section for $Wb\bar{b}$ production @ Tevatron by about 8% [Febres Cordero, Reina, Wackeroth 2006]

“The b-quark mass effects can impact the shape of the kinematic distributions in particular in phase space regions where the relevant kinematic observable is of the order of m_b. Apart from these regions, however, these effects can be approximated by rescaling the NLO cross section for m_b=0 with the ratio of LO cross sections for massive and massless bottom quarks.”

[Febres Cordero, Reina, Wackeroth 2009]

Reasonable evaluation time with basic **`Mathematica`** setup

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1)$$

$$p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1)$$

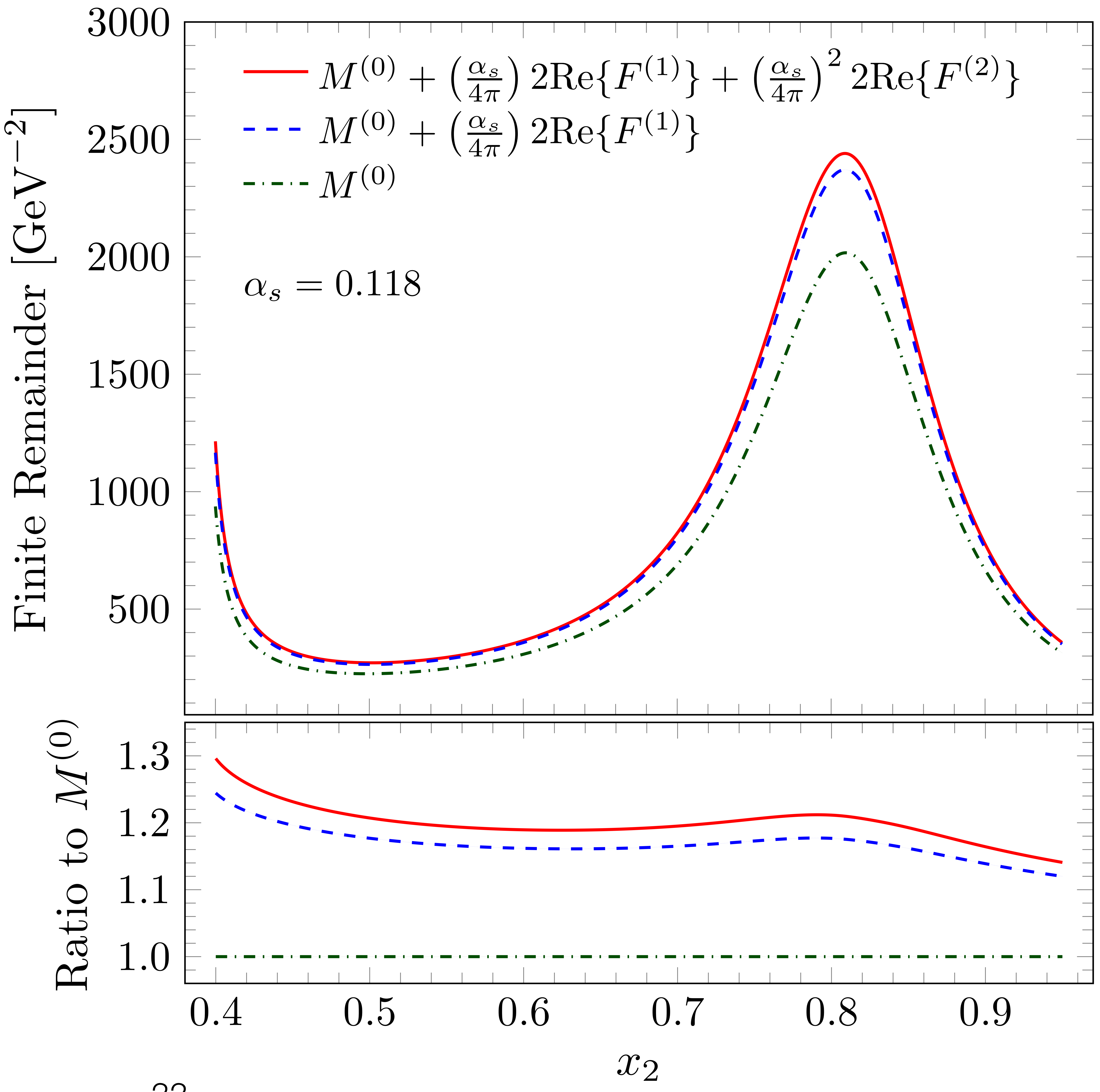
$$p_3 = x_1\frac{\sqrt{s}}{2}(1,1,0,0)\,,$$

$$p_4 = x_2\frac{\sqrt{s}}{2}(1,\cos\theta,-\sin\phi\sin\theta,-\cos\phi\sin\theta)\,,$$

$$p_5 = \sqrt{s}(1,0,0,0) - p_3 - p_4$$

$$s = 1\,,\; m_W^2 = 0.1\,,\; \phi = 0.1\,,\; x_1 = 0.6$$

1100 points → Average 260 s/point



“Tricks” to speed up the rational reconstruction

$$F^{(2)}(\{p\}) = \sum_i e_i\left(\{p\}\right)\ \mathbf{mon}_i\left(f_j^{(w)}\right)$$

- Determine linear relations among the rational coefficients + ansatz
- Reconstruct directly decomposed in partial fractions w.r.t. s_{23}

~ ×7 speed-up!

Univariate partial fractioning

"Black-box" evaluation

$$f(x,y) = \frac{N(x,y)}{\prod_{i=1}^{s} \ell_i^{e_i}(x,y)}$$

Known from univariate slice

$$d_N := \mathbf{deg}_y \left[N(x,y)\right]$$

$$d_i := \mathbf{deg}_y \left[\ell_i(x,y)\right]$$

$\Longrightarrow$ Ansatz (w.r.t. y): $$f(x,y) = \sum_{i=1}^{s} \sum_{j=1}^{e_i} \sum_{t=0}^{d_i-1} \frac{u_{ijt}(x)\, y^t}{\ell_i^j(x,y)} + r(x) + \sum_{h=1}^{d_N - \sum_{i=1}^{s} e_i d_i} v_h(x)\; y^h$$

Linear fit to reconstruct the unknown functions: $u_{ijt}(x)\,, R(x)\,, v_h(x)$