Causality and stability of third-order fluid dynamics

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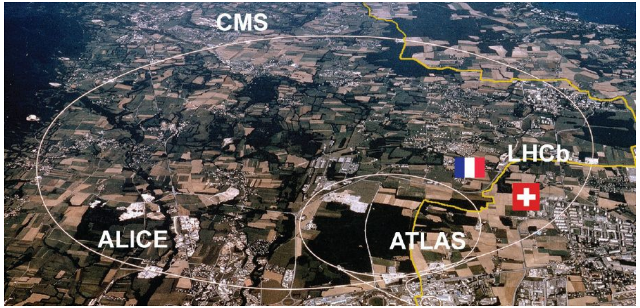
Overview

- Motivation
- Ideal fluid dynamics
- Dissipative fluid dynamics
- Third-order fluid dynamics
- Causality and stability
- Conclusions and perspectives

Motivation

Why do we study relativistic fluid dynamics? Heavy-ion collisions!



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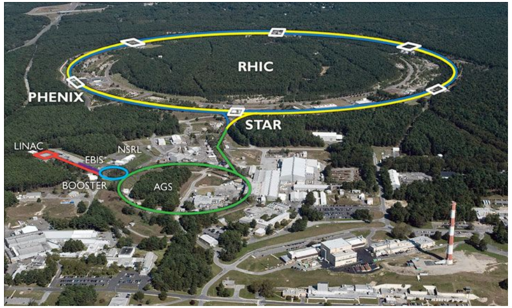

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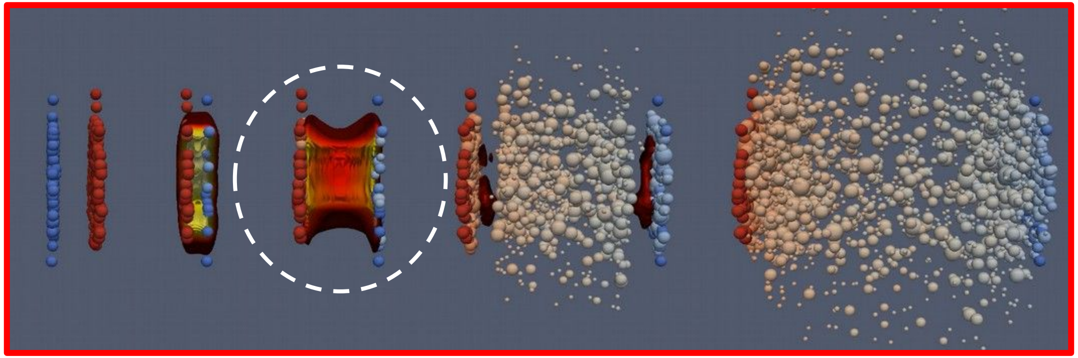


Building a theory

- What are the necessary ingredients?
    - Conservation laws
    - Equation of state
    - Relations for the dissipative currents
        - Phenomenology or kinetic theory

- What are the **minimal** conditions a formalism must satisfy?
    - (Linear) stability of the equilibrium state

Linear stability analysis

How can we conclude if a given formalism is suitable to describe relativistic fluids?

equilibrium state

perturbations

"Stable"

"Unstable"

Necessary conditions, although not always sufficient!

Ideal fluid dynamics

Conservation laws

$$\partial_\mu N^\mu = 0$$

net-charge conservation

$$\partial_\mu T^{\mu\nu} = 0$$

energy-momentum conservation

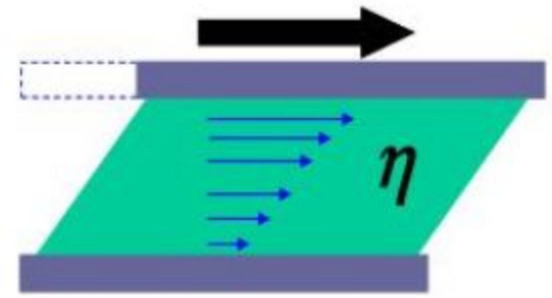
$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu} P$$

$$N^\mu = n u^\mu$$

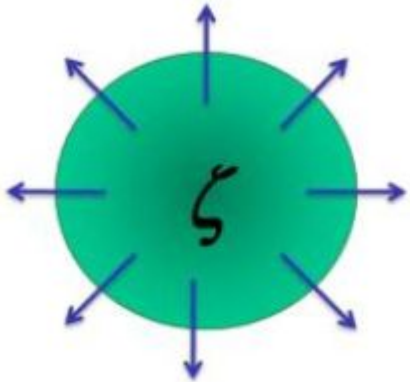
Complete description using conservation laws + equation of state

Ideal fluids?

In general, there are no ideal fluids



shear viscosity



bulk viscosity

diffusion

[Song @ CATHIE/TECHQM Workshop (2009)]

These effects must be taken into account in the equations

Dissipative fluid dynamics

Conserved currents

shear-stress tensor

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P + \Pi) + \pi^{\mu\nu},$$

bulk viscous pressure

$$N^\mu = n u^\mu + n^\mu,$$

net-charge diffusion

5 equations and 14 independent fields!

↳ Constitutive relations

Navier-Stokes theory

$$\pi^{\mu\nu} = 2\eta\Delta^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$

Eckart, Phys. Rev. (1940)
Landau & Lifshitz (1959)

- **First-order** theory
- Dissipative currents in terms of the fluid-dynamical variables
- **Acausal** and **unstable** in the linear regime

Hiscock & Lindblom, PRD (1983)
Hiscock, PRD (1986)
Hiscock & Lindblom, PRD (1987)

Israel-Stewart theory

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\Delta^{\mu\nu\alpha\beta}\partial_\alpha u_\beta + \cdots$$

→ **nonlinear terms**

Israel, Ann. Phys. (1976)
Israel & Stewart, Ann. Phys. (1979)

- **Second-order** theory
- Dissipative currents as dynamical variables
- Linearly **causal** and **stable** in certain conditions
- Widely used in numerical models

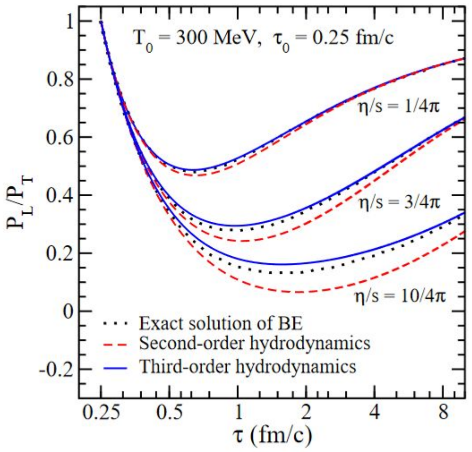
Olson & Hiscock, Ann. Phys (1989)
Olson, Ann. Phys (1990)
Denicol et. al., J. Phys. G (2008)
Pu et. al., PRD (2010)
CVB & Denicol, PRD (2020)

Third-order fluid dynamics

$$\begin{aligned}
\dot{\pi}^{\langle\mu\nu\rangle} =& -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} \\
& - \frac{4}{3}\pi^{\mu\nu}\theta + \frac{25}{7\beta_\pi}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} - \frac{1}{3\beta_\pi}\pi_\gamma^{\langle\mu}\pi^{\nu\rangle\gamma}\theta \\
& - \frac{38}{245\beta_\pi}\pi^{\mu\nu}\pi^{\rho\gamma}\sigma_{\rho\gamma} - \frac{22}{49\beta_\pi}\pi^{\rho\langle\mu}\pi^{\nu\rangle\gamma}\sigma_{\rho\gamma} \\
& - \frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot{u}_\gamma\tau_\pi\right) + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right) \\
& - \frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right) + \frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^{\langle\mu}\pi^{\nu\rangle\gamma}\right) \\
& - \frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\gamma}\pi^{\langle\mu\nu\rangle}\right) + \frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^{\gamma}\pi^{\langle\mu\nu\rangle}\right) \\
& - \frac{2}{7}\tau_\pi\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} - \frac{2}{7}\tau_\pi\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma} \\
& - \frac{10}{63}\tau_\pi\pi^{\mu\nu}\theta^2 + \frac{26}{21}\tau_\pi\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta.
\end{aligned}$$

Jaiswal, PRC (2013)

- Boltzmann equation
- *Relaxation Time Approximation* (RTA)
- Chapman-Enskog method
- Only shear viscosity



<u>Acausal</u> and <u>unstable</u>

Third-order fluid dynamics

Converting gradients of shear-stress into an independent variable

$$\nabla^{\langle\alpha}\pi^{\mu\nu\rangle} \longrightarrow \rho^{\alpha\mu\nu}$$

Equation of motion for the shear-stress tensor

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\langle\mu\nu\rangle} = 2\eta\sigma^{\mu\nu} - \tau_\pi \nabla_\alpha \rho^{\alpha\mu\nu} + \cdots$$

Introducing a relaxation equation

$$\boxed{\tau_\rho \dot{\rho}^{\langle\mu\nu\lambda\rangle}} + \rho^{\mu\nu\lambda} = \frac{3}{7}\eta_\rho \nabla^{\langle\mu}\pi^{\nu\lambda\rangle} + \text{nonlinear terms}$$

CVB & Denicol, [arXiv:2107.10319]

Causality and stability

Israel-Stewart:

Pu et. al., PRD (2010)

$$3(1-c_s^2)\tau_\pi \geq \frac{4\eta}{\varepsilon_0 + P_0}$$

Third-order:

CVB & Denicol, [arXiv:2107.10319]

$$\left[3\tau_\pi\left(1-c_s^2\right) - 4\frac{\eta}{\varepsilon_0 + P_0}\right]\tau_\rho > \frac{27}{35}\eta_\rho\tau_\pi\left(1-c_s^2\right),$$

$$3(1-c_s^2)\tau_\pi \geq \frac{4\eta}{\varepsilon_0 + P_0}.$$

Conclusions & Perspectives

- The original third-order theory is linearly unstable
- Stability implies the inclusion of a relaxation time scale
- Constraints for the transport coefficients

- Derivation of a complete nonlinear third-order theory
- Analyze the effects of including bulk viscosity
- Compare simulations with previous results

Thank you!