



# Storage Rings and Gravitational Waves

Jan. 17, 2022 @IAS-HEP2022

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Many thanks to F. Zimmermann for discussions.

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≈ Presented at SRGW2021  
<https://indico.cern.ch/event/982987/>  
<https://arxiv.org/abs/2105.00992>

# 1 Orbit deviation by gravitational wave

A celestial gravitational wave is a plane wave of tidal force. An object at distance  $X^\mu$  transverse to the wave from an origin receives an acceleration by gravitational radiation (GR) wave:

$$\frac{d^2 X_\mu}{dt^2} = \frac{1}{2} \ddot{h}_{\mu\nu} X^\nu, \quad (1)$$

where  $h_{\mu\nu}$  has the form

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2)$$

with two linear polarization components  $h_+$  and  $h_\times$ . For the time being, let us consider the case of linear polarization with  $h_+ = h$  and  $h_\times = 0$ .

# horizontal displacement of the beam particle

The acceleration can be expressed in terms of accelerators for a relativistic particle as:

$$\frac{dp_x}{ds} = \frac{1}{2c^2} \ddot{h} R \cos 2\theta, \quad (3)$$

where  $p_x \approx dx/ds$  is the horizontal (radial) momentum divided by the design momentum  $p_0$  of the ring. The coordinate  $s \approx ct$  is the path length along the ring. If the GR is a sinusoidal wave  $h \cos(\omega_{GR}t)$  with an angular frequency  $\omega_{GR} = 2\pi c/\lambda = ck$ , the above are written as

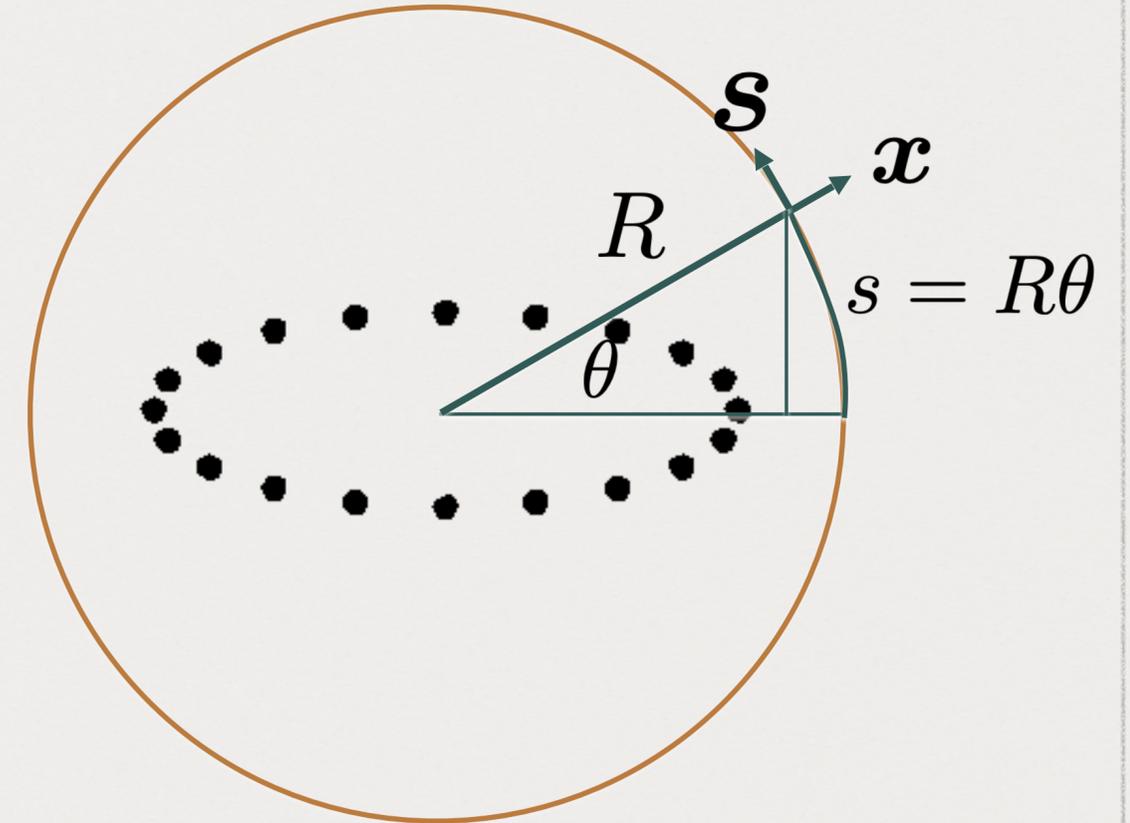
$$\frac{dp_x}{ds} = -\frac{k^2}{2} h R \cos 2\theta \cos(\omega_{GR}t). \quad (4)$$

Once we know  $dp_x/ds$ , the displacement  $x$  at an arbitrary location  $s$  in the beam line is obtained using  $R_{12}$ -component of the transfer matrix (Green's function) as:

$$x(s) = \int_0^s ds' R_{12}(s, s') \frac{dp_x}{ds}(s') \quad (5)$$

$$= \int_0^s ds' \sqrt{\beta_x(s)\beta_x(s')} \sin(\psi_x(s) - \psi_x(s')) \frac{dp_x}{ds}(s'), \quad (6)$$

where  $\beta_x(s)$  and  $\psi_x(s)$  are the horizontal  $\beta$ -function and the phase advance of the beam line.



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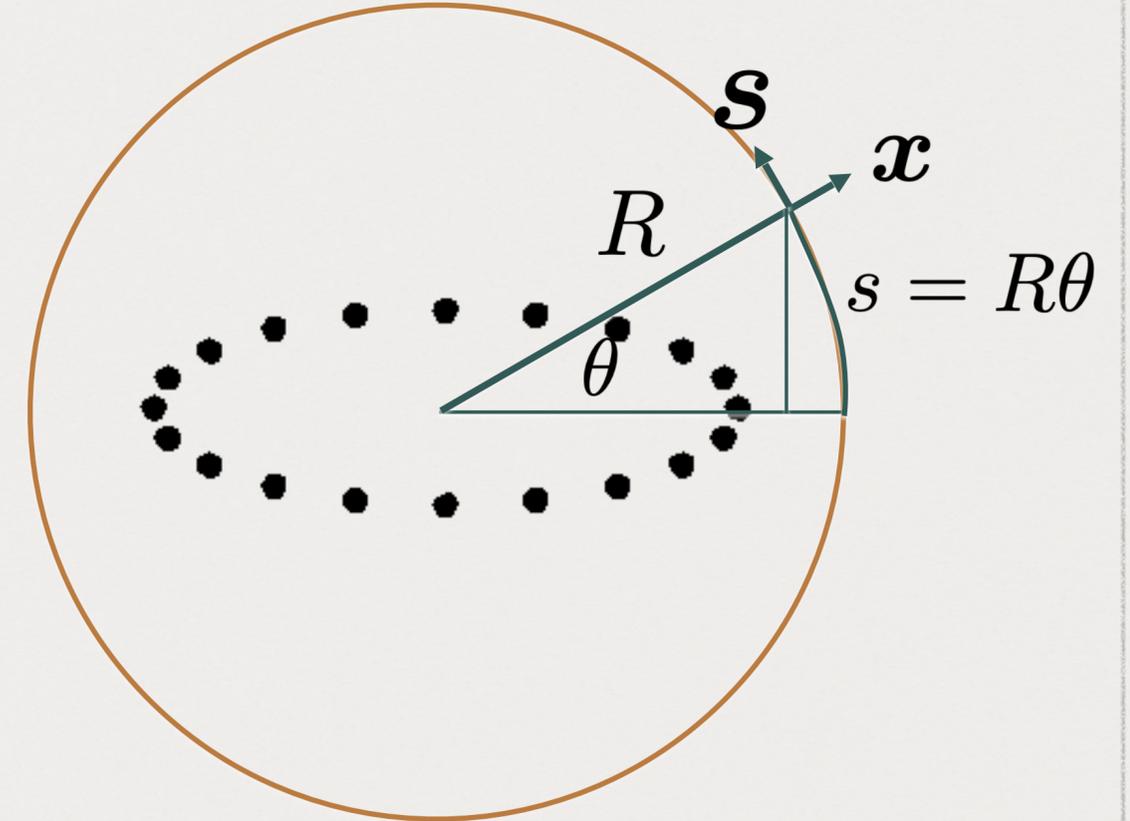
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# Resonance

Let us assume the ring is a perfect circle with a radius  $R$ , and uniformly focused in the horizontal plane, with the  $\beta$ -function  $\beta_x$ . If the GR incidents perpendicular to the ring plane, the horizontal orbit deviation at an angle  $\theta = s/R$  generated by the GR force (4) is written as

$$\Delta x = -\frac{k^2 R^2}{2} h \int_0^\theta \beta_x \sin(\nu_x(\theta - \theta')) \cos(kR\theta') \cos 2\theta' d\theta', \quad (7)$$

where  $\nu_x = R/\beta_x$  is the horizontal betatron tune.

Then the orbit can resonate to the GR if

$$kR \pm 2 = \nu_x \quad (8)$$

resulting an increase the horizontal amplitude by

$$\Delta x \approx \pi k^2 R^2 \beta_x h \approx \pi \nu_x R h \quad (9)$$

per every turn. It is 0.2 fm for  $h = 10^{-22}$ ,  $R = 5$  km and  $\nu_x = 130$  in the case of the model ring shown later.

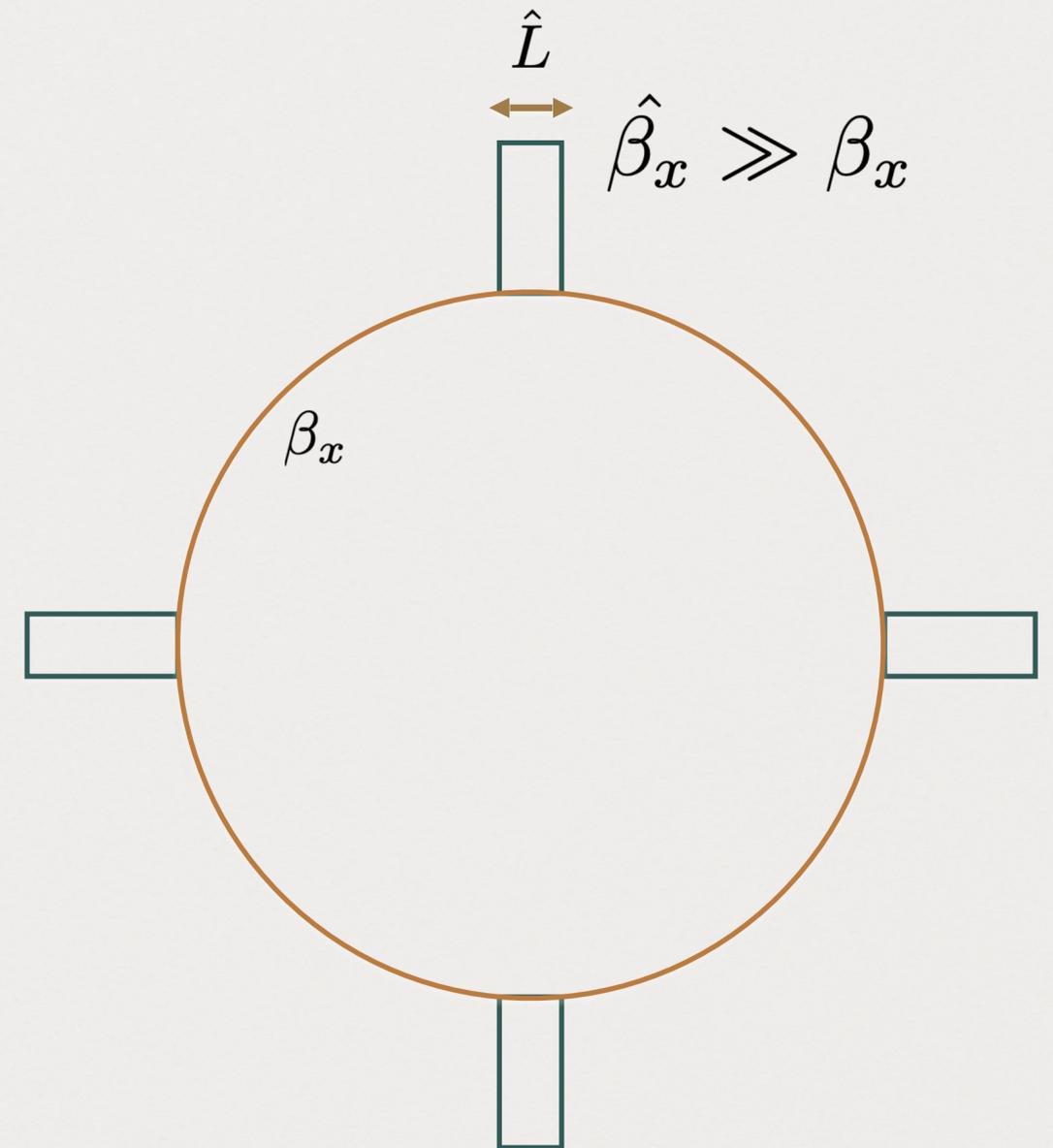


## 2 Enhancement of the response: Beam Antenna?

As we have seen, even at the exact resonance, the response of the betatron motion to GR seems pretty weak. Is there any way to enhance the response? If we look at Eq. 7, the amplitude of the oscillation is proportional to  $\beta_x$ . So what about having a special section in the ring with a very large  $\beta_x$ ? For instance, consider a ring with the circumference  $C = 30$  km and the average  $\beta_x = 50$  m in the ring. If we have four  $\hat{L} = 1$  km sections with  $\hat{\beta}_x = 1000$  km, the response of  $dp_x/ds$  to the GR from these special sections is much larger than that from the arc by a factor of

$$\sqrt{\frac{\hat{\beta}_x}{\beta_x} \frac{4\hat{L}}{C}} \approx 19. \quad (10)$$

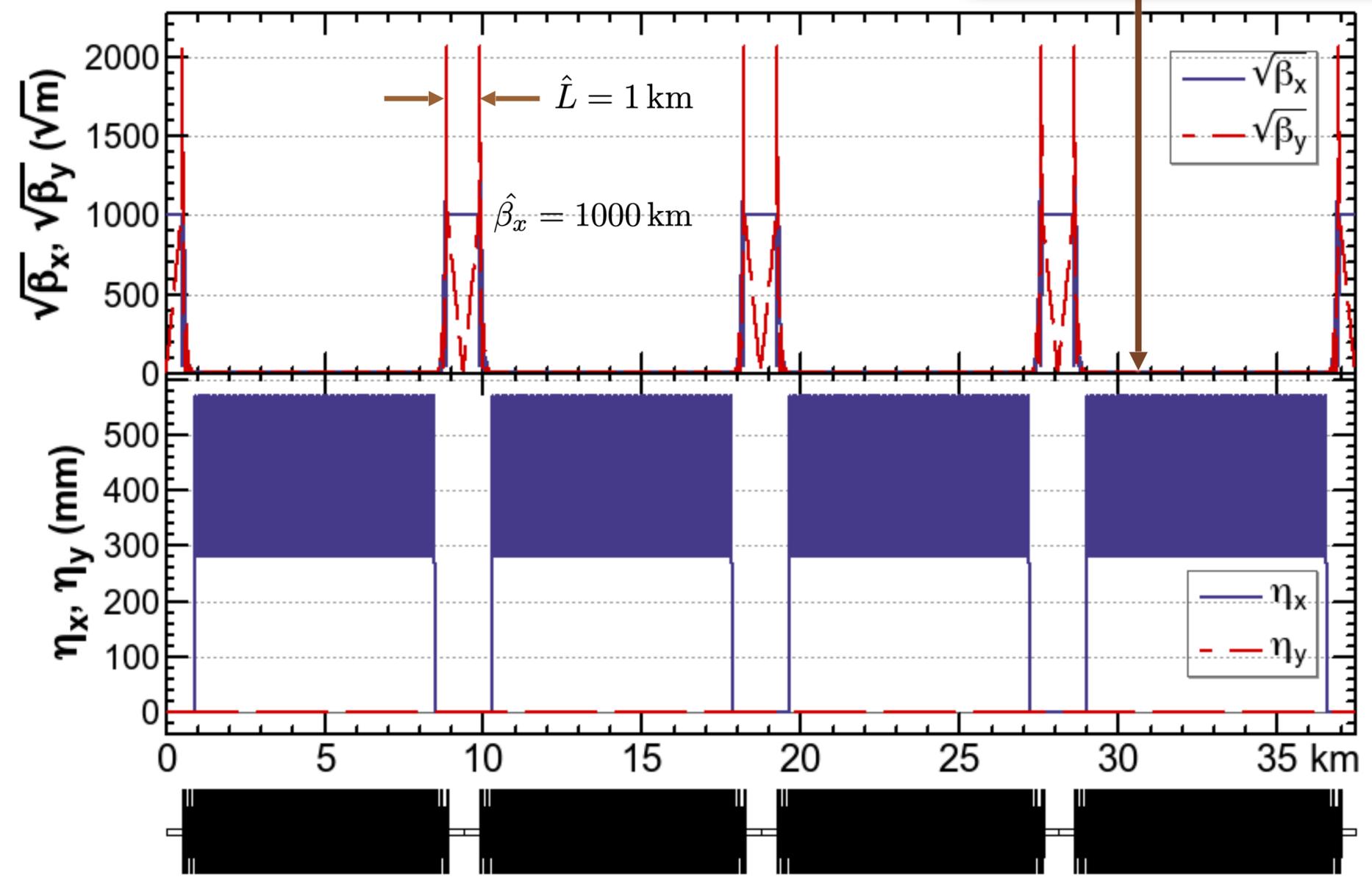
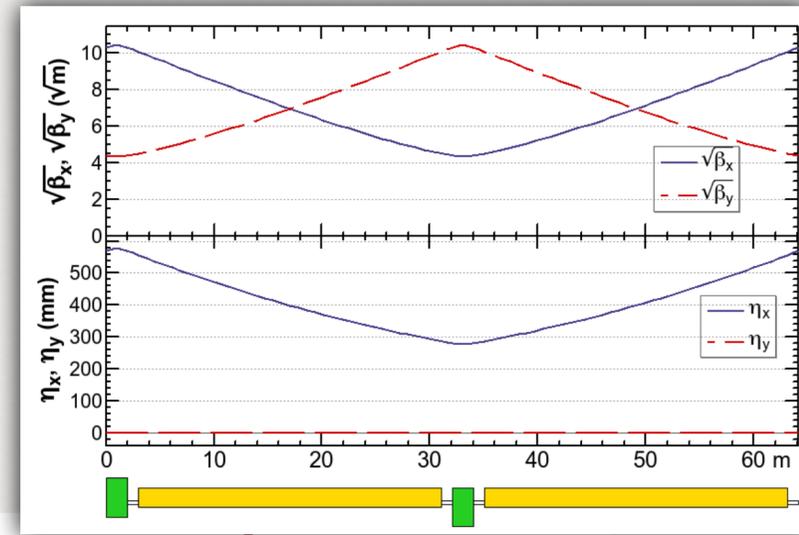
The response to the horizontal orbit deviation at the antenna is enhanced by a factor of 2700, multiplied by another  $\sqrt{\hat{\beta}_x/\beta_x}$ .



# An example ring optics for a GR antenna



$$\overline{\beta_x} \approx 50 \text{ m}$$



Parameters		
Particles		$p$
Beam energy	TeV	1.0
Circumference	km	37.4
Length of an IR, $\hat{L}$	km	1.0
Number of IRs		4
$\beta_x$ at the IR, $\hat{\beta}_x$	km	1000
$\beta_x$ ave. in the arc	m	50
Betatron tunes, $\nu_x/\nu_y$		130.8/131.3
SR damping time in $x$	s	73600
SR equiv. emittance	fm	0.198

# Response of "antenna"

In the case of such an "antenna", the response to GR in the arc is small compared to the  $\hat{\beta}_x$  sections, so we ignore below. Thus Eq. 7 is reduced to

$$\Delta\hat{x} = -\frac{k^2 R \hat{L}}{2} h \sum_{m=0}^n \hat{\beta}_x \sin(2(n-m)\pi \Delta\nu_x) \cos\left(\frac{m\pi}{2} kR\right) \cos m\pi, \quad (11)$$

where  $\Delta\mu_x$  is the phase advance between two  $\hat{L}$  sections, and we have used  $\theta = m\pi/2$  at the  $m$ -th  $\hat{L}$  section.

Equation 11 has resonances at

$$2\Delta\nu_x = \pm kR + N, \quad (12)$$

where  $N$  is an integer. At the resonance, the horizontal amplitude accumulates by

$$\Delta\hat{x} \approx -\frac{k^2 R \hat{L}}{2} \hat{\beta}_x h \quad (13)$$

per turn.

If we set  $\omega_{\text{GR}} = 2\pi \times 1 \text{ kHz}$  and with parameters above, Eq. 13 gives  $\Delta\hat{x} \approx 1000h \text{ m}$ , which seems still too small to detect. As the amplitude is proportional to  $\omega_{\text{GR}}^2$ , there may exist a possibility to detect a GR in  $\omega_{\text{GR}} = 2\pi \times 1 \text{ MHz}$  range, giving  $\Delta\hat{x} \approx 10^9 h \text{ m}$  (0.1 pm for  $h = 10^{-22}$ ), if there are sources.



### 3 Noise sources

Even if the signal is enhanced by the “antenna”, the problems exist on the noises in the betatron oscillation, which include:

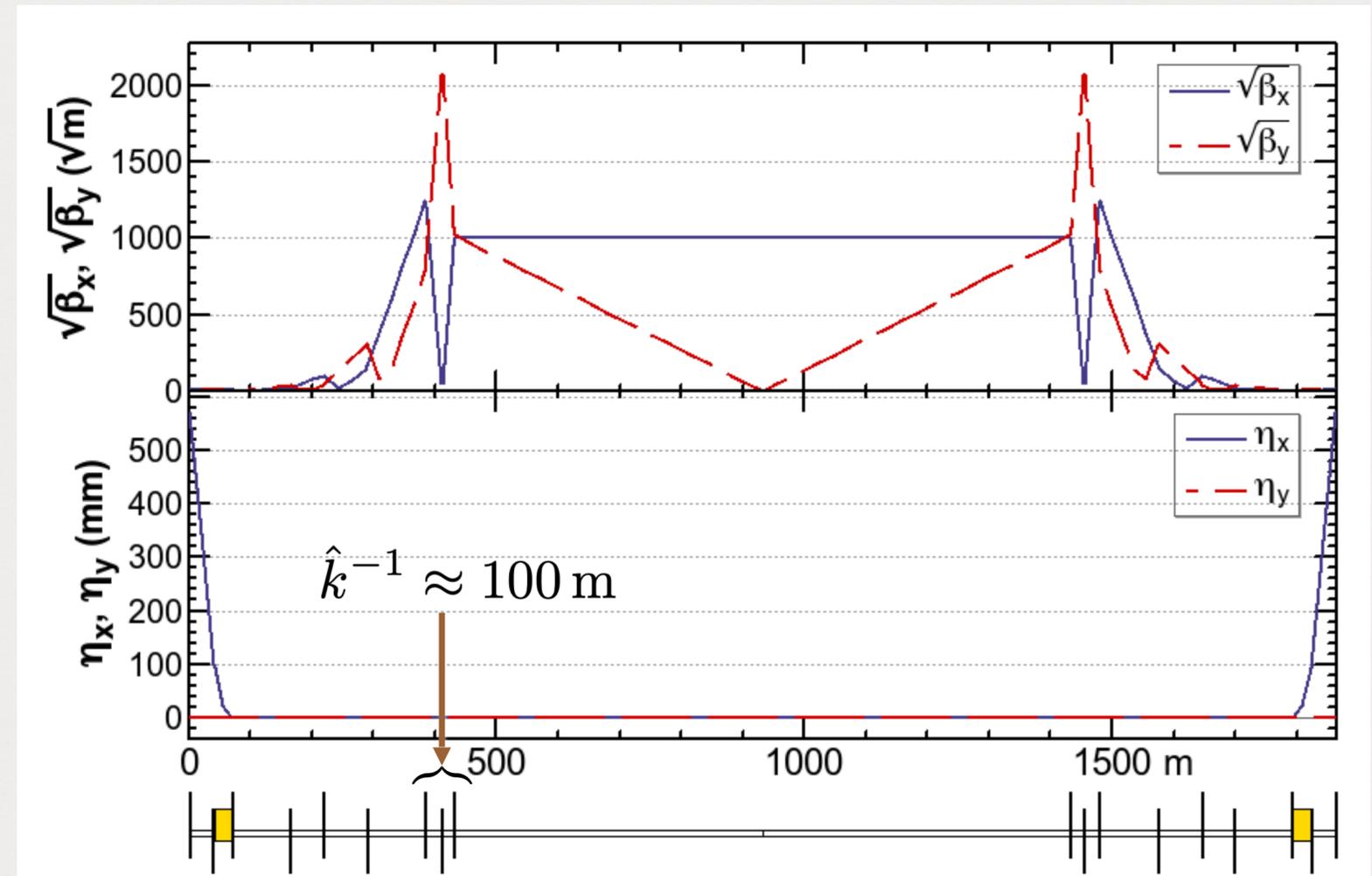
- The excitation by the vibration of the quads, esp. on the both sides of the  $\hat{L}$  sections. The triplet-like quad at each side has the focal length  $\hat{k}^{-1} \approx 100$  m, then its horizontal vibration by  $\Delta x_Q$  excites the betatron oscillation of

$$\Delta x_n \approx \sqrt{8\hat{\beta}\hat{k}}\Delta x_Q \approx 3 \times 10^4 \Delta x_Q, \quad (14)$$

as there are 8 units of such triplets in the ring. The minimum magnitude of vibration  $\Delta x_Q$  is given by the *thermal vibration* of the triplet:

$$\Delta x_Q^2 \approx \frac{4k_B T \omega_Q \Delta f}{M_Q \omega_{GR}^2 Q_M} \quad (15)$$

where  $M_Q$ ,  $\omega_Q$ ,  $Q_M$ ,  $T$ ,  $\Delta f$ , and  $k_B$  are the mass, eigen angular frequency, and the  $Q_M$ -value of the triplet, the temperature, the detection bandwidth, and the Boltzmann constant, respectively. If we plugin  $M_Q=1000$  kg,  $\omega_Q/2\pi = 10$  Hz,  $Q_M = 10$ ,  $T = 4.2$  K,  $\Delta f = 100$  kHz, obtain  $\Delta x_Q = 6$  pm, which generates the betatron amplitude  $\Delta x_n = 0.17 \mu\text{m}$  through Eq. 14.



# More noise sources

- The vibration may be reduced by making  $Q_M$  higher and lowering  $\omega_Q$ . This is somewhat similar to the mirror suspension system for the GR laser interferometers. Also the detection bandwidth is a subject to be refined.
- Similar effects arise from all quadrupoles in the entire ring.
- The emittance of the beam may accompany an intrinsic fluctuation of the orbit.:

$$\Delta \hat{x}_s = \sqrt{\hat{\beta}_x \varepsilon_x / N_p}, \quad (16)$$

where  $N_p$  is the number of particles in the bunch. If this beam has the same invariant emittance as the LHC,  $\gamma \varepsilon_x = 4 \mu\text{m}$  and the beam energy and  $N_p$  are 1 TeV and  $10^{11}$ , respectively, the orbit fluctuation becomes  $\Delta \hat{x}_s \approx 0.19 \mu\text{m}$ .

- As the LHC's emittance above seems too large for our purpose, we need much smaller one. One and the easiest way to use the synchrotron radiation. This ring has the damping time of about 20 hours and the equilibrium emittance 0.2 fm, which is  $5 \times 10^{-8}$  smaller than the LHC's. The corresponding orbit fluctuation is  $\Delta \hat{x}_s \approx 40 \text{ pm}$ .
- if we need the SR damping, an RF acceleration is inevitable. It will be another noise source for the beam, esp. if there are residual dispersions at the RF cavities.



# Global Antennas?

## 4 Two (or more) storage rings across the globe

As we discussed above, the orbit response to GR is proportional to the ring radius. Actually the beams are not necessarily in the same ring: if we place two identical storage rings on the opposite side of the earth, we should be able to utilize their distance  $2R \approx 2R_{\bullet}$  as

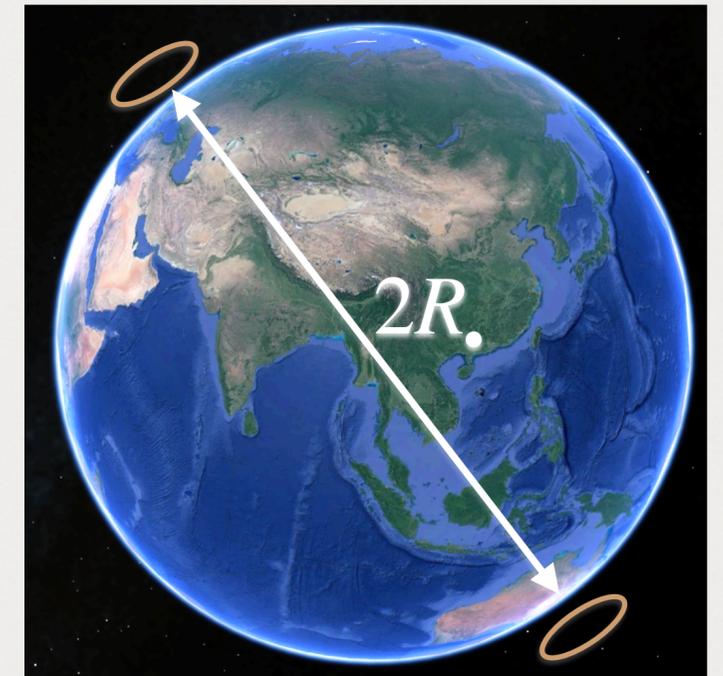
$$\Delta\hat{y} \approx -\frac{k^2 R_{\bullet} \hat{L}}{2} \hat{\beta}_y h. \quad (17)$$

Here the oscillation is in the vertical plane, if two rings are placed at the opposite side of the earth. Then the increment of oscillation amplitude per turn becomes

$$\Delta\hat{y} = 1.4 \times 10^6 h, \quad (18)$$

with  $k = 2\pi \times 1 \text{ kHz}/c$ ,  $\hat{\beta}_y = 1000 \text{ km}$ ,  $R_{\bullet} = 6400 \text{ km}$ ,  $\hat{L} = 1000 \text{ m}$ . So the amplitude seems still not enough for a 1 kHz source, but for 1 MHz, it might be well in a detectable range.

Note that in this case, each ring can be as small as the antenna section  $\hat{L}$ .



Google Earth Pro

## 5 Summary

- The beam in a storage ring can respond to a wave of gravitational radiation, somewhat similar to the Weber bar.
- The betatron motion of the beam in a ring can resonate to the GR.
- A special beam optics “beam antenna” may enhance the sensitivity,
- There will be many noise sources to overcome: thermal motion of quadrupoles, beam fluctuation due to emittance, synchrotron radiation and the acceleration, etc.
- Placing two (or more) storage rings across the Earth can improve the sensitivity.

