Correspondence between Israel-Stewart theory and first-order causal and stable hydrodynamics for the boost-invariant flow

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Relativistic Hydrodynamics

- Evolution of the macroscopic conserved quantities [1,2,3].

$$\partial_\mu (T^{\mu\nu}_{(0)} + \delta T^{\mu\nu}) = 0, \ \ \partial_\mu (N^{\mu}_{(0)} + \delta N^{\mu}) = 0. \quad (1)$$

- $T^{\mu\nu} = T^{\mu\nu}_{(0)} + \delta T^{\mu\nu}$ is the conserved energy momentum tensor and $N^{\mu} = N^{\mu}_{(0)} + \delta N^{\mu}$ is associated with conserved current, e.g. baryon number current associated with baryon number conservation.
- At each space time point x^μ we can assign temperature ($T(x)$), chemical potential ($\mu(x)$) and a collective four-velocity field ($u^\mu(x)$).
- An ideal fluid is defined by the assumption of local thermal equilibrium, i.e., all fluid elements must be exactly in thermodynamic equilibrium.
- Primary fluid-dynamical variables: $T(x)$, $\mu(x)$ and $u^\mu(x)$.

[1] Romatschke, P., Romatschke, U.., arXiv:1712.05815.
[2] A. Jaiswal, arXiv:1408.0867
[3] A. Jaiswal, V. Roy, arXiv:1605.08694.

- The conserved currents of an ideal fluid can then be expressed as,

$$T^{\mu\nu}_{(0)} = \epsilon u^{\mu} u^{\nu} + P\Delta^{\mu\nu}, \;\; N^{\mu}_{(0)} = nu^{\mu}; \;\; S^{\mu}_{(0)} = su^{\mu}. \quad (2)$$

- $\Delta^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu}$ is the projector orthogonal to u^{μ}.
- The deviation from local thermodynamic equilibrium results in dissipative effects. Generically all fluids are of dissipative nature.
- Dissipative effects in a fluid originate from irreversible thermodynamic processes that occur during the motion of the fluid.
- The earliest covariant formulation of dissipative fluid dynamics were due to Eckart in 1940 [4] and, later, by Landau and Lifshitz in 1959 [5].

$$T^{\mu\nu} = \epsilon u^{\mu} u^{\nu} + P\Delta^{\mu\nu} + \Pi\Delta^{\mu\nu} + 2u^{(\mu}h^{\nu)} + \pi^{\mu\nu} \quad (3)$$

$$N^{\mu} = nu^{\mu} + n^{\mu}. \quad (4)$$

- Choice of the fluid frame: Landau frame: $h^{\mu} = 0$.

[4] C. Eckart, Phys. Rev.58, 267 (1940).

[5] L.D. Landau and E.M. Lifshitz,Fluid Mechanics(Butterworth-Heinemann, Oxford,1987).

- In the presence of dissipative currents, the entropy is no longer a conserved quantity i.e. $\partial_\mu S^\mu \neq 0$.
- Entropy four current,

$$S^\mu = P\beta^\mu + \beta_\nu T^{\mu\nu} - \alpha N^\mu, \;\; \beta^\mu = u^\mu/T \tag{5}$$

- The relativistic Navier-Stokes theory can be obtained by applying the second law of thermodynamics to each fluid element.

$$\partial_\mu S^\mu = -\beta\Pi\Theta - n^\mu\nabla_\mu\alpha + \beta\pi^{\mu\nu}\sigma_{\mu\nu} \geq 0. \tag{6}$$

- Second law of thermodynamic can easily be satisfied if one identifies[6],

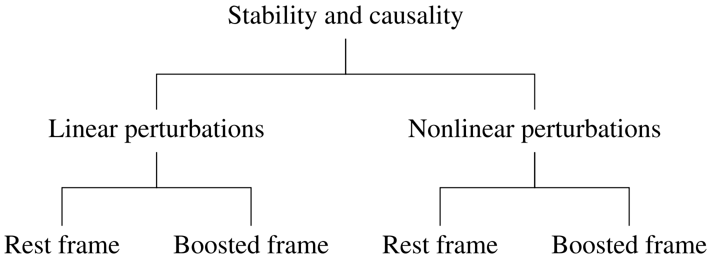
$$\Pi = -\zeta\Theta; \;\; n^\mu = \kappa\nabla^\mu\alpha, \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}. \tag{7}$$

- As long as $\zeta, \kappa, \eta \geq 0$, the entropy production is always positive.

[6] $\Theta \equiv \partial^\mu u_\mu; \; \nabla^\mu = \Delta^{\mu\alpha}\partial_\alpha; \; \sigma^{\mu\nu} \equiv [\frac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) - \frac{1}{3}(\nabla^\alpha u_\alpha)\Delta_{\mu\nu}]$

Stability and causality

- Entropy production is essential but not sufficient condition for a theory of dissipative relativistic hydrodynamics [7,8].
- Dynamics of departures of these fluids from their equilibrium states or stability and causality also important for a relativistic theory.



- If the stability and causality is preserved in all the boosted frames then we get an acceptable physical theories of relativistic dissipative hydrodynamics [9].

[7] W. A. Hiscock and L. Lindblom, ANNALS OF PHYSICS 151, 466-496 (1983)
[8] W. A. Hiscock and L. Lindblom,PHYSICAL REVIEW D, VOLUME 31, NUMBER 4, 725.
[9] Go to backup 1

IS theory

- It turns out that relativistic generalization of the Navier-Stokes theory is unstable i.e. presence of the exponentially growing modes.
- It is also been argued that Navier-Stokes theory is acausal.
- Israel and Stewart's formulation of causal relativistic dissipative fluid dynamics is the most popular and widely used.
- Up to second order in dissipative currents,

$$S^\mu = su^\mu - \alpha n^\mu - \left(\beta_0 \Pi^2 - \beta_1 n_\mu n^\mu + \beta_2 \pi_{\rho\sigma}\pi^{\rho\sigma}\right)\frac{u^\mu}{2T} \\ -\left(\alpha_0 \Pi \Delta^{\mu\nu} + \alpha_1 \pi^{\mu\nu}\right)\frac{n_\nu}{T} + \mathcal{O}(\delta^3). \tag{8}$$

- The existence of second-order contributions to the entropy four-current in leads to: [10,11].

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = \frac{1}{\beta_2}\left[\sigma^{\mu\nu} - \beta_{\pi\pi}\theta\pi^{\mu\nu}......\right]. \tag{9}$$

[10] W. Israel and J. M. Stewart, "Transient relativistic thermodynamics and kinetic theory," Annals Phys.118, 341 (1979).

[11] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev.D85(2012) 114047

- Energy momentum tensor and number current [12]:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \Pi\Delta^{\mu\nu} + 2u^{(\mu}h^{\nu)} + \pi^{\mu\nu}; \quad N^\mu = nu^\mu + n^\mu. \quad (10)$$

	Landau's theory	Israel-Stewart theory
Basic variables	T, μ, u^μ	$T, \mu, u^\mu, \Pi, \pi^{\mu\nu}, n^\mu$
Dissipative flux	$\Pi, \pi^{\mu\nu}, n^\mu$	$\Pi, \pi^{\mu\nu}, n^\mu$
Hydro Equations	$u_\nu\partial_\mu T^{\mu\nu} = 0,$ $\Delta^\alpha_\nu\partial_\mu T^{\mu\nu} = 0,$ $\partial_\mu N^\mu = 0,$	$u_\nu\partial_\mu T^{\mu\nu} = 0,$ $\Delta^\alpha_\nu\partial_\mu T^{\mu\nu} = 0,$ $\partial_\mu N^\mu = 0,$
$\partial_\mu S^\mu \geq 0$	$\Pi = -\zeta\partial_\mu u^\mu,$ $n^\mu = \kappa\nabla^\mu(\mu/T),$ $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}.$	$\tau_\Pi\dot{\Pi} + \Pi = f_\Pi(\Pi, \pi^{\mu\nu}, n^\mu),$ $\tau_n\dot{n}^{\langle\mu\rangle} + n^\mu = f_n(\Pi, \pi^{\mu\nu}, n^\mu),$ $\tau_\pi\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = f_\pi(\Pi, \pi^{\mu\nu}, n^\mu).$
Causality and stability	No	Yes (linear level)

[12] Amaresh Jaiswal, arXiv: 1408.0867.

First order causal and stable hydrodynamics

- Important questions are: [13,14]
  1. Is it possible to get relativistic viscous hydrodynamics which only deals with dynamical variables T, u^{α}, and μ?
  2. Does this theory gives rise to sensible physics, e.g. the equilibrium state is stable, and there is no superluminal propagation?
- Naive generalization of the Navier-Stokes equations does satisfy Q1. but Landau's theory is not complete as it give rise to unstable equilibrium.
- On the other hand Israel Stewart like theories preserve stability and causality but it deals with non Navier-Stokes degrees of freedom.
- The idea we would like to explore is whether both stability and causality might be maintained if one uses a certain out of equilibrium definition of the hydrodynamic variables which differs from the choice adopted by either Eckart or by Landau and Lifshitz.

[13] Bemfica, F.S., Disconzi, M.M., Noronha, J.arXiv:1708.06255.
[14] Kovtun, P.arXiv:1907.08191

- The physical objects $T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$, $J^{\mu} = \langle \hat{J^{\mu}} \rangle$ can still be expressed in terms of the quantities T, u^{α} and μ.
- In equilibrium, the quantities T, u^{α} and μ become the actual temperature, fluid velocity, and the chemical potential.
- However, out of equilibrium, T, u^{α} and μ have no first-principles microscopic definitions, and thus should be viewed as merely auxiliary variables used to parameterize the physical observables $T^{\mu\nu}$ and J^{μ}.
- The hydrodynamic expansion is a gradient expansion.

$$T^{\mu\nu} = \mathcal{O}(1) + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + ... + \mathcal{O}(\partial^k) + ..., \tag{11}$$

$$J^{\mu} = \mathcal{O}(1) + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + ... + \mathcal{O}(\partial^k) + ..., \tag{12}$$

- Given a time like unit vector u^{μ}, the energy-momentum tensor ($T^{\mu\nu}$) and the current (J^{μ}) may be decomposed as,

$$T^{\mu\nu} = \mathcal{E} u^{\mu} u^{\nu} + \mathcal{P} \Delta^{\mu\nu} + (Q^{\mu} u^{\nu} + Q^{\nu} u^{\mu}) + \mathcal{T}^{\mu\nu}, \quad J^{\mu} = \mathcal{N} u^{\mu} + \mathcal{J}^{\mu}; \tag{13}$$

where Q^{μ}, $\mathcal{T}^{\mu\nu}$, and $\mathcal{J}^{\mu}$ are transverse to u^{μ}, and $\mathcal{T}^{\mu\nu}$ is symmetric and traceless.

- To the first order in the derivative expansion,

$$\mathcal{E} = \epsilon + \varepsilon_1 \frac{u^\lambda \partial_\lambda T}{T} + \varepsilon_2(\partial.u) + \varepsilon_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{14}$$

$$\mathcal{P} = p + \pi_1 \frac{u^\lambda \partial_\lambda T}{T} + \pi_2(\partial.u) + \pi_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{15}$$

$$\mathcal{Q}^\mu = \theta_1 \dot{u}^\mu + \frac{\theta_2}{T}\Delta^{\mu\lambda}\partial_\lambda T + \theta_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{16}$$

$$\mathcal{T}^{\mu\nu} = -\eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2), \tag{17}$$

$$\mathcal{N} = n + \nu_1 \frac{\dot{T}}{T} + \nu_2(\partial.u) + \nu_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{18}$$

$$\mathcal{J}^\mu = \gamma_1 \dot{u}^\mu + \frac{\gamma_2}{T}\Delta^{\mu\lambda}\partial_\lambda T + \gamma_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2). \tag{19}$$

- In equilibrium, the quantities T, u^α and μ become the actual temperature, fluid velocity, and the chemical potential.
- It can be argued that in the 16 dimensional parameter space spanned by $\varepsilon_{1,2,3}$, $\pi_{1,2,3}$, $\theta_{1,2,3}$, $\nu_{1,2,3}$, $\gamma_{1,2,3}$, and η the theory is causal and stable.

- Linear mode analysis: look for plane wave solutions of the form, $e^{ik.x-i\omega t}$.
- Shear channel for uncharged fluid e.g.

$$\omega(k) = \frac{i(\epsilon_0 + p_0)\sqrt{1 - v_0^2}}{\eta v_0^2 - \theta} + O(k.v_0)\,, \tag{20}$$

- Stability of the shear channel fluctuations requires: $\theta > \eta > 0$.
- The Landau-Lifshitz convention sets $\theta = 0$ at non-zero $\eta \implies$ stability criteria is not satisfied.
- Sound channel of uncharged fluid e.g.

$$\omega(k) = -i\frac{\epsilon_0 + p_0}{\theta} + O(k^2). \tag{21}$$

- For the stability of the sound mode one requires $\theta > 0$.
- $\theta > 0$ contradicts the Landau-Lifshitz convention.
- Special frame choice of the most general first order hydrodynamics can give rise to unstable equilibrium state. This is just a bad choice of the frame. In general first order hydrodynamics is stable and causal.

IS-FOCS correspondence

- For Bjorken flow, $ds^2 = d\tau^2 - dx^2 - dy^2 - \tau^2 d\xi^2$, $\tau = \sqrt{t^2 - \tau^2}$, $\xi = Tanh^{-1}(z/t)$, $u^\mu = (1,0,0,0)$, $\pi^\mu_\nu = diag(0, -\pi/2, -\pi/2, \pi)$.
- For the IS theory hydrodynamic equation becomes, [15,16]

$$\frac{d\varepsilon}{d\tau} = -\frac{\varepsilon + p}{\tau} + \frac{\pi}{\tau}, \tag{22}$$

$$\tau_R \frac{d\pi}{d\tau} + \pi = \frac{4}{3}\frac{\eta}{\tau} - \left(\frac{4}{3} + \lambda\right)\tau_R \frac{\pi}{\tau}, \tag{23}$$

- For the FOCS approach, the evolution equations are reduced to the formula

$$\frac{d\mathcal{E}}{d\tau} + \frac{\mathcal{E} + \mathcal{P}}{\tau} - \frac{4}{3}\frac{\eta}{\tau^2} = 0, \tag{24}$$

where the following constitutive relations are assumed,

$$\mathcal{E} = \varepsilon + \varepsilon_1 \frac{dT}{Td\tau} + \frac{\varepsilon_2}{\tau}; \mathcal{P} = p + \pi_1 \frac{dT}{Td\tau} + \frac{\pi_2}{\tau}. \tag{25}$$

[15] Bjorken, J.D..Phys Rev1983,D27,140-151.

[16] G.S.Denicol and J. Noronha, arXiv:1711.01657

	IS theory	FOCS theory
Variables	$T, u^{\mu}, \pi^{\mu\nu}$	T, u^{μ}
EoS	$p = \frac{1}{3}\varepsilon = \frac{aT^4}{3}$	$p = \frac{1}{3}\varepsilon = \frac{aT^4}{3}$
Equations	$\frac{d\varepsilon}{d\tau} = -\frac{\varepsilon+p}{\tau} + \frac{\pi}{\tau},$ $\tau_R \frac{d\pi}{d\tau} + \pi = \frac{4}{3}\frac{\eta}{\tau} - \left(\frac{4}{3} + \lambda\right)\tau_R \frac{\pi}{\tau},$	$\frac{d\mathcal{E}}{d\tau} + \frac{\mathcal{E}+\mathcal{P}}{\tau} - \frac{4}{3}\frac{\eta}{\tau^2} = 0,$ $\mathcal{E} = aT^4 + \varepsilon_1 \frac{dT}{Td\tau} + \frac{\varepsilon_2}{\tau},$ $\mathcal{P} = \frac{aT^4}{3} + \pi_1 \frac{dT}{Td\tau} + \frac{\pi_2}{\tau},$
	$y = \frac{dT}{d\tau}$	$y = \frac{dT}{d\tau}$

- $\varepsilon_i = \varepsilon_i^0 T^n$; ε_i^0 is just a constant.
- $\pi_i = \pi_i^0 T^n$; π_i^0 is just a constant.
- Both ε_i^0 and π_i^0 can be dimensional. Therefore we keep a general scaling of the form T^n.

- For the IS theory

$$4a\tau_R T^3\frac{dy}{d\tau}+12\tau_R aT^2y^2+aT^3y\left[4+\left(\frac{28}{3}+4\left(\frac{4}{3}+\lambda\right)\right)\frac{\tau_R}{\tau}\right] +\frac{4aT^4}{3\tau}+\frac{4}{3}aT^4\left(\frac{4}{3}+\lambda\right)\frac{\tau_R}{\tau^2}-\frac{4}{3}\frac{\eta}{\tau^2}=0. \quad (26)$$

- FOCS hydrodynamic equation with $y = dT/d\tau$,

$$\varepsilon_1^0 T^{n-1}\frac{dy}{d\tau}+(n-1)\,\varepsilon_1^0\,T^{n-2}\,y^2+\left(4aT^3+(\varepsilon_1^0+\pi_1^0+n\,\varepsilon_2^0)\frac{T^{n-1}}{\tau}\right)y +\frac{4}{3\tau}aT^4+\frac{\pi_2\,T^n}{\tau^2}-\frac{4}{3}\frac{\eta}{\tau^2}=0. \quad (27)$$

- Note that Eq. (26) has the form of a Ricatti equation ($ay'+by^2+cy+d=0$, with $b/a\neq 0$ and $c/a\neq 0$), which was analyzed and may be possible to solve analytically. [17]

[17] G.S.Denicol and J. Noronha, arXiv:1711.01657

- Both IS and FOCS formalism has one common equation, $y = dT/d\tau$.
- After equating the terms with the same derivatives of the function y in Eq. (26) and (27) we find:

$$\varepsilon_1^0 = 4a\tau_R T^{4-n}, \tag{28}$$

$$\varepsilon_1^0 = \frac{12}{n-1} a\tau_R T^{4-n}, \tag{29}$$

$$\pi_1^0 = \frac{4}{3} a\tau_R (11 + 3\lambda) T^{4-n} - \varepsilon_1^0 - n\varepsilon_2^0, \tag{30}$$

$$\pi_2^0 = \frac{4}{9} a\tau_R \left(4 + 3\lambda\right) T^{4-n}. \tag{31}$$

- One can easily notice that in the strictly conformal case, $n = 3$, it is impossible to exactly match the FOCS and IS equations.
- An interesting situation takes place when $n = 4$. In this case Eqs. (28) and (29) are fully consistent.
- The kinetic coefficient ε_1^0 has dimension of fm and, thus, it can be treated as a fixed relaxation time related to τ_R (which is also constant).

- To uniquely determine the kinetic coefficient in the FOCS theory we use the traceless condition of the energy momentum tensor.

$$T^{\mu}_{\mu} = 0 \implies \pi_i = \epsilon_i/3 \implies \lambda = -1. \tag{32}$$

- In the FOCS approach, the bulk viscosity appears as a linear combination of the regulators and one can show that [18],

$$\zeta = (-\pi_2 + c_s^2(\varepsilon_2 + \pi_1) - c_s^4\varepsilon_1) = 0. \tag{33}$$

- For conformal equation of state with $c_s^2 = 1/3$ and the condition of vanishing of T^{μ}_{μ},

$$\begin{aligned} \varepsilon_1^0 &= 4a\tau_R, && (34)\\ \varepsilon_2^0 &= \frac{4}{3}a\tau_R, && (35)\\ \pi_1^0 &= \frac{4}{3}a\tau_R, && (36)\\ \pi_2^0 &= \frac{4}{9}a\tau_R. && (37) \end{aligned}$$

[18] Kovtun, P,arXiv:1907.08191.

Analytical solution of FOCS

- The general solution of the IS equations for Bjorken flow is [19],

$$\varepsilon(\hat{\tau}) = \varepsilon_0 \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{\frac{4}{3}+\frac{\lambda+1}{2}} \exp\left(-\frac{\hat{\tau}-\hat{\tau}_0}{2}\right) \times \left[\frac{M_{-\frac{\lambda+1}{2},\frac{\sqrt{\lambda^2+4\tilde{a}}}{2}}(\hat{\tau}) + \alpha\, W_{-\frac{\lambda+1}{2},\frac{\sqrt{\lambda^2+4\tilde{a}}}{2}}(\hat{\tau})}{M_{-\frac{\lambda+1}{2},\frac{\sqrt{\lambda^2+4\tilde{a}}}{2}}(\hat{\tau}_0) + \alpha\, W_{-\frac{\lambda+1}{2},\frac{\sqrt{\lambda^2+4\tilde{a}}}{2}}(\hat{\tau}_0)}\right] \quad (38)$$

  where $\hat{\tau} = \tau/\tau_R$, $\tilde{a} = 16/(9\tau_R T)(\eta/s)$,$\hat{\tau}_0$ is the initial time, ε_0 and α are constants that define the initial value problem, and $M_{k,\mu}(x)$ and $W_{k,\mu}(x)$ are Whittaker functions.
- The matching to IS theory shown here implies that the general solution for the energy density in IS also holds for the FOCS theory (for appropriate values of λ).
- This is the first analytical solution of the FOCS theory.

[19] Denicol, G.S., Noronha, J.. arXiv:1711.01657.

- For the type of Israel-Stewart theory considered here, causality and stability around equilibrium hold when $\eta/(s\tau_R T) \leq 1/2$ (where $s = 4\varepsilon/3T$)[20,21].
- The parameter λ does not appear contribute in a linearized analysis. So it can not be constrained.
- Mapping between IS theory and FOCS theory is only well defined if the IS parameter λ takes values that are distinct from the standard 14-moment result.
- For boost-invariant, baryon-free systems with a conformal equation of state, If the regulator sectors of the theories are determined by a constant relaxation time, there exists a mapping between the FOCS and IS approaches that makes their dynamics exactly the same.
- Even for finite mass but in the absence of any conserved charge there is a one to one correspondence.
- If we consider finite baryon density, even then a correspondence exists for an ideal gas equation of state.

[22].

[20] Denicol, G.S., Niemi, H., Molnar, E., Rischke, D.H..,arXiv:1202.4551.

[21] Pu, S., Koide, T., Rischke, D.H.. arXiv:0907.3906.

[22] If no time Go to the end

- In the presence of baryon number density

$$T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - p\Delta^{\mu\nu} + \pi^{\mu\nu}; \;\; N^{\mu} = n u^{\mu} + n^{\mu}. \tag{39}$$

- IS theory in the presence of baryon number density (for Bjorken flow)

$$\frac{d\varepsilon}{d\tau} = -\frac{1}{\tau}\left[(\varepsilon + p) - \pi\right]; \; \tau_{\pi}\frac{d\pi}{d\tau} = \frac{4}{3}\frac{\eta}{\tau} - \pi - \beta\frac{\tau_{\pi}}{\tau}\pi, \tag{40}$$

$$\frac{dn}{d\tau} + \frac{n}{\tau} = 0. \tag{41}$$

- Equation:

$$\left(\frac{\nu_1}{T} - \frac{\nu_3\mu}{T^2}\right)\ddot{T} + \left(\frac{\partial\nu_1}{\partial T}\frac{1}{T} - \frac{\nu_1}{T^2} - \frac{\partial\nu_3}{\partial T}\frac{\mu}{T^2} + 2\nu_3\frac{\mu}{T^3}\right)\dot{T}^2$$
$$+ \left(\frac{\partial\nu_1}{\partial\mu}\frac{1}{T} + \frac{\partial\nu_3}{\partial T}\frac{1}{T} - \frac{\nu_3}{T^2} - \frac{\partial\nu_3}{\partial\mu}\frac{\mu}{T^2} - \frac{\nu_3}{T^2}\right)\dot{T}\dot{\mu}$$
$$+ \left(\frac{\nu_3}{T}\right)\ddot{\mu} + \left(\frac{\partial\nu_3}{\partial\mu}\frac{1}{T}\right)\dot{\mu}^2 + \left(\frac{\partial n}{\partial T} + \frac{1}{\tau}\frac{\partial\nu_2}{\partial T}\right)\dot{T}$$
$$+ \left(\frac{\partial n}{\partial\mu} + \frac{1}{\tau}\frac{\partial\nu_2}{\partial\mu}\right)\dot{\mu} + \frac{n}{\tau} + \frac{\nu_1}{\tau}\frac{\dot{T}}{T} + \frac{\nu_3}{\tau}\frac{\dot{\mu}}{T} - \frac{\nu_3}{\tau}\frac{\mu}{T^2}\dot{T} = 0. \tag{42}$$

- In the FOCS sector,

$$\frac{d\mathcal{E}}{d\tau}+\frac{\mathcal{E}+\mathcal{P}}{\tau}-\frac{4}{3}\frac{\eta}{\tau^2}=0;\quad \frac{d\mathcal{N}}{d\tau}+\frac{\mathcal{N}}{\tau}=0. \tag{43}$$

- Here one considers the following constitutive relations for a charged fluid:

$$\mathcal{E}=\varepsilon+\varepsilon_1\frac{\dot{T}}{T}+\varepsilon_2\frac{1}{\tau}+\varepsilon_3\frac{d}{d\tau}\left(\frac{\mu}{T}\right);\mathcal{P}=p+\pi_1\frac{\dot{T}}{T}+\pi_2\frac{1}{\tau}+\pi_3\frac{d}{d\tau}\left(\frac{\mu}{T}\right), \tag{44}$$

$$\mathcal{N}=n+\nu_1\frac{\dot{T}}{T}+\nu_2\frac{1}{\tau}+\nu_3\frac{d}{d\tau}\left(\frac{\mu}{T}\right). \tag{45}$$

$$\begin{aligned}
&\left(\frac{\varepsilon_1}{T}-\frac{\varepsilon_3\mu}{T^2}\right)\ddot{T}+\left(\frac{\partial\varepsilon_1}{\partial T}\frac{1}{T}-\frac{\varepsilon_1}{T^2}-\frac{\partial\varepsilon_3}{\partial T}\frac{\mu}{T^2}+2\varepsilon_3\frac{\mu}{T^3}\right)\dot{T}^2\\
&+\left(\frac{\partial\varepsilon_1}{\partial\mu}\frac{1}{T}+\frac{\partial\varepsilon_3}{\partial T}\frac{1}{T}-\frac{\varepsilon_3}{T^2}-\frac{\partial\varepsilon_3}{\partial\mu}\frac{\mu}{T^2}-\frac{\varepsilon_3}{T^2}\right)\dot{T}\dot{\mu}\\
&+\left(\frac{\varepsilon_3}{T}\right)\ddot{\mu}+\left(\frac{\partial\varepsilon_3}{\partial\mu}\frac{1}{T}\right)\dot{\mu}^2+\left(\frac{\partial\varepsilon}{\partial T}+\frac{1}{\tau}\frac{\partial\varepsilon_2}{\partial T}\right)\dot{T}\\
&+\left(\frac{\partial\varepsilon}{\partial\mu}+\frac{1}{\tau}\frac{\partial\varepsilon_2}{\partial\mu}\right)\dot{\mu}+\frac{\varepsilon+p}{\tau}+\frac{\varepsilon_1+\pi_1}{\tau}\frac{\dot{T}}{T}+\frac{\pi_2}{\tau^2}\\
&+\frac{\varepsilon_3+\pi_3}{\tau}\frac{\dot{\mu}}{T}-\frac{\varepsilon_3+\pi_3}{\tau}\frac{\mu}{T^2}\dot{T}-\frac{4}{3}\frac{\eta}{\tau^2}=0.
\end{aligned} \tag{46}$$

- The number conservation equation boils down to,

$$\left(\frac{\nu_1}{T}-\frac{\nu_3\mu}{T^2}\right)\ddot{T}+\left(\frac{\partial\nu_1}{\partial T}\frac{1}{T}-\frac{\nu_1}{T^2}-\frac{\partial\nu_3}{\partial T}\frac{\mu}{T^2}+2\nu_3\frac{\mu}{T^3}\right)\dot{T}^2$$
$$+\left(\frac{\partial\nu_1}{\partial\mu}\frac{1}{T}+\frac{\partial\nu_3}{\partial T}\frac{1}{T}-\frac{\nu_3}{T^2}-\frac{\partial\nu_3}{\partial\mu}\frac{\mu}{T^2}-\frac{\nu_3}{T^2}\right)\dot{T}\dot{\mu}$$
$$+\left(\frac{\nu_3}{T}\right)\ddot{\mu}+\left(\frac{\partial\nu_3}{\partial\mu}\frac{1}{T}\right)\dot{\mu}^2+\left(\frac{\partial n}{\partial T}+\frac{1}{\tau}\frac{\partial\nu_2}{\partial T}\right)\dot{T}$$
$$+\left(\frac{\partial n}{\partial\mu}+\frac{1}{\tau}\frac{\partial\nu_2}{\partial\mu}\right)\dot{\mu}+\frac{n}{\tau}+\frac{\nu_1}{\tau}\frac{\dot{T}}{T}+\frac{\nu_3}{\tau}\frac{\dot{\mu}}{T}-\frac{\nu_3}{\tau}\frac{\mu}{T^2}\dot{T}=0. \tag{47}$$

- If we compare the dynamical equations in these two theories along with the conditions,
  1. Zero bulk viscosity for ideal gas EoS.
  2. No trace of the energy-momentum.

$$\begin{aligned}\varepsilon_1 &= \tau_R T\frac{\partial \varepsilon}{\partial T} + \tau_R\mu\frac{\partial \varepsilon}{\partial \mu} = 3\tau_R T\frac{\partial p}{\partial T} + 3\tau_R\mu\frac{\partial p}{\partial \mu} \\ &= 3\tau_R Ts + 3\tau_R\mu n = 3\tau_R(\varepsilon + p) = 4\tau_R\varepsilon,\end{aligned} \tag{48}$$

$$\varepsilon_2 = \tau_R(\varepsilon + p) = \frac{4}{3}\tau_R\varepsilon, \tag{49}$$

$$\varepsilon_3 = \tau_R T\frac{\partial \varepsilon}{\partial \mu} = 3\tau_R T\frac{\partial p}{\partial \mu} = 3\tau_R Tn, \tag{50}$$

$$\pi_1 = \tau_R(\varepsilon + p) = \frac{4}{3}\tau_R\varepsilon, \tag{51}$$

$$\pi_2 = \frac{1}{3}\tau_R(\varepsilon + p) = \frac{4}{9}\tau_R\varepsilon, \tag{52}$$

$$\pi_3 = \tau_R Tn. \tag{53}$$

Thank You!

- Assume the background equilibrium state is homogeneous in space and the background is Minkowski space time.
- Background field variables have vanishing gradients.
- Look only for exponential plane wave solutions to the perturbation equations,

$$\delta Q = \delta Q_0 \exp(ikx + \Gamma t). \tag{54}$$

- The set of perturbation equation takes the form,

$$M^A_B \delta Y^B = 0 \tag{55}$$

δY^B represents the list of fields which describe the perturbation of the fluid. M^A_B complex- valued matrix which describes the linearized equations of motion.

- There will exist exponential plane-wave solutions whenever Γ and k have values which satisfy the dispersion relation [23],

$$\det M = 0. \tag{56}$$

[23]Go back to main text

IS theory

- The main idea behind the Israel-Stewart formulation was to apply the second law of thermodynamics to a more general expression of the non-equilibrium entropy four-current.
- A more realistic description of the entropy four-current can be obtained by considering it to be a function not only of the primary fluid-dynamical variables, but also of the dissipative currents.

$$S^{\mu} = P\beta^{\mu} + \beta_{\nu}T^{\mu\nu} - \alpha N^{\mu} - Q^{\mu}(\delta N^{\mu}, \delta T^{\mu\nu}) \tag{57}$$

- Up to second order in dissipative currents,

$$S^{\mu} = su^{\mu} - \alpha n^{\mu} - (\beta_0\Pi^2 - \beta_1 n_{\mu}n^{\mu} + \beta_2\pi_{\rho\sigma}\pi^{\rho\sigma})\frac{u^{\mu}}{2T} - (\alpha_0\Pi\Delta^{\mu\nu} + \alpha_1\pi^{\mu\nu})\frac{n_{\nu}}{T} + \mathcal{O}(\delta^3). \tag{58}$$

- The existence of second-order contributions to the entropy four-current in leads to constitutive relations for the dissipative quantities which are different from relativistic Navier-Stokes theory.

- In IS theory dissipative currents satisfy dynamical equations [24,25],

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\frac{1}{\beta_0}\Big[\theta + \beta_{\Pi\Pi}\Pi\theta + \alpha_0\nabla_\mu n^\mu + \psi\alpha_{n\Pi}n_\mu\dot{u}^\mu + \psi\alpha_{\Pi n}n_\mu\nabla^\mu\alpha\Big], \quad (59)$$

$$\dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} = \frac{1}{\beta_1}\Big[T\nabla_\mu\alpha - \beta_{nn}n_\mu\theta + \alpha_0\nabla_\mu\Pi + \alpha_1\nabla_\nu\pi^\nu_\mu + \tilde{\psi}\alpha_{n\Pi}\Pi\dot{u}_\mu + \tilde{\psi}\alpha_{\Pi n}\Pi\nabla_\mu\alpha + \tilde{\chi}\alpha_{\pi n}\pi^\nu_\mu\nabla_\nu\alpha + \tilde{\chi}\alpha_{n\pi}\pi^\nu_\mu\dot{u}_\nu\Big] \quad (60)$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = \frac{1}{\beta_2}\Big[\sigma_{\mu\nu} - \beta_{\pi\pi}\theta\pi_{\mu\nu} - \alpha_1\nabla_{\langle\mu}n_{\nu\rangle} - \chi\alpha_{\pi n}n_{\langle\mu}\nabla_{\nu\rangle}\alpha - \chi\alpha_{n\pi}n_{\langle\mu}\dot{u}_{\nu\rangle}\Big]. \quad (61)$$

- These relaxation times indicate the time scales within which the dissipative currents react to hydrodynamic gradients, in contrast to the relativistic Navier-Stokes theory.

[24] W. Israel and J. M. Stewart, "Transient relativistic thermodynamics and kinetic theory," Annals Phys.118, 341 (1979).

[25] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev.D85(2012) 114047

- The hydrodynamic expansion is a gradient expansion.

$$T^{\mu\nu} = \mathcal{O}(1) + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + ... + \mathcal{O}(\partial^k) + ..., \tag{62}$$

$$J^{\mu} = \mathcal{O}(1) + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + ... + \mathcal{O}(\partial^k) + ..., \tag{63}$$

where $\mathcal{O}(\partial^k)$ denotes the terms with k derivatives of T, u^α, μ, for example the $\mathcal{O}(\partial^2)$ contributions contain terms proportional to $\partial^2 T$, $(\partial T)^2$, $(\partial T)(\partial u)$ etc.

- Given a time like unit vector u^μ, the energy-momentum tensor ($T^{\mu\nu}$) and the current (J^μ) may be decomposed as,

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (Q^\mu u^\nu + Q^\nu u^\mu) + \mathcal{T}^{\mu\nu}, \quad J^\mu = \mathcal{N}u^\mu + \mathcal{J}^\mu; \tag{64}$$

where Q^μ, $\mathcal{T}^{\mu\nu}$, and $\mathcal{J}^\mu$ are transverse to u^μ, and $\mathcal{T}^{\mu\nu}$ is symmetric and traceless.

$$Q^\mu u_\mu = 0; \mathcal{T}^{\mu\nu} u_\mu = 0 = \mathcal{T}^{\mu\nu} u_\nu; \mathcal{J}^\mu u_\mu = 0; \mathcal{T}^\mu_{\ \mu} = 0; \Delta^{\mu\nu} u_\nu = 0. \tag{65}$$

- To the first order in the derivative expansion,

$$\mathcal{E} = \epsilon + \varepsilon_1 \frac{u^\lambda \partial_\lambda T}{T} + \varepsilon_2(\partial.u) + \varepsilon_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{66}$$

$$\mathcal{P} = p + \pi_1 \frac{u^\lambda \partial_\lambda T}{T} + \pi_2(\partial.u) + \pi_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{67}$$

$$\mathcal{Q}^\mu = \theta_1 \dot{u}^\mu + \frac{\theta_2}{T}\Delta^{\mu\lambda}\partial_\lambda T + \theta_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{68}$$

$$\mathcal{T}^{\mu\nu} = -\eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2), \tag{69}$$

$$\mathcal{N} = n + \nu_1 \frac{\dot{T}}{T} + \nu_2(\partial.u) + \nu_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{70}$$

$$\mathcal{J}^\mu = \gamma_1 \dot{u}^\mu + \frac{\gamma_2}{T}\Delta^{\mu\lambda}\partial_\lambda T + \gamma_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2), \tag{71}$$

- At zero-derivative order, the constitutive relations are determined by the three independent parameters ϵ, p, and n which in general all depend on T and μ.
- At one-derivative order, there are sixteen transport coefficients (seven for uncharged fluids) $\varepsilon_{1,2,3}$, $\pi_{1,2,3}$, $\theta_{1,2,3}$, $\nu_{1,2,3}$, $\gamma_{1,2,3}$, and η, which in general all depend on T and μ. Not all of the one-derivative transport coefficients are physical.

- Out of equilibrium, the T, u^α and μ are just auxiliary variables without first-principles microscopic definition.
- One can choose a different out-of-equilibrium choices of T, u^α and μ.
- Only criteria is that all these choices agree in equilibrium.
- Given any choice of T, u^α and μ, one can always redefine,

$$T' = T + \delta T; \mu' = \mu + \delta\mu; u'^\alpha = u^\alpha + \delta u^\alpha. \tag{72}$$

- In terms of the redefinition of the fundamental hydrodynamic variables, $\mathcal{E}$, $\mathcal{P}$, Q^μ, $\mathcal{N}$, $\mathcal{T}^{\mu\nu}$ and $\mathcal{J}^\mu$ all changes, but what remains unchanged are $T^{\mu\nu}$ and J^μ.

$$\mathcal{E}' = u'_\mu u'_\nu T^{\mu\nu} = \mathcal{E} + \mathcal{O}(\partial^2) \tag{73}$$

$$\mathcal{P}'(T', \mu') = \mathcal{P}(T, \mu) + O(\partial^2)\,, \tag{74}$$

$$\mathcal{Q}'_\mu(T', u', \mu') = \mathcal{Q}_\mu(T, u, \mu) - (\epsilon{+}p)\delta u_\mu + O(\partial^2)\,, \tag{75}$$

$$\mathcal{T}'_{\mu\nu}(T', u', \mu') = \mathcal{T}_{\mu\nu}(T, u, \mu) + O(\partial^2)\,, \tag{76}$$

$$\mathcal{N}'(T', \mu') = \mathcal{N}(T, \mu) + O(\partial^2)\,, \tag{77}$$

$$\mathcal{J}'_\alpha(T', u', \mu') = \mathcal{J}_\alpha(T, u, \mu) - n\,\delta u_\alpha + O(\partial^2)\,. \tag{78}$$

- We can also perform the most general first-order field redefinition,

$$\delta T = a_1 \dot{T}/T + a_2 \partial\cdot u + a_3 u^\lambda \partial_\lambda(\mu/T)\,, \tag{79}$$

$$\delta u^\mu = b_1 \dot{u}^\mu + b_2/T\,\Delta^{\mu\nu}\partial_\nu T + b_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T)\,, \tag{80}$$

$$\delta\mu = c_1 \dot{T}/T + c_2 \partial\cdot u + c_3 u^\lambda \partial_\lambda(\mu/T)\,, \tag{81}$$

- Under the most general first order field redefinition,

$$\begin{aligned}
&\mathcal{E}'(T',\mu',u') = \mathcal{E}(T,\mu,u) + \mathcal{O}(\partial^2) \\
&\implies \epsilon(T',\mu') + f'_\varepsilon = \epsilon(T,\mu) + f_\varepsilon + \mathcal{O}(\partial^2) \\
&\implies f'_\varepsilon = f_\varepsilon - \left(\frac{\partial\epsilon}{\partial T}\right)\delta T - \left(\frac{\partial\epsilon}{\partial\mu}\right)\delta\mu + \mathcal{O}(\partial^2) \\
&\implies \varepsilon_1 \frac{\dot{T}'}{T'} + \varepsilon_2 \partial_\lambda u'^\lambda + \epsilon_3 u'^\lambda \partial_\lambda(\mu'/T') = \varepsilon_1 \frac{\dot{T}}{T} + \varepsilon_2 \partial_\lambda u^\lambda + \epsilon_3 u^\lambda \partial_\lambda(\mu/T) \\
&\quad - \left(\frac{\partial\epsilon}{\partial T}\right)\left[a_1 \frac{\dot{T}}{T} + a_2(\partial.u) + a_3 u^\lambda \partial_\lambda(\mu/T)\right] \\
&\quad - \left(\frac{\partial\epsilon}{\partial\mu}\right)\left[c_1 \frac{\dot{T}}{T} + c_2(\partial.u) + c_3 u^\lambda \partial_\lambda(\mu/T)\right].
\end{aligned} \tag{82}$$

- The constitutive relations for $T^{\mu\nu}$ and J^{μ}, written in terms of the new fields T', u', μ', look the same as the constitutive relations in terms of the old fields T, u, μ, with the following change:

$$\varepsilon_i \to \varepsilon_i - \epsilon_{,T} a_i - \epsilon_{,\mu} c_i \,, \tag{83}$$

$$\pi_i \to \pi_i - p_{,T} a_i - p_{,\mu} c_i \,, \tag{84}$$

$$\nu_i \to \nu_i - n_{,T} a_i - n_{,\mu} c_i \,, \tag{85}$$

$$\theta_i \to \theta_i - (\epsilon + p) b_i \,, \tag{86}$$

$$\gamma_i \to \gamma_i - n b_i \,, \tag{87}$$

$$\eta \to \eta \,, \tag{88}$$

- It is clear that ε_i, π_i, ν_i, θ_i, γ_i are not invariant under redefinition of the fundamental variables. But one can construct some invariant quantities, e.g.

$$f_i = \pi_i - \left(\frac{\partial p}{\partial \epsilon}\right)_n \varepsilon_i - \left(\frac{\partial p}{\partial n}\right)_\epsilon \nu_i. \tag{89}$$

$$l_i \equiv \gamma_i - \frac{n}{\epsilon + p}\theta_i. \tag{90}$$

- Only invariant transport quantities are, f_i, l_i and η.

- It is clear that one can go to a new frame (by choosing a_i and c_i appropriately) in which $\mathcal{E} = \epsilon$ and $\mathcal{N} = n$.

$$\varepsilon_i \to \varepsilon_i - \left(\frac{\partial \epsilon}{\partial T}\right) a_i - \left(\frac{\partial \epsilon}{\partial \mu}\right) c_i = 0, \tag{91}$$

$$\nu_i \to \nu_i - \left(\frac{\partial n}{\partial T}\right) a_i - \left(\frac{\partial n}{\partial \mu}\right) c_i = 0. \tag{92}$$

- In this frame,

$$\pi_i \to \pi_i - \left(\frac{\partial p}{\partial T}\right) a_i - \left(\frac{\partial p}{\partial \mu}\right) c_i = \pi_i - \varepsilon_i \left(\frac{\partial p}{\partial \epsilon}\right)_n - \nu_i \left(\frac{\partial p}{\partial n}\right)_\epsilon = f_i. \tag{93}$$

- If we choose $b_i = \frac{\theta_i}{\epsilon+p}$, then in the Landau frame $\theta_i = 0, \implies Q^\mu = 0$ and in this frame $\gamma_i \to \gamma_i - nb_i = \gamma_i - \frac{n}{\epsilon+p}\theta_i = l_i$.
- In the Landau frame,

$$T^{\mu\nu} = \epsilon u^{\mu\nu} + \left(p + f_1 \frac{\dot{T}}{T} + f_2 \partial.u + f_3 u^\lambda \partial_\lambda(\mu/T)\right) \Delta^{\mu\nu} - \eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2), \tag{94}$$

$$J^\mu = nu^\mu + l_1 \dot{u}^\mu + \frac{l_2}{T}\Delta^{\mu\lambda}\partial_\lambda T + l_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2). \tag{95}$$

- Till now we only looked at the constitutive relations, but have not used on shell conditions. the hydrodynamic equations themselves, $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu J^\mu = 0$.
- The conservation equations $\partial_\mu(nu^\mu) + O(\partial^2) = 0$ and $u_\nu \partial_\mu(\epsilon u^\mu u^\nu + p\Delta^{\mu\nu}) + O(\partial^2) = 0$ imply two "on-shell" relations among the scalars $\dot{T}$, $\partial\cdot u$, and $\dot{\mu}$, up to $O(\partial^2)$ terms.
- Similarly, the projected energy-momentum conservation $\Delta^\alpha_\nu \partial_\mu(\epsilon u^\mu u^\nu + p\Delta^{\mu\nu}) + O(\partial^2) = 0$ implies one "on-shell" relation among the vectors $\dot{u}^\alpha$, $\Delta^{\alpha\lambda}\partial_\lambda T$, and $\Delta^{\alpha\lambda}\partial_\lambda(\mu/T)$, up to $O(\partial^2)$ terms

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + [p - \zeta(\partial.u)]\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2), \tag{96}$$

$$J^\mu = nu^\mu - \sigma T\Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \chi_T\Delta^{\mu\lambda}\partial_\lambda T + \mathcal{O}(\partial^2), \tag{97}$$

with,

$$\sigma = \frac{n}{\epsilon + p}l_1 - \frac{l_3}{T};\ \ \chi_T = \frac{1}{T}(l_2 - l_1). \tag{98}$$

- Study the linearized stability, $T = T_0 + \delta T$, $v = v_0 + \delta v$, $\mu = \mu_0 + \delta\mu$.
- Look for plane wave solutions of the form, $e^{ik.x - i\omega t}$.
- Solving the hydrodynamic equations $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu J^\mu = 0$ with the general constitutive relations, one finds the eigen frequencies $\omega(k)$ which depend on T_0, v_0 and μ_0, as well as on all the transport coefficients.
- First-order hydrodynamics of uncharged fluids in the general frame we have six transport coefficients: $\varepsilon_{1,2}$, $\pi_{1,2}$, $\theta \equiv \theta_1 = \theta_2$, and η.
- For charged fluids in the general frame one has a fourteen-dimensional parameter space of transport coefficients, $\varepsilon_{1,2,3}$, $\pi_{1,2,3}$, $\nu_{1,2,3}$, $\theta \equiv \theta_1 = \theta_2$, θ_3, $\gamma \equiv \gamma_1 = \gamma_2$, γ_3, η.
- One can also find a subspace in the fourteen-dimensional parameter space of transport coefficients where a class of stable frames can be defined.

- In the IS sector we consider following equation for the Bjorken flow [26,27]

$$D\epsilon + (\epsilon + p)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0, \tag{99}$$

$$(\epsilon + p)Du^{\mu} - \Delta^{\mu}_{\lambda}\nabla^{\lambda}P + \Delta^{\mu}_{\lambda}\nabla_{\mu}\pi^{\mu\lambda} = 0, \tag{100}$$

$$\tau_R\Delta^{\mu\nu}_{\alpha\beta}D\pi^{\alpha\beta} + \delta_{\pi\pi}\theta\pi^{\mu\nu} + \tau_{\pi\pi}\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha\lambda}\sigma^{\beta}_{\lambda} - 2\tau_R\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha}_{\lambda}\omega^{\beta\lambda} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}. \tag{101}$$

$$D = u^{\mu}\nabla_{\mu}, \theta = \nabla_{\mu}u^{\mu}, \sigma_{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\nabla_{\alpha}u_{\beta}, \omega_{\mu\nu} = (\Delta^{\lambda}_{\mu}\nabla_{\lambda}u_{\nu} - \Delta^{\lambda}_{\nu}\nabla_{\lambda}u_{\mu})/2.$$

- For Bjorken flow, $ds^2 = d\tau^2 - dx^2 - dy^2 - \tau^2 d\xi^2$, $\tau = \sqrt{t^2 - \tau^2}$, $\xi = Tanh^{-1}(z/t)$, $u^{\mu} = (1,0,0,0)$, $\pi^{\mu}_{\nu} = diag(0, -\pi/2, -\pi/2, \pi)$, $\delta_{\pi\pi} = 4/3\tau_R$, $\tau_{\pi\pi} = \lambda\tau_R$.
- For the IS theory hydrodynamic equation becomes,

$$\frac{d\varepsilon}{d\tau} = -\frac{\varepsilon + p}{\tau} + \frac{\pi}{\tau}, \tag{102}$$

$$\tau_R\frac{d\pi}{d\tau} + \pi = \frac{4}{3}\frac{\eta}{\tau} - \left(\frac{4}{3} + \lambda\right)\tau_R\frac{\pi}{\tau}, \tag{103}$$

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[26] Bjorken, J.D..Phys Rev1983,D27,140-151.

[27] G.S.Denicol and J. Noronha, arXiv:1711.01657

More general case

- IS theory in the presence of baryon number density,

$$\begin{aligned} \dot{\varepsilon} + (\varepsilon + p)\partial_\mu u^\mu - \pi^{\mu\nu}\sigma_{\mu\nu} &= 0, \\ (\varepsilon + p)\dot{u}^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \pi^{\mu\nu} &= 0, \\ \dot{n} + n\partial_\mu u^\mu + \partial_\mu n^\mu &= 0. \end{aligned} \tag{104}$$

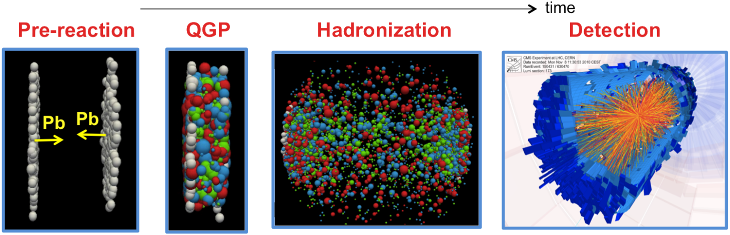
$$\begin{aligned} \dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta \\ - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}, \end{aligned} \tag{105}$$

$$\begin{aligned} \dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} = \beta_n \nabla^\mu \alpha - n_\nu \omega^{\nu\mu} - n^\mu \theta - \frac{3}{5} n_\nu \sigma^{\nu\mu} \\ - \frac{3\beta_n}{\varepsilon + p}\pi^{\mu\nu}\nabla_\nu \alpha. \end{aligned} \tag{106}$$

- Here $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ is the anti-symmetric vorticity tensor, $\sigma^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\theta\Delta^{\mu\nu}$, $\theta = \partial_\mu u^\mu$ is the expansion scalar, $\alpha = \mu/T$, $\tau_\pi = \eta/\beta_\pi$, and $\tau_n = \kappa_n/\beta_n$. The quantity η is the shear viscosity coefficient, μ and T denote baryon chemical potential and temperature respectively.

Choice of fluid four velocity

- The definition of fluid four velocity is crucial in dissipative fluids due to the presence of both energy and particle diffusion,
  1. Eckart definition: velocity is defined by the flow of particles:
  $$N^\mu = nu^\mu, \;\; n^\mu = 0. \tag{107}$$
  2. Landau definition: velocity is specified by the flow of the energy.
  $$u_\mu T^{\mu\nu} = \epsilon u^\nu \implies h^\mu = 0. \tag{108}$$
  3. In the Landau frame,
  $$T^{\mu\nu} = \epsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \tag{109}$$
  $$N^\mu = nu^\mu + n^\mu. \tag{110}$$
- In order to derive the complete set of equations for dissipative fluid dynamics along with the conservation equation we also need the dynamical or constitutive relations satisfied by the dissipative tensors, Π, $\pi^{\mu\nu}$ and n^μ.



- Different stages[28]:
  1. Modelling the initial stages.
  2. Bulk evolution of the locally thermalized medium.
  3. Hadronic freezeout.
- To model the bulk evolution of the strongly interacting matter produced in relativistic heavy-ion collisions, relativistic dissipative hydrodynamics has become the basic theoretical tool. [29], [30].

[28] https://wl33.web.rice.edu/research.html
[29] C. Gale, S. Jeon, B. Schenke, Int.J.Mod.Phys.A 28 (2013) 1340011
[30] S. Jeon, U. Heinz, Int.J.Mod.Phys.E 24 (2015) 10, 1530010.