New exact solutions of non relativistic, viscous hydrodynamics

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ZIMÁNYI SCHOOL 2021, BUDAPEST

6/12/2021

PARTIALLY SUPPORTED BY:
NKFIH K-133046, FK-123842, FK-123959
EFOP 3.6.1-16-2016-00001
THOR, COST Action CA15213

Introduction

New, exact solutions of relativistic viscous hydrodynamics have been found recently

T. Csörgő, G. K.: arXiv:2003.08859

→ Spherically symmetric Hubble-flow: great amount of freedom of dissipative coefficients

→ Perfect fluid attractor: how to extract the effect of bulk viscosity in final state measurments?

→ Indirect description of experimental data has been succesfully done

→ The non relativistic limit leads new exact solution of viscous, non relativistic hydro

Motivation of this work

- The same effects can be understood in a much simpler formalism
- Perfect fluid attractor: is it a general property of hydro?
- ***The basic equations of non relativistic, viscous hydro are fully clarified***

T. Csörgő, G. K.: manuscript in preparation

Non relativistic, viscous hydrodynamics

Local conservation of the particle number, energy and momentum:

$$\partial_t n + \nabla(n\vec{v}) = 0$$

$$\partial_t \varepsilon + \nabla(\varepsilon\vec{v}) + p\nabla\vec{v} = \zeta(\nabla\vec{v})^2 + 2\eta\left[\mathrm{Tr}\left(D^2\right) - \frac{1}{3}(\nabla\vec{v})^2\right]$$

$$(\varepsilon + p)(\partial_t + \vec{v}\nabla)\vec{v} + \nabla p = \zeta\nabla(\nabla\vec{v}) + \eta\left[\Delta\vec{v} + \frac{1}{3}\nabla(\nabla\vec{v})\right]$$

To close the equation system:

EoS: ε=κp

Balance equation of entropy:

$$\partial_t \sigma + \nabla(\sigma\vec{v}) = \frac{\zeta}{T}(\nabla\vec{v})^2 + \frac{2\eta}{T}\left[\mathrm{Tr}\left(D^2\right) - \frac{1}{3}(\nabla\vec{v})^2\right] \geq 0$$

ζ: bulk viscosity
η: shear viscosity

where: $D_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i}\right)$

Interpretation of κ

Relationship to the speed of sound

→ κ is constant: $$c_s^2(T) = \left(1 + \kappa^{-1}\right) \frac{T}{m}$$

→ κ is temperature dependent (lQCD): $$c_s^2(T) = \left[1 + \left(\kappa + T\frac{d\kappa}{dT}\right)^{-1}\right] \frac{T}{m}$$

Interpretation of κ

Relationship to the speed of sound

→ κ is constant:

$$c_s^2(T) = \overbrace{\left(1 + \kappa^{-1}\right)}^{\gamma\text{: adiabatic index}} \frac{T}{m}$$

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$$c_s^2(T) = \underbrace{\left[1 + \left(\kappa + T\frac{d\kappa}{dT}\right)^{-1}\right]}_{\gamma(T)\text{: temperature dependent adiabatic index}} \frac{T}{m}$$

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Relationship to the heat capacities

→ κ is constant:

$$\gamma = \frac{C_p}{C_V} = 1+\kappa^{-1}$$

→ κ is temperature dependent (lQCD):

$$\gamma(T) = \frac{C_p(T)}{C_V(T)} = 1+\left(\kappa + T\frac{d\kappa}{dT}\right)^{-1}$$

Interpretation of κ

Even if the mass is temperature dependent, the formulae remain unchanged:

$$c_s^2 = \frac{C_p(T)}{C_V(T)} \frac{T}{m(T)} = \left[1 + \left(\kappa + T\frac{d\kappa}{dT}\right)^{-1}\right] \frac{T}{m(T)}$$

Conjecture for multicomponent hadronic matter:

$$c_s^2 = \frac{C_p(T)}{C_V(T)} \frac{T}{\langle m(T)\rangle}$$

$$\langle m(T)\rangle = \frac{\sum_i n_i m_i(T)}{\sum_i n_i}$$

This conjecture based on the results of arXiv:1610.02197

Spherically symmetric fireball

Hubble flow: $$\vec{v} = \frac{\dot{R}}{R}(r_x, r_y, r_z)$$

The scale of the fireball: $R(t)$

Self similarity $$(\partial_t + \vec{v}\nabla)s = 0$$

and scale variable: $$s = \frac{r^2}{R^2}$$

Spheroidally symmetric, rotating fireball

Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$

Hubble flow: $v_H(\vec{r}, t) = \left(\frac{\dot{R}}{R} r_x, \frac{\dot{Y}}{Y} r_y, \frac{\dot{R}}{R} r_z \right)$

Rotational term: $v_{rot}(\vec{r}, t) = \omega (r_z, 0, -r_x)$

The scales of the fireball: $R(t),\ Y(t)$

Angular velocity: $\dot{\vartheta} = \omega(t)$

Scale variable: $s = \frac{r_x^2 + r_z^2}{R^2} + \frac{r_y^2}{Y^2}$

M. I. Nagy, T. Csörgő: arXiv:1309.4390
M. I. Nagy, T. Csörgő: arXiv:1606.09160
T. Csörgő, M. I. Nagy, I. F. Barna: arXiv:1511.02593

Ellipsoidally symmetric, rotating fireball

Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$

Hubble flow:
$$v_H(\vec{r},t) = \begin{pmatrix} \left(\frac{\dot{X}}{X}\cos^2\vartheta + \frac{\dot{Z}}{Z}\sin^2\vartheta\right) r_x \\ \frac{\dot{Y}}{Y} r_y \\ \left(\frac{\dot{X}}{X}\sin^2\vartheta + \frac{\dot{Z}}{Z}\cos^2\vartheta\right) r_z \end{pmatrix} + \left(\frac{\dot{Z}}{Z} - \frac{\dot{X}}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix} r_z \\ 0 \\ r_x \end{pmatrix}$$

Rotational term:
$$v_{rot}(\vec{r},t) = \dot{\vartheta}\begin{pmatrix} r_z \\ 0 \\ -r_x \end{pmatrix} + \dot{\vartheta}\begin{pmatrix} \left(\frac{X}{Z}\cos^2\vartheta + \frac{Z}{X}\sin^2\vartheta\right) r_z \\ 0 \\ -\left(\frac{X}{Z}\sin^2\vartheta + \frac{Z}{X}\cos^2\vartheta\right) r_x \end{pmatrix} + \dot{\vartheta}\left(\frac{X}{Z} - \frac{Z}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix} r_x \\ 0 \\ -r_z \end{pmatrix}$$

The scales of the fireball: $X(t),\ Y(t),\ Z(t)$ Average transverse scale: $R = \frac{X+Z}{2}$

Angular velocity:
$$\dot{\vartheta} = \frac{\omega(t)}{2} = \frac{\omega_0}{2}\frac{R_0^2}{R(t)^2}$$

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Scale variable:
$$s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2} + \left(\frac{1}{Z^2} - \frac{1}{X^2}\right)\left[(r_x^2 - r_z^2)\sin^2\vartheta + r_x r_z \sin(2\vartheta)\right]$$

General form of the new solutions

The temperature and particle density can be given in a „symmetry independent” form:

$$n(\vec{r},t) = n_0 f_T^{1/\kappa}(t)\mathcal{V}(s)$$

$$T(\vec{r},t) = T_0 f_T(t)\mathcal{T}(s)$$

$$p(\vec{r},t) = p_0 f_T^{1+\frac{1}{\kappa}}(t)\mathcal{V}(s)\mathcal{T}(s)$$

$$\mathcal{V}(s) = \frac{1}{\mathcal{T}(s)}\exp\left(-\frac{C_E}{2}\int_0^s \frac{du}{\mathcal{T}(u)}\right)$$

These expressions are valid only for ***constant κ***

The $f_T(t)$ function carries the symmetry and includes the dissipative effects

Spherically symmetric, dissipative fireball solution
- with homogeneous pressure -

If the pressure is homogeneous, then $v(s)\tau(s)=1$, $C_E=0$ so the Euler equation and ζ are:

$$\ddot{R}=0 \longrightarrow \dot{R}=\text{const.} \longrightarrow R=\dot{R}t+R_0\sim\dot{R}t$$

$$\zeta\equiv\zeta(p(t))$$

With that, the energy conservation becomes:

$$\kappa\partial_t\ln(f_T)+\frac{d}{t}=\frac{\zeta(p(t))}{p(t)}\frac{d^2}{t^2}$$

If the bulk viscosity is linear in pressure: $\zeta(p(t))=\zeta_0\dfrac{p(t)}{p_0}$

$$p(t)=p_0\left(\frac{t_0}{t}\right)^{d\left(1+\frac{1}{\kappa}\right)}\exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\left[1-\frac{t_0}{t}\right]\right)$$

$$T(t,s)=T_0\left(\frac{t_0}{t}\right)^{\frac{d}{\kappa}}\exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\left[1-\frac{t_0}{t}\right]\right)\mathcal{T}(s)$$

$$\xrightarrow{\tau_0 \ll \tau}$$

Late time approximation:

perfect fluid asymptote

$$T(t)\sim T_A\left(\frac{t_0}{t}\right)^{\frac{d}{\kappa}}\mathcal{T}(s)$$

$$p(t)\sim p_A\left(\frac{t_0}{t}\right)^{d\left(1+\frac{1}{\kappa}\right)}$$

$$T_A=T_0\exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\right)$$

$$p_A=p_0\exp\left(\frac{d^2\zeta_0}{\kappa p_0 t_0}\right)$$

Spherically symmetric, dissipative fireball solution
- with inhomogeneous pressure -

If the pressure is inhomogeneous, then ζ has to be linear in pressure: $\zeta(t,s) = \zeta_0 \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t) \left(\frac{R_0}{R} \right)^{\frac{d}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{g}_T}{g_T} = \underbrace{\frac{\zeta_0 d^2}{\kappa p_0} \left(\frac{\dot{R}}{R} \right)^2}_{\text{effect of bulk viscosity}}$$

The Euler equation is:

$$R\ddot{R} = C_E \frac{T_0}{m} \left(\frac{R_0}{R} \right)^{\frac{d}{\kappa}} \underbrace{g_T(t)}_{\text{effect of bulk viscosity}}$$

Spheroidally symmetric, dissipative fireball solution
- with inhomogeneous pressure -

If the pressure is inhomogeneous, then ζ and η has to be linear in pressure: $\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t)\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot g_T}{g_T} = \underbrace{\frac{\zeta_0}{\kappa p_0}\left(\frac{2\dot R}{R} + \frac{\dot Y}{Y}\right)^2}_{\text{effect of bulk viscosity}} + \underbrace{\frac{2\eta_0}{\kappa p_0}\left[\frac{2\dot R^2}{R^2} + \frac{\dot Y^2}{Y^2} - \frac{1}{3}\left(\frac{2\dot R}{R} + \frac{\dot Y}{Y}\right)^2\right]}_{\text{effect of shear viscosity}}$$

The Euler equation is:

$$R\ddot R = Y\ddot Y = C_E \frac{T_0}{m}\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa}} \underbrace{g_T(t)}_{\text{effect of bulk and shear viscosity}}$$

Spheroidally symmetric, dissipative fireball solution
- with inhomogeneous pressure and rotation -

If the pressure is inhomogeneous, then ζ and η has to be linear in pressure: $\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t)\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{g}_T}{g_T} = \underbrace{\frac{\zeta_0}{\kappa p_0}\left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y}\right)^2}_{\text{effect of bulk viscosity}} + \underbrace{\frac{2\eta_0}{\kappa p_0}\left[\frac{2\dot{R}^2}{R^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{3}\left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y}\right)^2\right]}_{\text{effect of shear viscosity}}$$

The Euler equation is:

$$R\ddot{R} - \underbrace{R^2\omega^2}_{\text{effect of rotation}} = Y\ddot{Y} = C_E \frac{T_0}{m}\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa}} \underbrace{g_T(t)}_{\text{effect of bulk and shear viscosity}}$$

Ellipsoidally symmetric, dissipative fireball solution
- with inhomogeneous pressure and rotation -

Pressure is inhomogeneous → ζ and η are linear in pressure: $\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$

Assumption:

$$f_T(t) = g_T(t)\left(\frac{X_0 Y_0 Z_0}{XYZ}\right)^{\frac{1}{\kappa}}$$

With that, the energy conservation becomes:

$$\frac{\dot{g}_T}{g_T} = \underbrace{\frac{\zeta_0}{\kappa p_0}\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}\right)^2}_{\text{effect of bulk viscosity}} + \overbrace{\underbrace{\frac{2\eta_0}{\kappa p_0}\left[\frac{\dot{X}^2}{X^2}+\frac{\dot{Y}^2}{Y^2}+\frac{\dot{Z}^2}{Z^2}-\frac{1}{3}\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}\right)^2\right]}_{\text{the only effect of shear viscosity for spheroidally symmetric fireball}} + \underbrace{\frac{\eta_0 {\omega_0}^2}{4\kappa p_0}\frac{(X_0+Z_0)^4}{(X+Z)^4}\left(\frac{X}{Z}-\frac{Z}{X}\right)^2}_{\text{effect of rotation, if } \eta\neq 0}}^{\text{effect of shear viscosity for ellipsoidally symmetric fireball}}$$

The Euler equation is (where $R=(X+Z)/2$):

$$X\left(\ddot{X}-\underbrace{R\omega^2}_{\text{effect of rotation}}\right) = Y\ddot{Y} = Z\left(\ddot{Z}-\underbrace{R\omega^2}_{\text{effect of rotation}}\right) = C_E\frac{T_0}{m}\left(\frac{X_0Y_0Z_0}{XYZ}\right)^{\frac{1}{\kappa}}\underbrace{g_T(t)}_{\text{effect of rotation, bulk and shear viscosity}}$$

Ellipsoidally symmetric, dissipative fireball solution
- with inhomogeneous pressure, rotation, and *T* dependent *κ(T)* -

Pressure is inhomogeneous → ζ and η are linear in pressure: $$\frac{\zeta(t,s)}{\zeta_0}=\frac{\eta(t,s)}{\eta_0}=\frac{p(t,s)}{p_0}$$

No assumption for $f_T(t)$ and we define the scale volume:

$$T(\vec{r},t)=T_0 f_T(t)\mathcal{T}(s) \qquad \cancel{f_T(t)=g_T(t)\left(\frac{X_0Y_0Z_0}{XYZ}\right)^{\frac{1}{\kappa}}} \qquad V\equiv V(t)=(2\pi)^{3/2}XYZ$$

The energy conservation becomes:

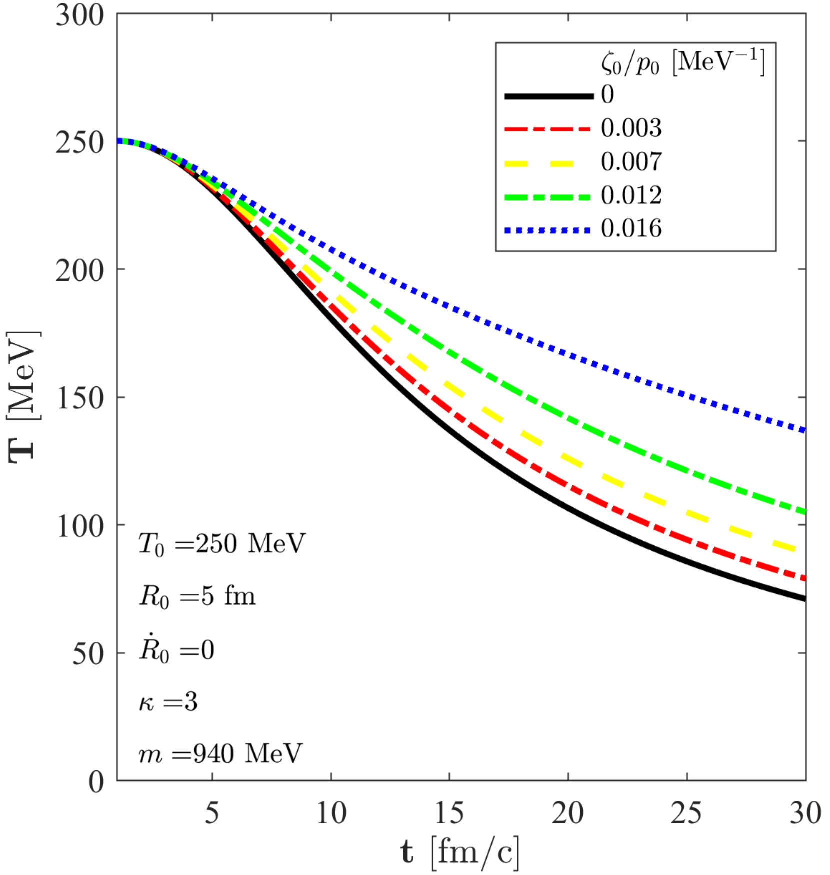
effect of shear viscosity for ellipsoidally symmetric fireball

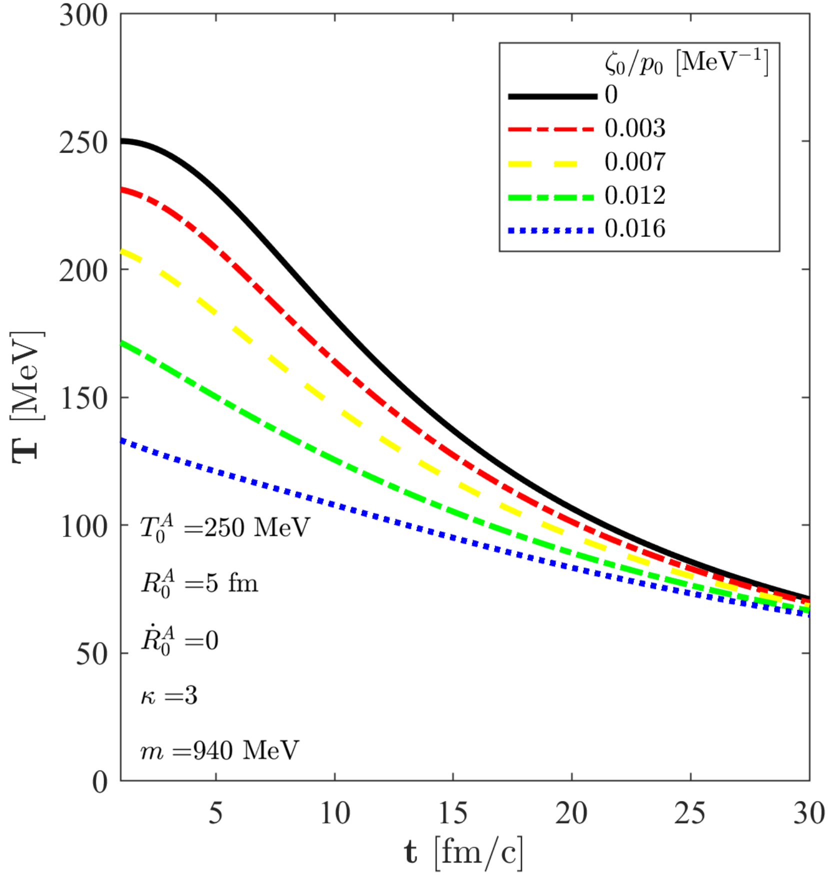
$$\left(\frac{1}{\gamma(T)-1}\right)\frac{\dot{f}_T}{f_T}+\frac{\dot{V}}{V}=\underbrace{\frac{\zeta_0}{p_0}\left(\frac{\dot{V}}{V}\right)^2}_{\text{effect of bulk viscosity}}+\underbrace{\frac{2\eta_0}{p_0}\left[\frac{\dot{X}^2}{X^2}+\frac{\dot{Y}^2}{Y^2}+\frac{\dot{Z}^2}{Z^2}-\frac{1}{3}\left(\frac{\dot{V}}{V}\right)^2\right]}_{\text{the only effect of shear viscosity for spheroidally symmetric fireball}}+\underbrace{\frac{\eta_0{\omega_0}^2}{4p_0}\frac{(X_0+Z_0)^4}{(X+Z)^4}\left(\frac{X}{Z}-\frac{Z}{X}\right)^2}_{\text{effect of rotation, if }\eta\neq 0}$$

The Euler equation:

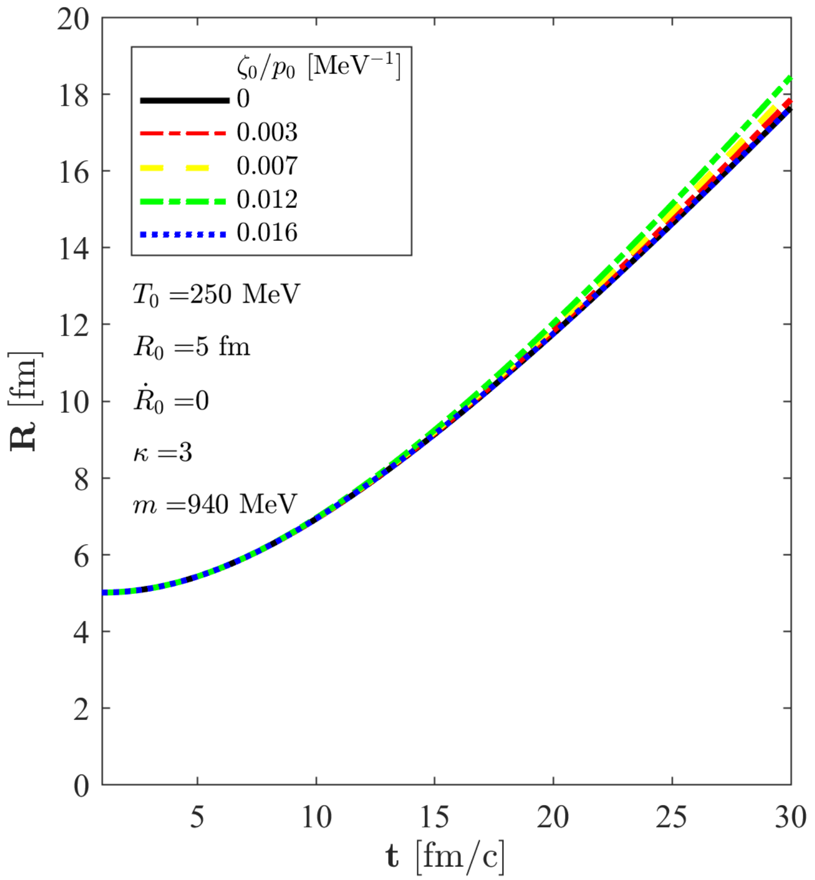
$$X\left(\ddot{X}-\underbrace{R\omega^2}_{\text{effect of rotation}}\right)=Y\ddot{Y}=Z\left(\ddot{Z}-\underbrace{R\omega^2}_{\text{effect of rotation}}\right)=C_E\frac{T_0}{m}\underbrace{f_T(t)}_{\text{effect of rotation, bulk and shear viscosity}}$$

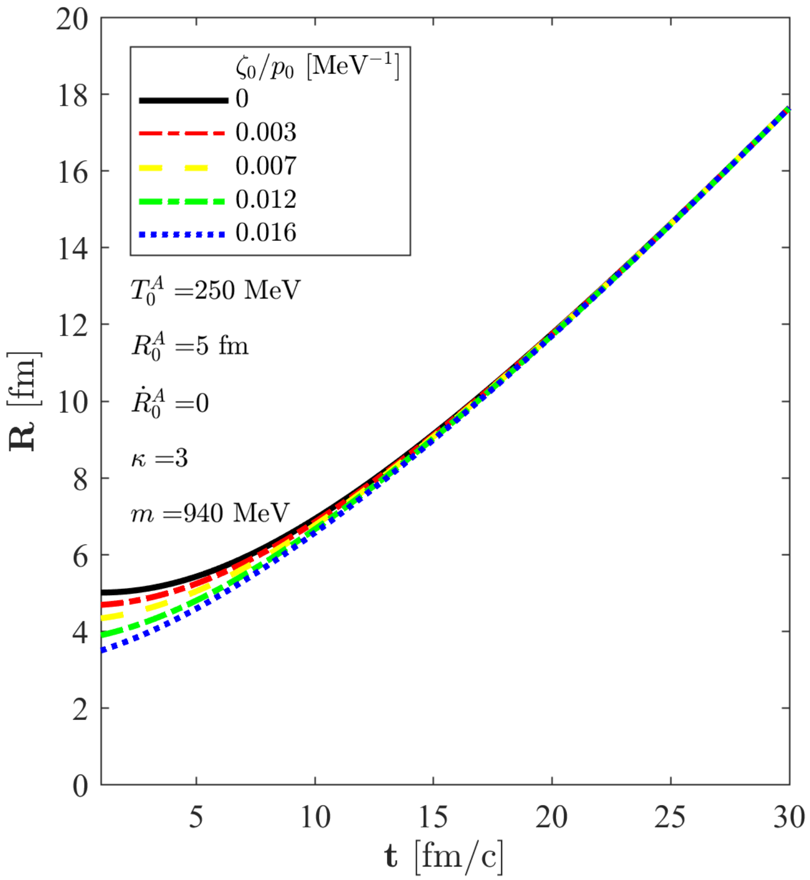
Spherically symmetric, dissipative fireball solution - with inhomogeneous pressure -



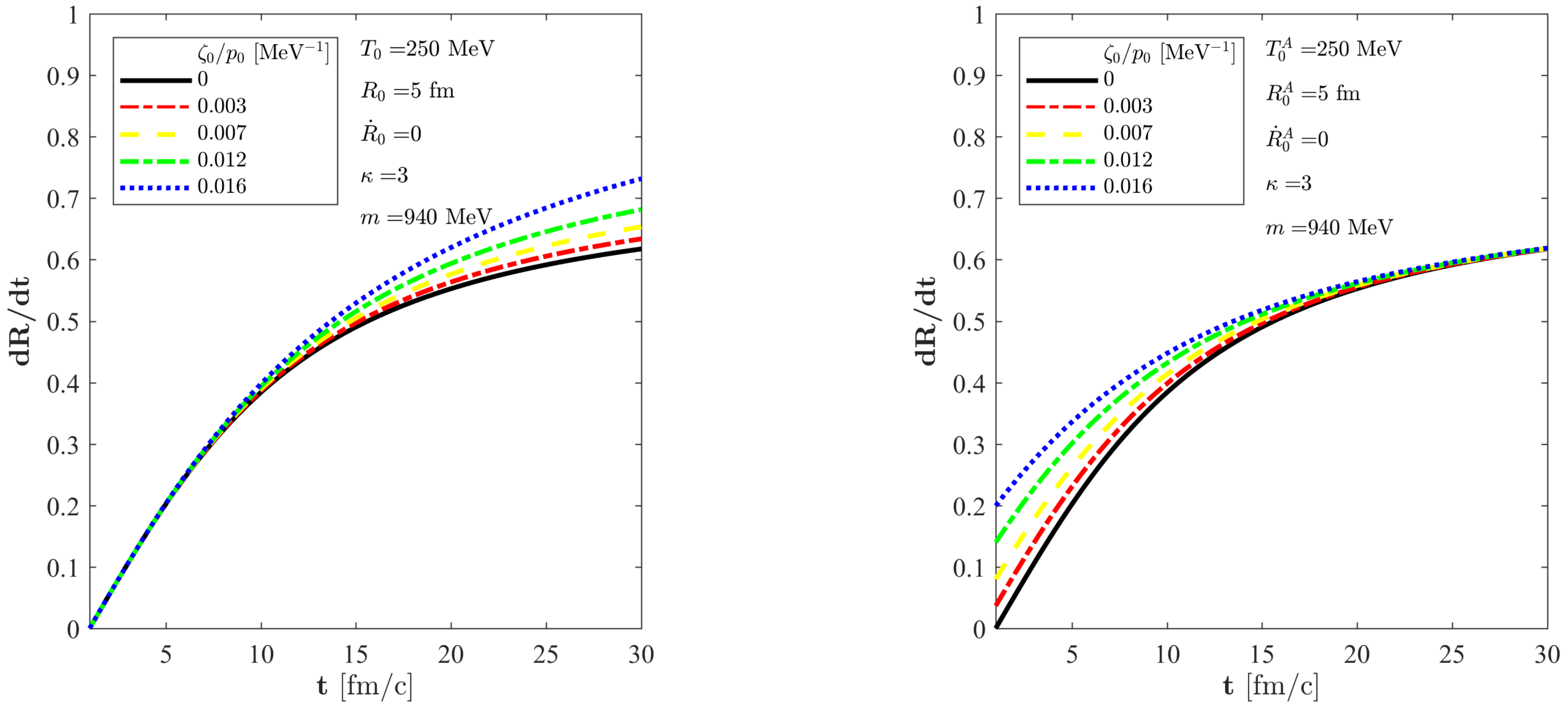


Spherically symmetric, dissipative fireball solution
- with inhomogeneous pressure -





Spherically symmetric, dissipative fireball solution
- with inhomogeneous pressure -



Summary

An application of our new relativistic, viscous solutions:

→ ***New, analytic, exact solutions of non relativistic Navier-Stokes equations with Hubble-flow***

Only academic results, *not plan to describe measurements*

The effects of viscosities and rotation are vanishing for late times

The solutions are asymptotically perfect both for a finite and vanishing μ

These exact solutions tend to perfect fluid solutions

Thank you for your attention!