
Radial and non-radial oscillation modes of non-rotating compact stars

Dániel Barta

barta.daniel@wigner.hu

Wigner Research Centre for Physics, Budapest



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In collaboration with:

Balázs Kacskovics (Wigner RCP), **Mátyás Vasúth** (Wigner RCP)

Related papers: [1908.02808](#) & [1904.00907](#)

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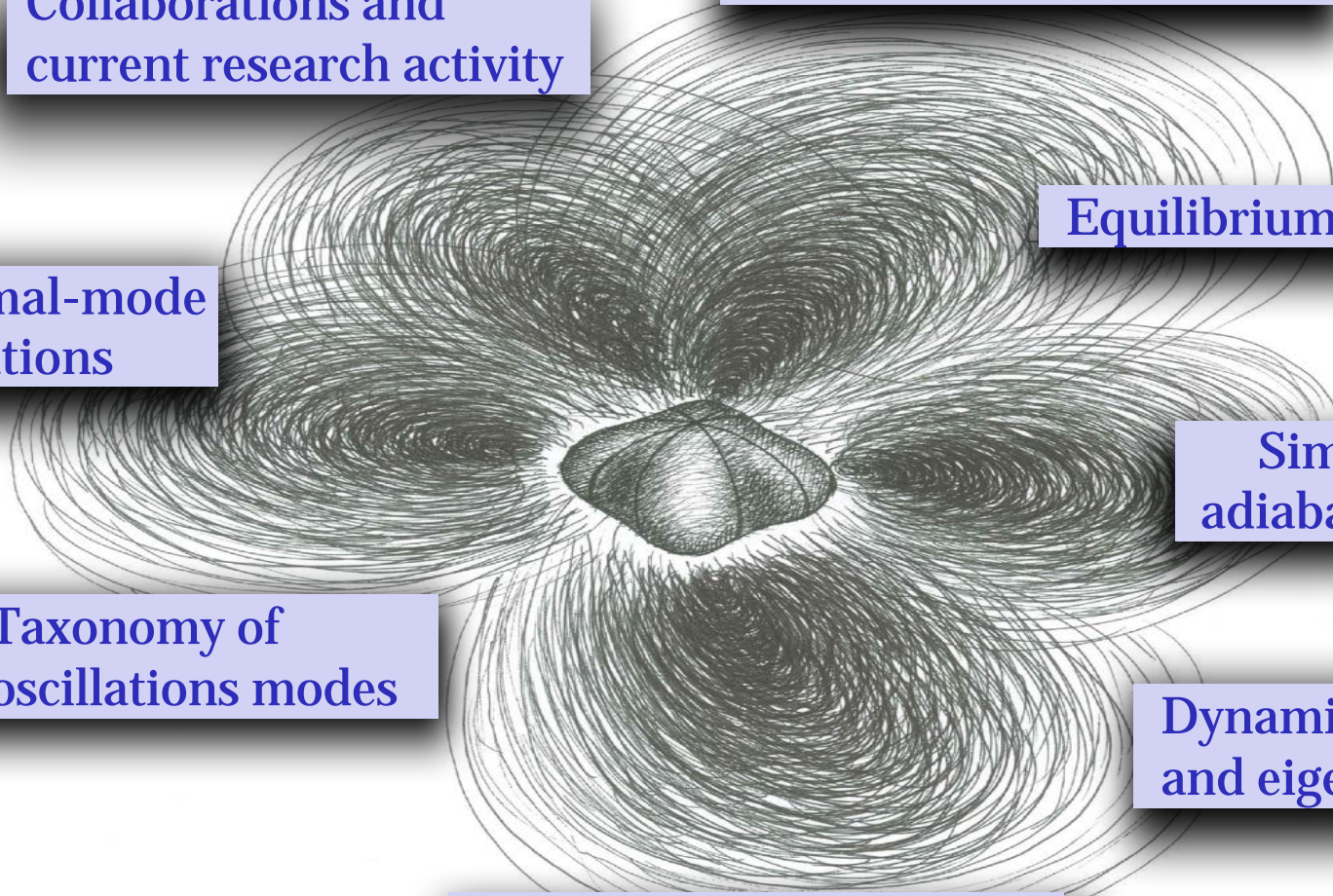
Polar normal-mode
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adiabatic radial oscillation

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& their parity



SOURCES OF OSCILLATIONS

- **Supernova explosion:** triggers all kinds of oscillation modes
- **Starquakes** caused by cracks in the crust or magnetic reconfiguration
- **Accretion** triggers oscillations
- **Tidal forces** in binary mergers
- **Oscillation modes** are *unstable* to gravitational wave emission → ***r*-mode** or ***f*-mode** oscillations



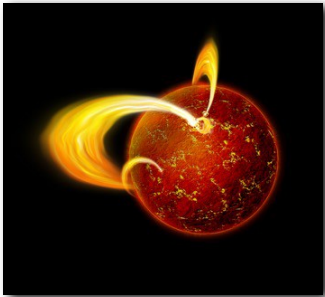
NEUTRON STARS AS GW SOURCES

“Burst” emission

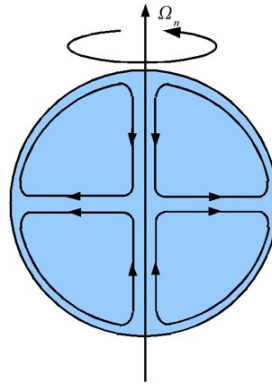
Binary neutron star mergers
(our safest bet for detection)



Magnetar flares
(likely too weak)



Pulsar glitches
(likely too weak)

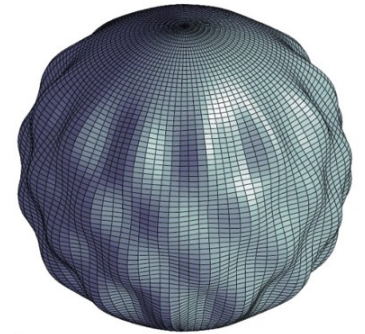


Continuous emission

Non-axisymmetric mass
quadrupole (“mountains”)



Fluid part (oscillations)



EQUILIBRIUM STELLAR MODEL

Einstein equations: $G_{\mu\nu} = 8\pi T_{\mu\nu}$

Energy–momentum conservation: $\nabla_\mu T^{\mu\nu} = 0$

Energy–momentum tensor (perfect fluid):

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$$

The energy density and the pressure of the fluid are related by an *equation of state*: $p = p(\rho)$ (zero temperature)

THERMODYNAMIC PROPERTIES

Gravitational mass:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

Gravitational potential:

$$\frac{dv}{dr} = \frac{2m + 8\pi r^3 p}{r(r - 2m)}$$

STRUCTURE

Hydrostatic equilibrium:

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r^2(1 - 2M/r)}$$

(Tolman–Oppenheimer–Volkoff equation)

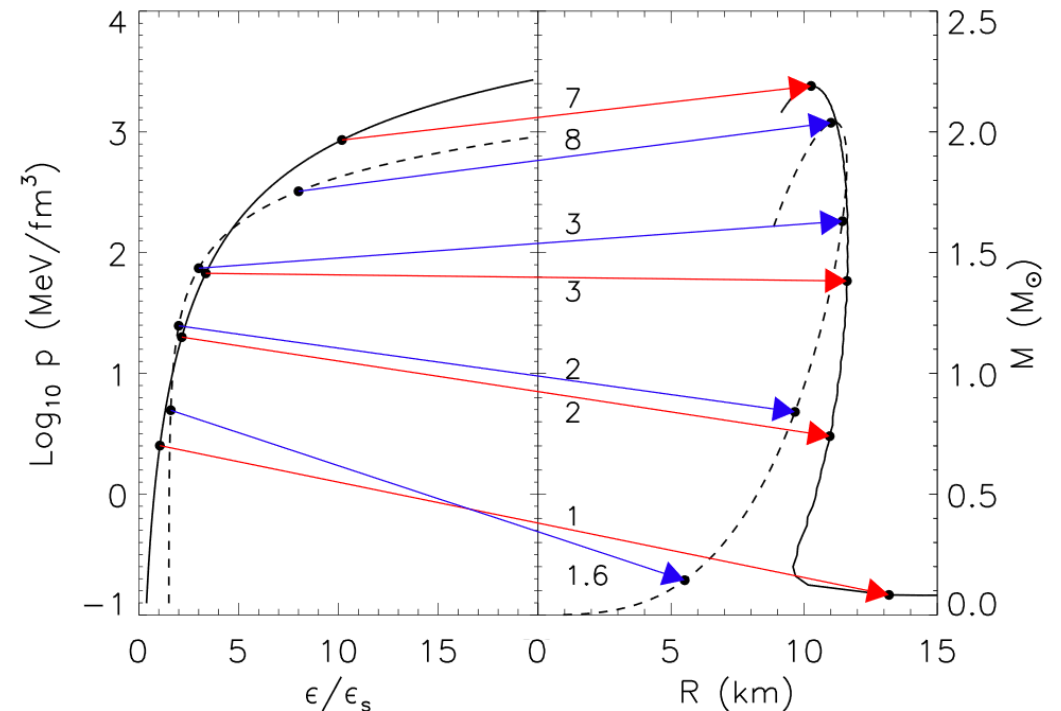
At the stellar center ($r = 0$):

- $M(0) = 0$: the mass function vanish
- $\rho_0 \equiv \rho(0)$: central density is freely specified

At the stellar surface ($r = R$):

- $M \equiv m(R)$: total mass of the star
- $p(R) = 0$: the isotropic pressure vanishes
- $e^{v(R)} = 1 - 2M/R$: normalizing the time coordinate at spatial infinity

BOUNDARY CONDITIONS



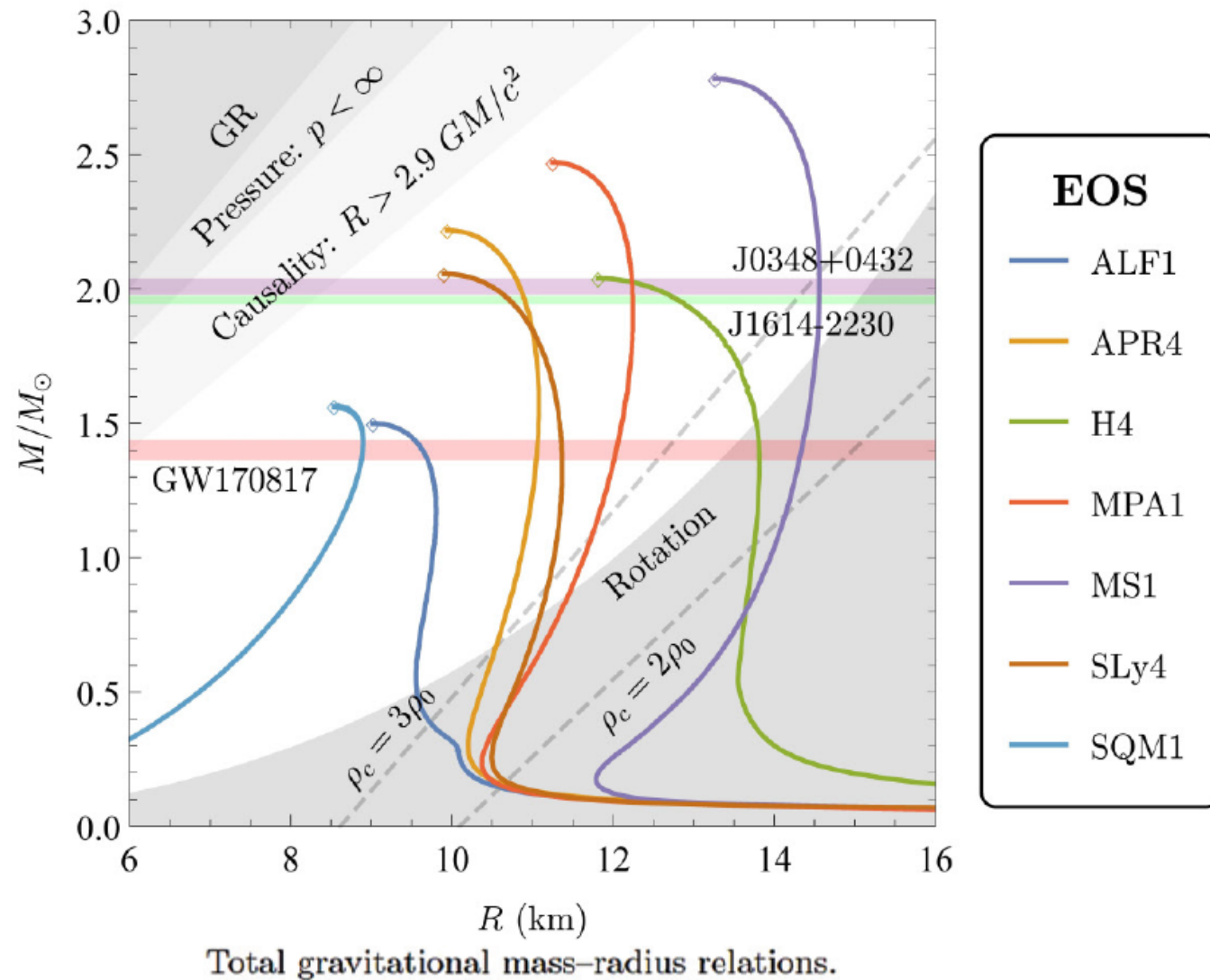
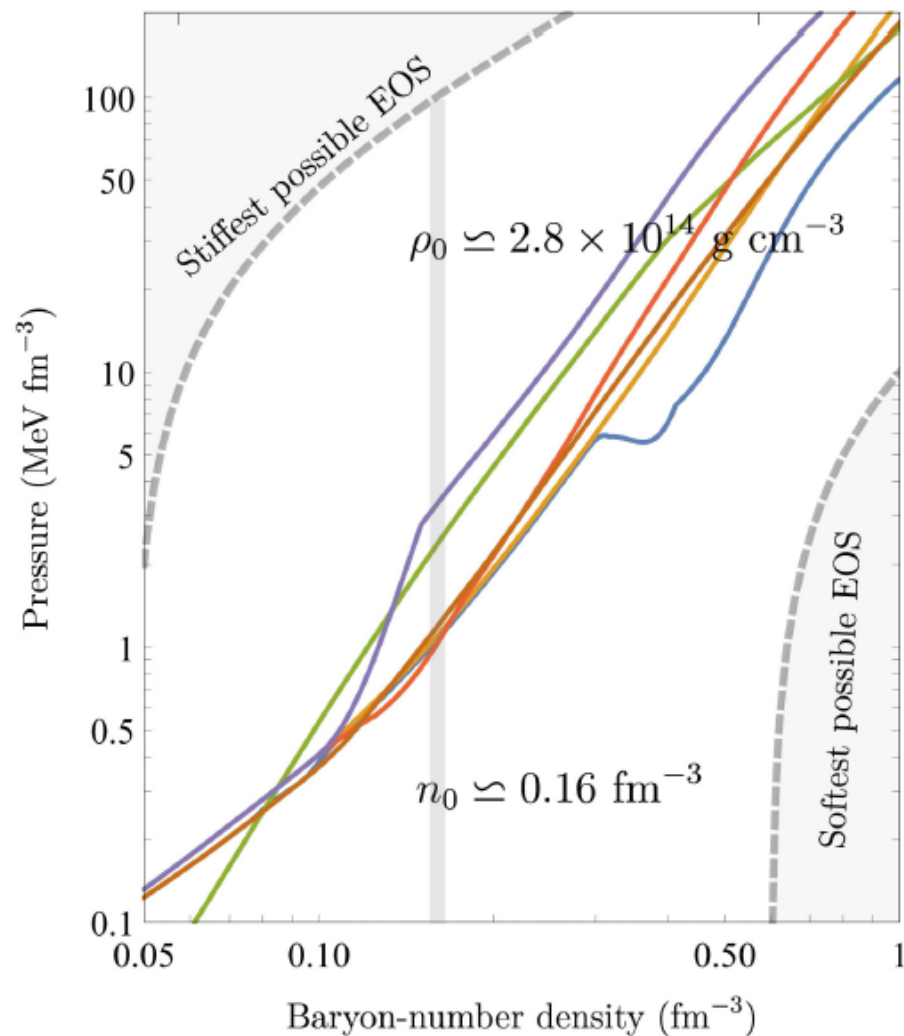
Metric tensor: $(ds^2)_0 = -e^v dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$
 where $m(r) \equiv r(1 - e^{-\lambda})/2$ is the „gravitational mass” inside radius r

Comoving coordinates: $u^\mu = e^{v/2}[-1, 0, 0, 0]$

Normalization condition: $g_{\mu\nu} u^\mu u^\nu = 1$

FLUID 4-VELOCITY

REALISTIC TABULATED EOS MODELS AND ASSOCIATED NEUTRON STARS



SIMPLEST CASE: LINEAR ADIABATIC RADIAL OSCILLATIONS

- **Perturbations in the fluid 4-velocity** can be expressed by

$$\delta u_{\text{radial}}^{\mu} = [e^{\nu_0/2}, -e^{\lambda_0 - \nu_0/2} \delta u_r, 0, 0] \quad \text{where} \quad \delta u_r = dr/dt$$

is associated with a displacement field in the Lagrangian representation: $\partial \xi / \partial t = \delta u_r$.

1. The perturbation equations are obtained from $\delta(\nabla_{\mu} T^{\mu\nu}) = 0$, $\delta(G_{\mu\nu} - 8\pi T_{\mu\nu}) = 0$. Then, it is straightforward to compute the linear perturbations of any equilibrium quantity ($\delta\rho, \delta p, \dots$).
2. With the assumption of harmonic time dependence, [Chandrasekhar \(1964\)](#) showed that:

FUNDAMENTAL EQUATION
FOR RADIAL PULSATION

$$\omega^2 e^{\lambda_0 - \nu_0} (p_0 + \varepsilon_0) \xi = \left[\frac{4}{r} \frac{dp_0}{dr} + 8\pi e^{\lambda_0} p_0 (p_0 + \varepsilon_0) - \frac{1}{p_0 + \varepsilon_0} \left(\frac{dp_0}{dr} \right)^2 \right] \xi - e^{-(\lambda_0 + 2\nu_0)/2} \frac{d}{dr} \left[e^{(\lambda_0 + 3\nu_0)/2} \frac{\Gamma p_0}{r^2} \frac{d}{dr} \underbrace{\left(r^2 e^{-\nu_0/2} \xi \right)}_X \right]$$

a) The fluid at the center of the star is assumed to remain at rest: $X = 0$ at $r = 0$

BOUNDARY CONDITIONS

b) The Lagrangian change in the pressure vanishes at the surface: $\delta p = e^{\nu_0/2} r^{-2} \Gamma p_0 \frac{d}{dr} (r^2 e^{-\nu_0/2} X) \equiv 0$ at $r = R$

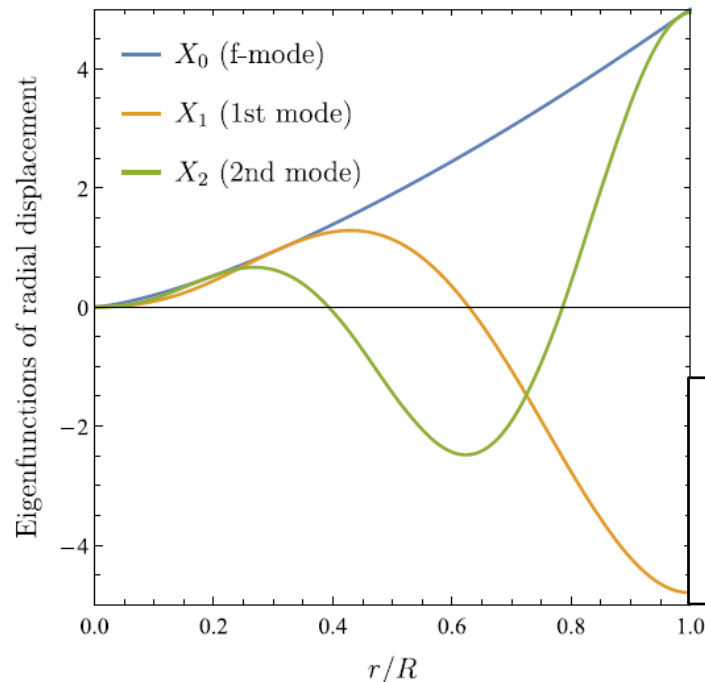
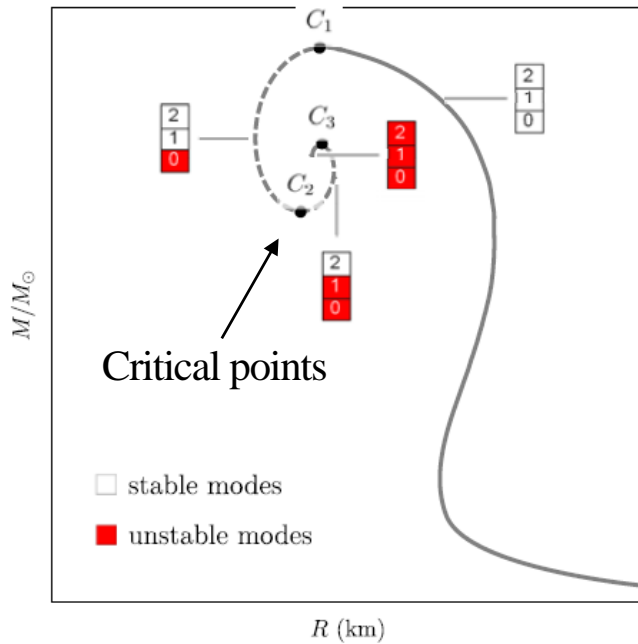
The fundamental equation together with its boundary conditions constitutes a [Sturm–Liouville eigenvalue problem \(SL-EVP\)](#) for a discrete set of scalar-valued eigenfunctions of radial displacement $\{X_0(r), X_1(r), \dots, X_j(r), \dots\}$ with their respective eigenvalues $\{\omega_0^2, \omega_1^2, \dots, \omega_j^2, \dots\}$.

3. To find the eigenfrequencies, we convert the boundary value problem to an initial value problem by "[shooting](#)" method!

DYNAMICAL STABILITY

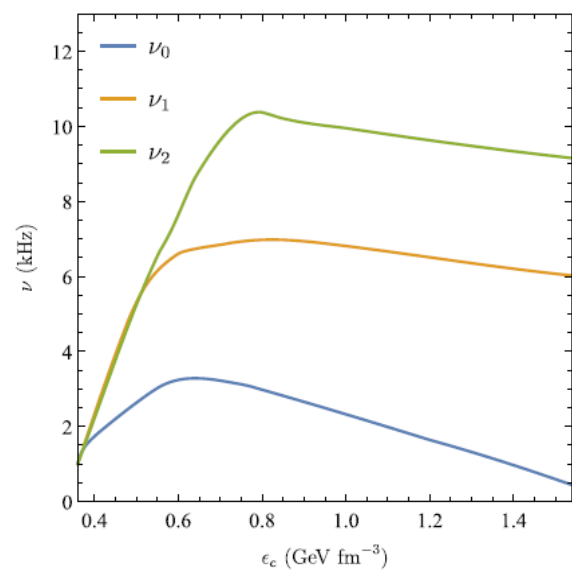
- *If any of these ω_j^2 is negative* for a particular star, the frequency is purely imaginary and therefore any perturbation of the star ($\sim e^{i\omega t}$) will grow exponentially in time. \Rightarrow **dynamically unstable**
- If $\omega_j^2 > 0$, **the star is stable** against adiabatic radial perturbations (up to the j th excited oscillation mode)

Schematic illustration of the unstable branch of the mass–radius relation. [Barta 2021, CQG **38**, 185002]

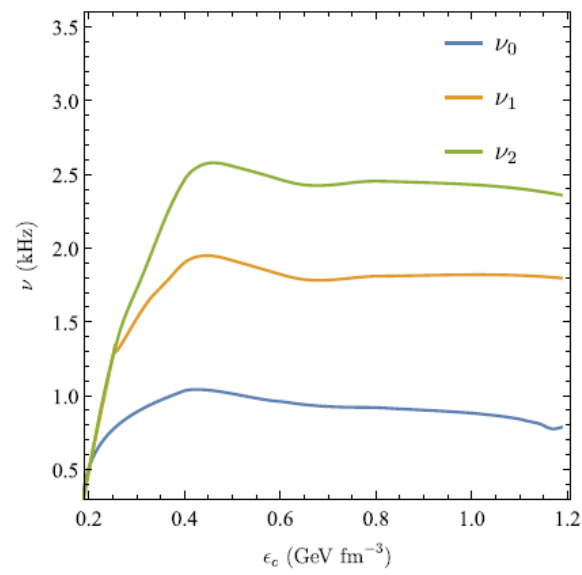


- The smallest eigenvalue ω_0^2 is associated with the *fundamental-mode frequency* of radial oscillations which *has no nodes* between the center and the stellar surface, whereas the first excited mode ($j = 1$) has a node, the second one ($j = 2$) has two, and so forth.

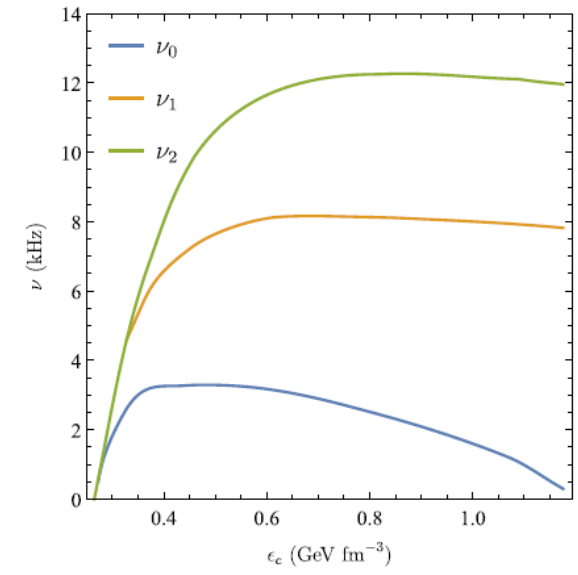
Eigenfunctions functions of radial displacement for the first three lowest-frequency oscillation modes $\{X_0(r), X_1(r), X_2(r)\}$ as a function of the fractional radius r/R obtained for SLy4 EoS at a central density $\rho_c = 0.547 \text{ GeV fm}^{-3}$. The displacement amplitude has been renormalized such that $X_0 = 1$. [Barta 2021, CQG **38**, 185002]



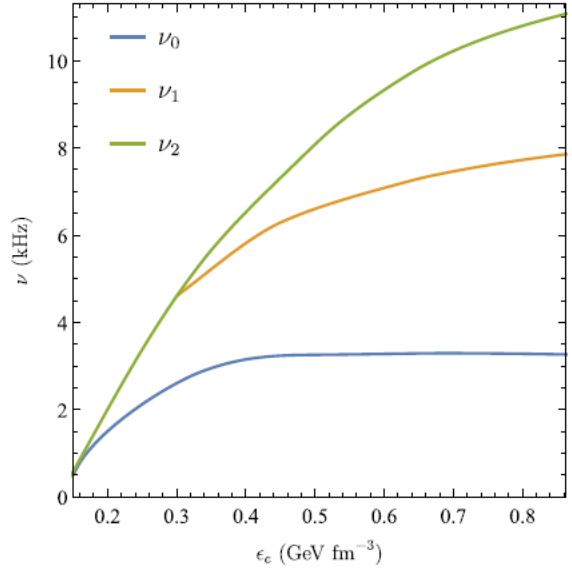
(a) APR4



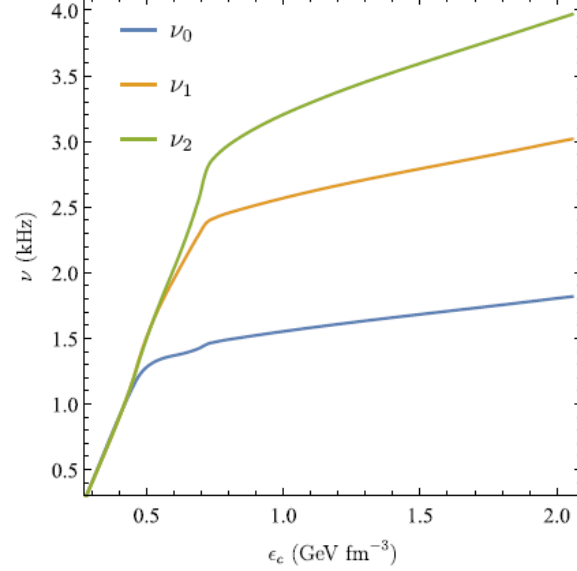
(b) H4



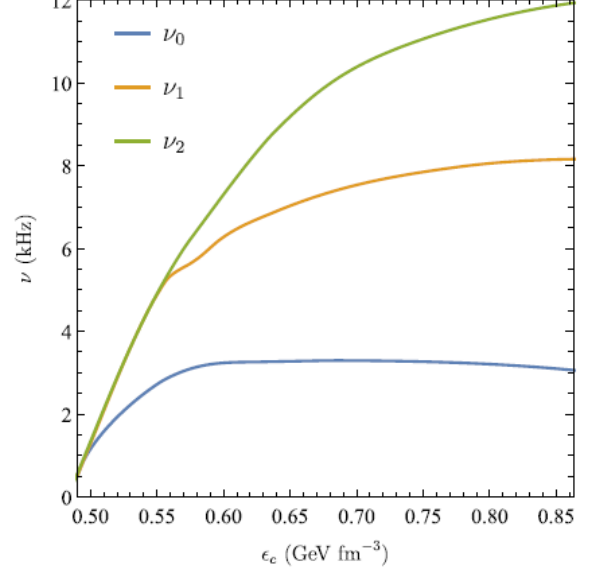
(c) MPA1



(d) MS1



(e) SLy4



(f) SQM1

The frequencies of the fundamental mode (ν_0) and the first two lowest-frequency excited modes (ν_1 and ν_2) of radial oscillation as functions of central density (ϵ_c) for each EoS of nucleonic state (APR4, MPA1, MS1, SLy4) and hybrid nucleon–hyperon–quark state (H4, SQM1).

INTERPRETATION OF RESULTS (RADIAL OSCILLATION)

1. *The decay of the lowest-frequency eigenmodes is a general feature* (irrespective of the particular EoS): The f-mode frequency drops toward zero as the particular stellar model approaches its dynamical stability limit which, indeed, is indicated by the presence of an eigenmode with zero-frequency.
 - The dynamical instability in stars with MPA1 and APR4 is exposed by the presence of a very low frequency of the *f*-mode, which has dropped to less than 5% of that of the first excited mode, at central energy densities associated with the maximal-mass stable configurations.
2. *The oscillation frequency of higher modes is always larger than that of a lower stable mode* and for all modes it appears to decrease as the central energy density approaches the smallest possible value ε_{\min} of the particular stellar model
 - When the central energy density of NSs is approaching ε_{\min} , such compact objects *become approximately homogeneous* and due to their small mass.
3. *Stellar models of softer EoSs have higher frequencies in the f-mode than the stiffer ones for the same central density.*
 - Stellar models of softer EoSs are generally associated with more centrally condensed stars with larger average densities.

NON-RADIAL FLUID DISPLACEMENT AND PERTURBATION

- The perturbation equations are obtained in the following way:

$$\delta(\nabla_\mu T^{\mu\nu}) = 0, \quad \delta(G_{\mu\nu} - 8\pi T_{\mu\nu}) = 0$$

- Perturbations in the 4-velocity of a fluid element δu^μ** (associated with a mode) can be decomposed in radial and angular parts:

$$\delta \mathbf{u} = \sum_{l,m} \underbrace{[W_l \hat{\mathbf{r}} + V_l \nabla Y_m^l]}_{\text{polar part: parity } (-1)^l} + \underbrace{U_l (\hat{\mathbf{r}} \times \nabla Y_m^l)}_{\text{axial part: parity } (-1)^{l+1}} e^{i\omega t}$$

Parity is defined to be the *change in sign* under a combination of reflection in the equatorial plane and rotation by π .

where $W_l(r)$, $V_l(r)$, $U_l(r)$ are radial eigenfunctions.

- A linear perturbation of scalar quantities ($\delta\rho$, δp , etc.) can be written as a sum of *quasi-normal modes* that are characterized by the indices (l,m) of the spherical harmonic functions Y_m^l and harmonic time dependence $e^{i\omega t}$.

The **frequency** ω is a *complex number*:

1. real part corresponding to the frequency of oscillations:

$$\text{Re}(\omega) = \omega_n \sqrt{1 - \zeta_d^2}, \text{ where } \omega_n \text{ is the natural frequency}$$

2. imaginary part to the relaxation time: $1/\tau = \text{Im}(\omega) = -\omega_n \zeta_d$

PARITY OF PERTURBATIONS

- A general non-stationary asymmetric spacetime:

$$ds^2 = -e^\nu dt^2 + e^{\mu_2} dr^2 + e^{\mu_3} d\theta^2 + e^\psi (d\varphi - \omega dt - q_2 dr - q_3 d\theta)^2$$

- Two different types (or parity) of perturbations of the spherically symmetric metric:
 1. **Polar (or “magnetic-type”) perturbation** has “even parity” $\pi = (-1)^l$. It gives small increments to the already nonzero metric coefficients $(e^\nu, e^{\mu_2}, e^{\mu_3}, e^\psi)$.
 2. **Axial (or “electric-type”) perturbation** has “odd parity” $\pi = (-1)^{l+1}$. This perturbation induces *frame dragging* and *imparts a rotation* to the compact star. It gives small values to the metric coefficients (ω, q_2, q_3) that were zero in $(ds^2)_0$.

In non-rotating stars (i.e. up to $O(\Omega)$) the **polar** and **axial perturbations** remain completely *decoupled*.

- Further more, for small-amplitude motions *there is no coupling* between the various spherical harmonics
- The equations of motion are considerably simplified by making an appropriate choice of “gauge” – *the Regge–Wheeler choice* – for the gravitational field!

TAXONOMY OF NS OSCILLATION MODES (I)

1. Purely polar fluid modes are slowly damped modes analogous to the Newtonian fluid pulsations:

- **Pressure (p) modes:** pressure is the restoring force

There are infinitely many of them. It experiences substantial fluctuations when these modes are excited. They have nodes inside the star. Higher frequencies 4 – 7 kHz (p_1 mode), damping time – a few seconds. *The radial displacement is much bigger than the tangential.*

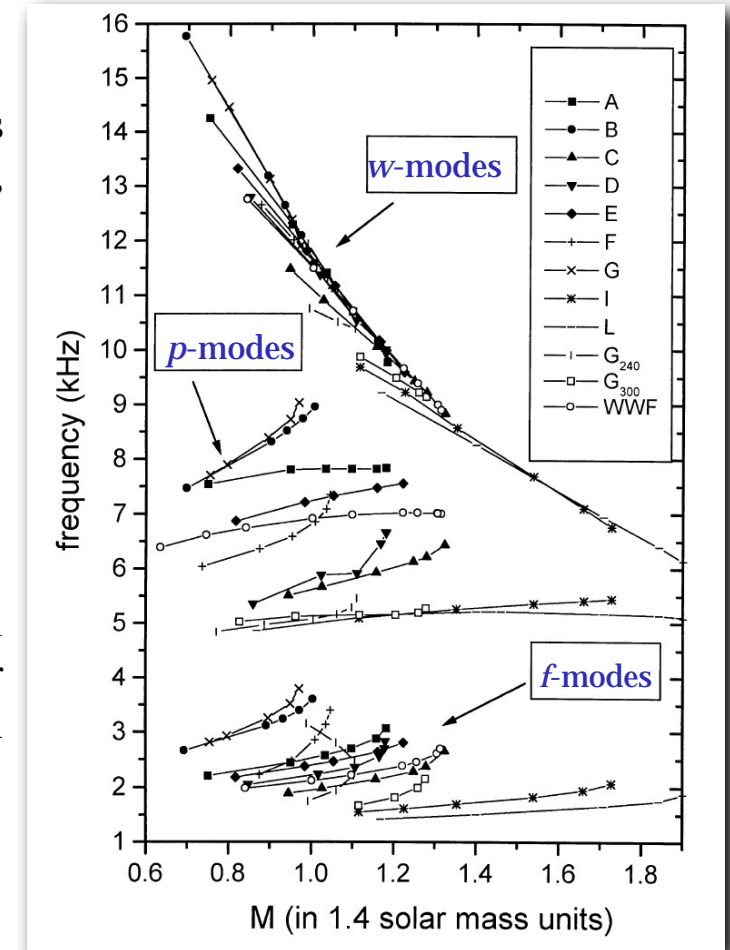
- **Fundamental (f) mode:** the first (nodeless) p -mode

The f -mode eigenfunctions have *no nodes* inside the star, and they grow towards the surface. Frequency of 1.5–3 kHz, Damping time – second (0.1 – 0.5 sec).

- **Gravity (g) modes:** driven by buoyancy (thermal/composition gradients)

Arise because gravity tends to smooth out material inhomogeneities along equipotential level-surfaces and buoyancy is the restoring force. Present only if we have temperature or composition gradients. Frequencies – normally below 100Hz, damping times – can even reach years. *The tangential displacement is much bigger than the radial.*

Typical spectra of *non-rotating* NSs without stratification. [Kokkotas et al. 2001]



TAXONOMY OF NS OSCILLATION MODES (II)

2. Purely axial fluid modes are not characterized by pulsations;

- a) rather, by a stationary differential rotation of the fluid inside the star.
- b) by gravitational waves which do not couple to the star at all.

- **Inertial (*i*) modes:** driven by rotation (Coriolis force)

Degenerate at zero frequency in nonrotating stars. In a rotating star, some inertial modes are generically unstable to the CFS instability; they have frequencies from zero to kHz and growth times inversely proportional to a high power of the star's angular velocity.

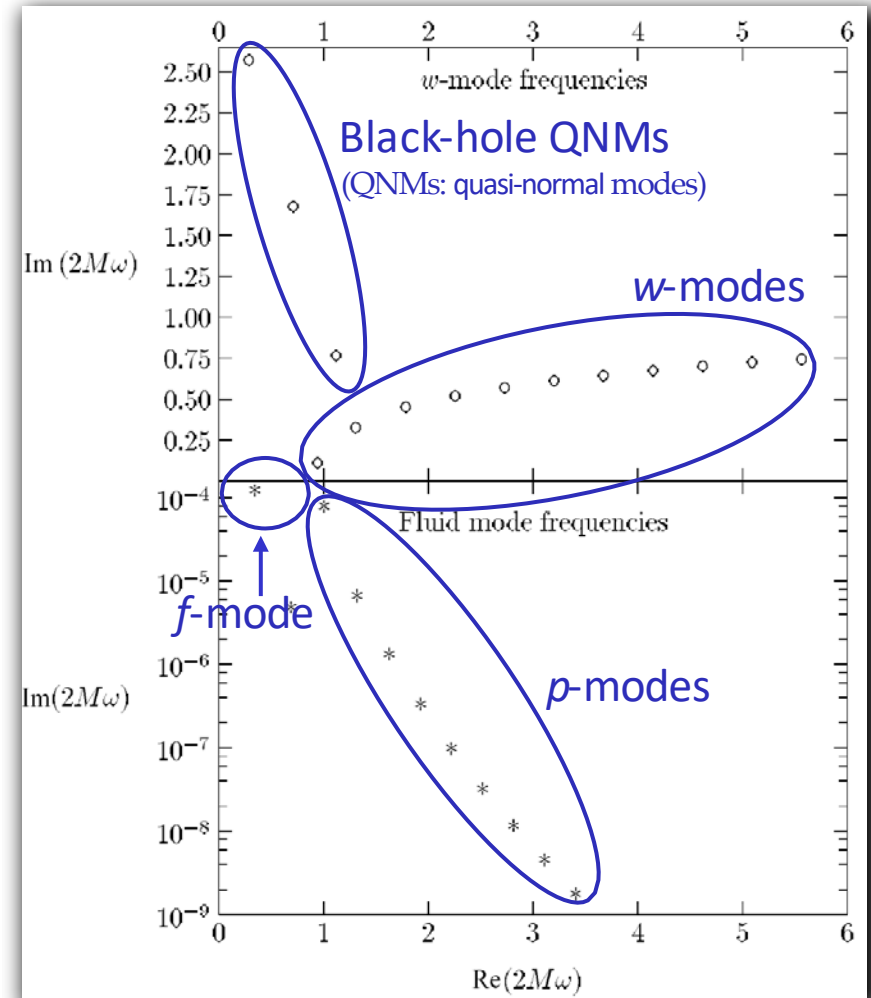
- **Rotation (*r*) mode:** special case of *i*-modes

A special case of inertial modes that reduce to the classical axial *r*-modes in the Newtonian limit. Generically unstable to the CFS instability with growth times as short as a few seconds at high rotation rates. (Chandrasekhar 1970; Friedman & Schutz 1978)

3. Coupled spacetime modes: both polar and axial modes.

- **Wave (*w*) modes:** dissipating energy through gravitational waves

Analogous to the quasi-normal modes of a black hole. They need dynamical spacetime. High frequency > 6 kHz, strongly damped modes with damping times of the order of 0.1 ms. *Very weak coupling to the fluid* and do not induce significant fluid motion.



[Andersson, Kokkotas, Schütz 1996, *MNRAS* **280**, 1230]

EVEN-PARITY (POLAR) NORMAL MODES OF PULSATION

- The geometry of spacetime inside and around the equilibrium configuration fluctuates in a manner described by **10 functions**, $h_{\mu\nu} = h_{\nu\mu}$, of $(t, r, \theta, \varphi) = (x^0, x^1, x^2, x^3)$.

$$\Rightarrow ds^2 = (ds^2)_0 + h_{\mu\nu} dx^\mu dx^\nu$$

- Using an appropriate gauge (Regge–Wheeler):

$$h_{\nu\mu}^{\text{polar}} = \begin{pmatrix} e^\nu H_0 & H_1 & 0 & 0 \\ H_1 & e^\lambda H_0 & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta K \end{pmatrix} Y_m^l$$

- For even-parity harmonics the equations of motion are a *set of coupled equations* for
 - the fluid-displacement functions $V(r, t), W(r, t)$ and
 - the metric perturbation functions $H_0(r, t), H_1(r, t), K(r, t)$.

\Rightarrow reduced to only 5 functions!

The entire theory of non-radial even-parity-mode pulsations consists of the study of the “equations of motion” which govern the **8 functions**, ξ^i , and $h_{\mu\nu}$, of (t, r, θ, φ) .

With the perturbed (polar mode) metric tensor:

$$ds^2 = -e^\nu(1 + r^\ell H_0 Y_m^\ell e^{i\omega t})dt^2 - 2i\omega r^{\ell+1} H_1 Y_m^\ell e^{i\omega t} dt dr + \\ + e^\lambda(1 - r^\ell H_0 Y_m^\ell e^{i\omega t})dr^2 + r^2(1 - r^\ell K Y_m^\ell e^{i\omega t})(d\theta^2 + \sin^2 \theta d\phi^2)$$

The small-amplitude motion of the perturbed configuration is described by the **Lagrangian displacement vector field** $\xi^i(t, r, \theta, \varphi)$. In an appropriate gauge $\xi^t = 0$, and the other components of the displacement 3-vector are given by

$$\xi^r = r^{l-1} e^{-\lambda/2} W Y_m^l e^{i\omega t}, \quad \xi^\theta = -r^{l-2} V \partial_\theta Y_m^l e^{i\omega t}, \quad \xi^\varphi = -r^l (r \sin \theta)^{-2} V \partial_\theta Y_m^l e^{i\omega t}$$

and the corresponding **perturbations** in the **fluid 4-velocity** can be expressed by

$$\delta u_{\text{polar}}^\mu = e^{-\nu/2} [H_0 P_l(\cos \theta), e^{-\lambda/2} \partial_t W P_l(\cos \theta), -r^{-2} \partial_t V \partial_\theta P_l(\cos \theta), 0].$$

lead to a ODE system of a set of 4 equations:

$$\begin{aligned} H_1' &= -r^{-1} [l + 1 + 2M e^\lambda r^{-1} + 4\pi r^2 e^\lambda (p - \rho)] H_1 + r^{-1} e^\lambda [H_0 + K - 16\pi(\rho + p)V], \\ K' &= r^{-1} H_0 + \frac{1}{2} l(l + 1) r^{-1} H_1 - [(l + 1) r^{-1} - \frac{1}{2} v'] K - 8\pi(\rho + p) e^{\lambda/2} r^{-1} W, \\ W' &= -(l + 1) r^{-1} W + r e^{\lambda/2} [\gamma^{-1} p^{-1} e^{-\nu/2} X - l(l + 1) r^{-2} V + \frac{1}{2} H_0 + K], \\ X' &= -l r^{-1} X + (\rho + p) e^{\nu/2} \left\{ \frac{1}{2} (r^{-1} - \frac{1}{2} v') H_0 + \frac{1}{2} [r \omega^2 e^{-\nu} + \frac{1}{2} l(l + 1) r^{-1}] H_1 \right. \\ &\quad \left. + \frac{1}{2} (\frac{3}{2} v' - r^{-1}) K - \frac{1}{2} l(l + 1) v' r^{-2} V - r^{-1} [4\pi(\rho + p) e^{\lambda/2} + \omega^2 e^{\lambda/2 - \nu} - \frac{1}{2} r^2 (r^{-2} e^{-\lambda/2} v')'] W \right\} \end{aligned}$$

The five perturbation function H_0, K, V , and W are not all independent!

These functions must satisfy the following relationship as a consequence of Einstein's equation:

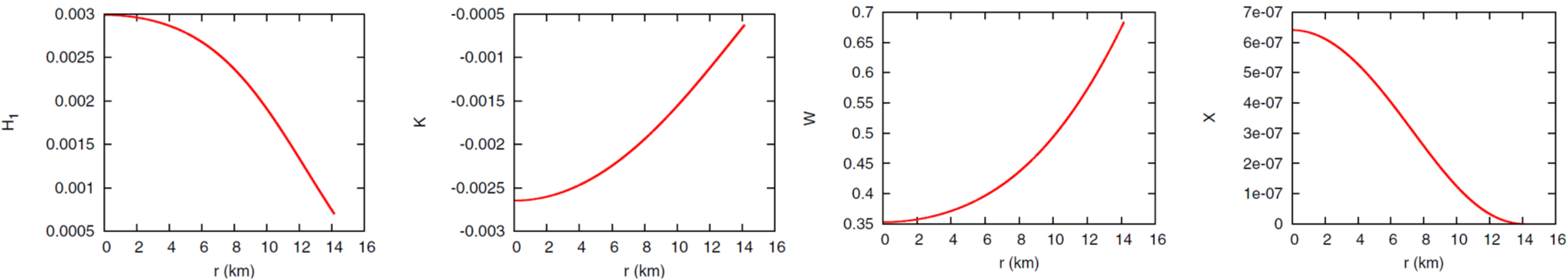
$$[3M + \frac{1}{2}(l+2)(l-1)r + 4\pi r^3 p]H_0 = 8\pi r^3 e^{-\nu/2}X - [\frac{1}{2}l(l+1)(M + 4\pi r^3 p) - \omega^2 r^3 e^{-(\lambda+\nu)}]H_1 \\ \text{algebraic relation} \quad + [\frac{1}{2}(l+2)(l-1)r - \omega^2 r^3 e^{-\nu} - r^{-1}e^\lambda(M + 4\pi r^3 p)(3M - r + 4\pi r^3 p)]K$$

In this equation the perturbation function X is defined by

$$X = \omega^2(\epsilon + p)e^{-\nu/2}V - \frac{p'}{r}e^{(\nu-\lambda)/2}W + \frac{1}{2}(\epsilon + p)e^{\nu/2}H_0$$

and V is to be thought of as the linear combination of H_1, K, W , and X obtained by eliminating H_0 :




$$V = \omega^{-2}(\rho + p)^{-1}e^\nu [e^{-\nu/2}X + r^{-1}p'e^{-\lambda/2}W - \frac{1}{2}(\rho + p)H_0]$$



Behavior of the perturbation functions of the ODE system inside the star.

[Chirenti, Silveira, Aguiar 2012, *IJMP: Conf. Ser.* **18** 48-52]

PAST AND PRESENT COLLABORATIONS

Period	Partner institution	Collaborators	Research topic and scientific activity
April 2018	 <p>EBERHARD KARLS UNIVERSITÄT TÜBINGEN</p>	Kostas Kokkotas	Linear adiabatic radial oscillations of neutron stars: <ul style="list-style-type: none"> • Study of "shooting" method for finding quasi-normal modes. • Comparison of preliminary numerical results.
Oct. 2021 – present	 <p>L'Observatoire de Paris LUTH Laboratoire de l'Univers et de ses Théories</p>	Philippe Grandclément Jérôme Novak Éricourgoulhon	For NS models (including fast-rotating or magnetized) and import EoS tables directly from CompOSE: <ul style="list-style-type: none"> • LORENE (set of C++ classes) to solve partial differential equations by means of multi-domain spectral methods. • KADATH library (a more generic spectral solver), designed to describe functions as a finite sum of orthogonal functions known as the basis functions.
Sept. 2021 – present	 <p>WIGNER</p>	György Wolf Gyula Fodor	Research project “ <i>Nuclear matter properties from heavy-ion collisions to compact stars</i> ”, supported by OTKA grant agreement No. K138277

Close collaboration with: Balázs Kacs Kovics and Mátyás Vasúth

Thank you very much for your kind
attention!

Dániel Barta

E-mail: barta.daniel@wigner.hu

