Instabilities in Classical Chromodynamics

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Overview

Motivation

Yang-Mills equations

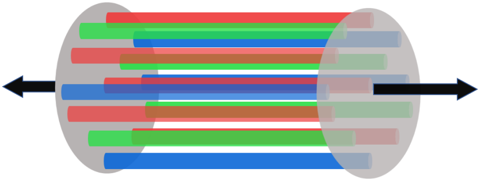
Stability of constant fields in classical chromodynamics: Abelian vs. non-Abelian configurations

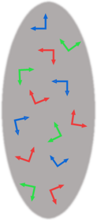
Summary

Based on https://arxiv.org/abs/2111.11396

Motivation

The earliest phase of heavy-ion collisions is described in terms of classical fields.





It was found, that early configurations are not stable, but the character of the instabilities is not clear (P. Romatschke and R. Venugopalan, Phys. Rev. Lett. 96, 062302 (2006)).

We want to find a difference between solutions in Abelian and non-Abelian configurations.

Yang-Mills equations & linearized QCD

Yang-Mills equations in adjoint representation

$$\partial_\mu F_a^{\mu\nu} + g f^{abc} A_\mu^b F_c^{\mu\nu} = J_a^\nu, \qquad F_a^{\mu\nu} = \partial^\mu A_a^\nu \quad \partial^\nu A_a^\mu + g f^{abc} A_b^\mu A_c^\nu$$

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Linearized QCD

$A_a^\mu(t, \mathbf{r}) = A_a^\mu(t, \mathbf{r}) + a_a^\mu(t, \mathbf{r})$, where $jA(t, \mathbf{r})j \quad ja(t, \mathbf{r})j$

Yang-Mills equations & linearized QCD

Yang-Mills equations in adjoint representation

$$\partial_\mu F^{\mu\nu}_a + g f^{abc} A^b_\mu F^{\mu\nu}_c = J^\nu_a, \qquad F^{\mu\nu}_a = \partial^\mu A^\nu_a \quad \partial^\nu A^\mu_a + g f^{abc} A^\mu_b A^\nu_c$$

Linearized QCD

$A^\mu_a(t,\mathbf{r}) = A^\mu_a(t,\mathbf{r}) + a^\mu_a(t,\mathbf{r})$, where $jA(t,\mathbf{r})j \quad ja(t,\mathbf{r})j$

Background gauge condition

$$D^\mu_{ab} a^a_\mu = \partial^\mu a^a_\mu + g f^{abc} A^\mu_b a^c_\mu = 0$$

Yang-Mills equations & linearized QCD

Yang-Mills equations in adjoint representation

$$\partial_\mu F^{\mu\nu}_a + g f^{abc} A^b_\mu F^{\mu\nu}_c = J^\nu_a, \qquad F^{\mu\nu}_a = \partial^\mu A^\nu_a \quad \partial^\nu A^\mu_a + g f^{abc} A^\mu_b A^\nu_c$$

Linearized QCD

$$A^\mu_a(t,\mathbf{r}) = A^\mu_a(t,\mathbf{r}) + a^\mu_a(t,\mathbf{r}), \qquad \text{where } jA(t,\mathbf{r})j \quad ja(t,\mathbf{r})j$$

Background gauge condition

$$D^\mu_{ab} a^a_\mu = \partial^\mu a^a_\mu + g f^{abc} A^\mu_b a^c_\mu = 0$$

Yang-Mills equations in the background gauge

$$\Big[g^{\mu\nu} \big(D_\rho D^\rho \big)_{ac} + 2 g f^{abc} F^{\mu\nu}_b \Big] a^c_\nu = J^\mu_a$$

Stability of Abelian chromomagnetic configuration

S. J. Chang and N. Weiss, Phys. rec. D **20**, 869 (1979), P. Sikivie, Phys. Rev. D **22**, 877 (1979)

Constant homogeneous chromomagnetic field

$A_a^\mu(t, \mathbf{r}) = (0, 0, 0, yB)\delta^{a1}$

Potential $A_a^\mu(t, \mathbf{r})$ satisfies YM equations with vanishing current

The color component a_1 satisfies Abelian equation and decouples from the remaining two components: $a_1^\nu = 0$

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$a_a^\mu(t, x, y, z) = e^{\;\; i(\omega t \;\; k_x x \;\; k_z z)} a_a^\mu(y)$

Mixing

colors 2 and 3 ! T $(y) = a_2^0(y) \quad ia_3^0(y)$,

coordinates y and z ! U $(y) = Y^+(y) \quad iZ^+(y)$

Stability of Abelian chromomagnetic configuration

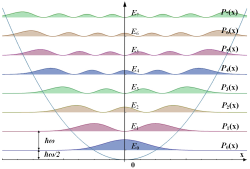
$$\Big(\quad \omega^2 + k_x^2 \quad \frac{d^2}{dy^2} + (k_z + gBy)^2 \quad 2gB \Big) W \quad = 0$$

Stability of Abelian chromomagnetic configuration

$$\Big(\quad \omega^2 + k_x^2 \quad \frac{d^2}{dy^2} + (k_z + gBy)^2 \quad 2gB\Big)W \quad = 0$$

Non-relativistic Schrödinger equation of harmonic oscillator

$$\Big(\quad 2mE + m^2\omega^2(y_0 \quad y)^2 \quad \frac{d^2}{dy^2}\Big)\varphi(y) = 0$$



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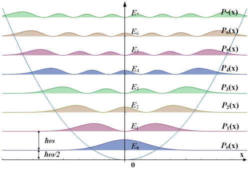
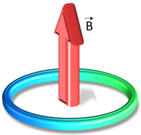


Unstable solution

$\omega^2 = k_x^2 \quad gB < 0$ for $gB > k_x^2 \quad ! \quad a \quad e^{\,p\overline{gB \quad k_x^2}\,t}$

Nielsen Olesen instability

The result is purely classical!



Stability of non-Abelian chromomagnetic configuration

T. N. Tudron, Phys. Rev. D **22**, 2566 (1980)

Constant homogeneous chromomagnetic field

$$A_a^\mu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{B/g} \\ 0 & 0 & \sqrt{B/g} & 0 \end{bmatrix}, \quad J_a^\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{gB^3} \\ 0 & 0 & \sqrt{gB^3} & 0 \end{bmatrix}.$$

Assumption

$a_a^\mu(t,\mathbf{x}) = a_a^\mu e^{\ i(\omega t \ \ \mathbf{kx})}$

Matrix equations

12x12 matrix in block form ! 2 equal matrices 3x3 and one 6x6

The equations are homogeneous, so the solution exists if matrix's determinant is equal to zero.

Stability of non-Abelian chromomagnetic configuration

$$\hat{M}_{B_t} = \hat{M}_{B_x} = \begin{bmatrix} \omega^2 + \mathbf{k}^2 + 2g^2A^2 & 2igAk_y & 2igAk_z \\ 2igAk_y & \omega^2 + \mathbf{k}^2 + g^2A^2 & 0 \\ 2igAk_z & 0 & \omega^2 + \mathbf{k}^2 + g^2A^2 \end{bmatrix}$$

$$\det \hat{M}_{B_t} = (\quad \omega^2 + \mathbf{k}^2 + g^2A^2)\Big(\omega^4 \quad \omega^2(2\mathbf{k}^2 + g^2A^2) + g^2A^2(3\mathbf{k}^2 \quad 2k_T^2) + \mathbf{k}^4 + 2g^4A^4\Big) = 0,$$

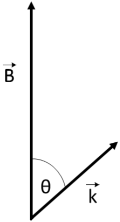
$$k_T = \sqrt{k_y^2 + k_z^2}, \qquad k_T = k \sin\theta$$

Solutions

$$\omega^2 = \mathbf{k}^2 + gB$$

$$\omega^2 = \mathbf{k}^2 + \tfrac{3}{2}gB \quad \tfrac{1}{2}\sqrt{16gBk_T^2 + g^2B^2}$$

The solutions are stable.



Stability of non-Abelian chromomagnetic configuration

$$\hat{M}_{B_{y,z}} = \begin{bmatrix} \omega^2+\mathbf{k}^2+2g^2A^2 & 2igAk_y & 2igAk_z & 0 & 0 & 0 \\ 2igAk_y & \omega^2+\mathbf{k}^2+g^2A^2 & 0 & 0 & 0 & 2g^2A^2 \\ 2igAk_z & 0 & \omega^2+\mathbf{k}^2+g^2A^2 & 0 & 2g^2A^2 & 0 \\ 0 & 0 & 0 & \omega^2+\mathbf{k}^2+2g^2A^2 & 2igAk_y & 2igAk_z \\ 0 & 0 & 2g^2A^2 & 2igAk_y & \omega^2+\mathbf{k}^2+g^2A^2 & 0 \\ 0 & 2g^2A^2 & 0 & 2igAk_z & 0 & \omega^2+\mathbf{k}^2+g^2A^2 \end{bmatrix}$$

$$\begin{aligned} \det \hat{M}_{B_{y,z}} &= \Big(\quad \omega^6 + (3\mathbf{k}^2 + 4gB)\omega^4 \quad (3\mathbf{k}^4 + 8gB\mathbf{k}^2 \quad 4gBk_T^2 + g^2B^2)\omega^2 \\ &+ \quad (\mathbf{k}^6 + 4gB\mathbf{k}^4 + g^2B^2\mathbf{k}^2 \quad 4gB\mathbf{k}^2k_T^2 \quad 4g^2B^2k_T^2 \quad 6g^3B^3)\Big)^2 \end{aligned}$$

Cubic equation

$$x^3 + a_2x^2 + a_1x + a_0 = 0,$$

where a_2, a_1, a_0 are real numbers.

Character of the equation's roots depends on discriminant's value.

$$= 18a_0a_1a_2 \quad 4a_2^3a_0 + a_1^2a_2^2 \quad 4a_1^3 \quad 27a_0^2$$

There are three possibilities:

if >0, the roots are real and distinct;

if $= 0$, the roots are real and at least two coincide;

if <0, one root is real and remaining two are complex.

Cubic equation

For $\quad > 0$ the solutions can be written in a Viète's trigonometric form:

$$x_n = 2\sqrt{\quad \frac{p}{3}} \cos\left[\frac{1}{3} \arccos\left(\frac{3q}{2p}\sqrt{\quad \frac{3}{p}}\right) \quad \frac{2\pi(n \quad 1)}{3}\right] \quad \frac{a_2}{3},$$

where $n = 1, 2, 3$ and $p \quad \frac{3a_1 \quad a_2^2}{3}$, $q \quad \frac{2a_2^3 \quad 9a_2a_1 + 27a_0}{27}$.

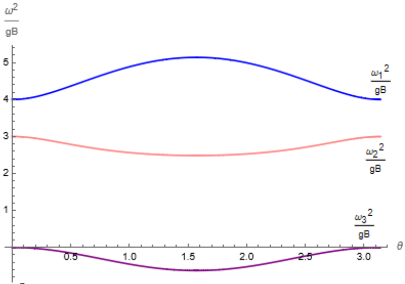
When $\quad < 0$ the solutions can be found according to Cardano formula.

$$x_1 = \quad \frac{1}{2}(u + v) + \frac{i\sqrt{3}}{2}(u \quad v) \quad \frac{1}{3}a_2,$$
$$x_2 = \quad \frac{1}{2}(u + v) \quad \frac{i\sqrt{3}}{2}(u \quad v) \quad \frac{1}{3}a_2,$$
$$x_3 = u + v \quad \frac{1}{3}a_2,$$

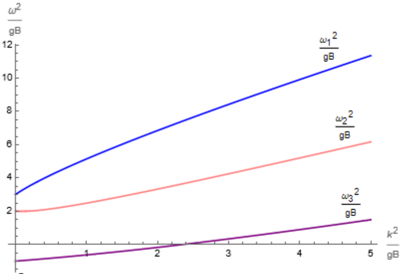
where

$$u \quad \sqrt[3]{\quad \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \qquad v \quad \sqrt[3]{\quad \frac{q}{2} \quad \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Stability of non-Abelian chromomagnetic configuration

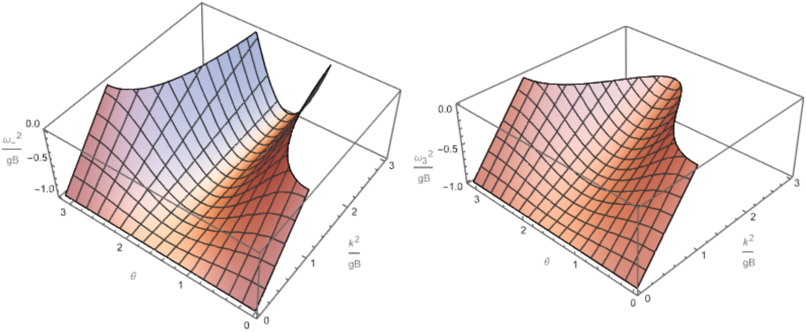


$\frac{\omega_n^2}{gB}$ as a function of θ for $\mathbf{k}^2 = gB$



$\frac{\omega_n^2}{gB}$ as a function of $\mathbf{k}^2$ for $\theta = \frac{\pi}{2}$

Instability: Abelian vs. non-Abelian chromomagnetic configuration



Domain of instability $\omega^2 < 0$ of Abelian (left) and non-Abelian (right) configurations

Stability of Abelian chromoelectric configuration

S. J. Chang and N. Weiss, Phys. rec. D **20**, 869 (1979), P. Sikivie, Phys. Rev. D **22**, 877 (1979)

Constant homogeneous chromoelectric field

$A^{\mu}_{a}(t, \mathbf{r}) = (\quad xE, 0, 0, 0)\delta^{a1}$
Potential $A^{\mu}_{a}(t, \mathbf{r})$ satisfies YM equations with vanishing current

The color component a_1 satisfies Abelian equation and decouples from the remaining two components: $a^{\nu}_{1} = 0$

$a^{\mu}_{a}(t, x, y, z) = e \quad ^{i(\omega t \quad k_y y \quad k_z z)} a^{\mu}_{a}(x)$

Mixing

colors 2 and 3 ! T $(x) = a^{0}_{2}(x) \quad i a^{0}_{3}(x)$,
coordinates t and x ! G $(x) = T^{+}(x) \quad X^{+}(x)$

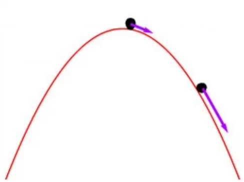
Stability of Abelian chromoelectric configuration

$$\left(\quad g^2 x^2 E^2 + k_y^2 + k_z^2 \quad \frac{d^2}{dx^2} \right) Y \quad (x) = 0$$

Stability of Abelian chromoelectric configuration

$$\Big(\quad g^2x^2E^2 + k_y^2 + k_z^2 \quad \frac{d^2}{dx^2}\Big)Y \quad (x) = 0$$

Coincidence with non-relativistic Schrödinger equation of inverted harmonic oscillator



Solutions

run-away solutions ! unstable

Stability of non-Abelian chromoelectric configuration

Constant homogeneous chromoelectric field

$$A_a^\mu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{E/g} & 0 & 0 & 0 \\ 0 & \sqrt{E/g} & 0 & 0 \end{bmatrix}, \qquad J_a^\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{gE^3} & 0 & 0 & 0 \\ 0 & \sqrt{gE^3} & 0 & 0 \end{bmatrix}.$$

Assumption

$a_a^\mu(t, \mathbf{x}) = a_a^\mu e^{\ i(\omega t \quad \mathbf{kx})}$

Matrix equations

12x12 matrix in block form ! 2 equal matrices 3x3 and one 6x6

The equations are homogeneous, so the solution exists if matrix's determinant is equal to zero.

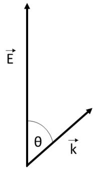
Stability of non-Abelian chromoelectric configuration

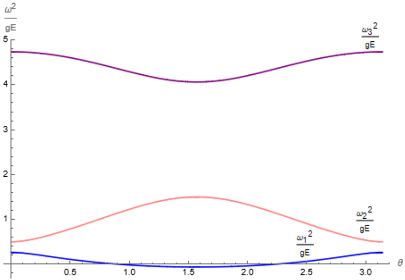
$$\hat{M}_{E_y} = \hat{M}_{E_z} = \begin{bmatrix} \omega^2 + \mathbf{k}^2 & 2igAk_x & 2igA\omega \\ 2igAk_x & \omega^2 + \mathbf{k}^2 + g^2A^2 & 0 \\ 2igA\omega & 0 & \omega^2 + \mathbf{k}^2 \quad g^2A^2 \end{bmatrix}$$

$$\det \hat{M}_{E_y} = \det \hat{M}_{E_z} = \quad \omega^6 + (4gE + 3k^2)\omega^4 \quad (3g^2E^2 + 4gB(k^2 \quad k_x^2) + 3k^4)\omega^2$$

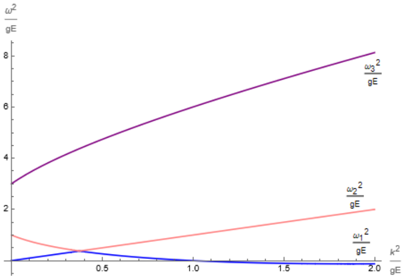
$$4gEk^2k_x^2 + 4g^2E^2k_x^2 \quad g^2B^2k^2 + k^6 = 0$$

Stability of non-Abelian chromoelectric configuration



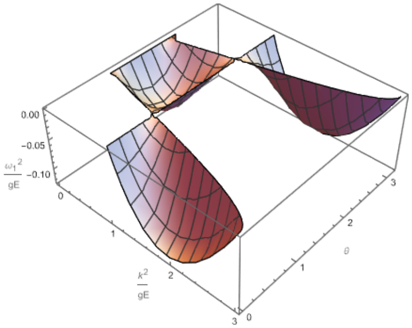


$\frac{\omega_n^2}{gE}$ as a function of θ for $\mathbf{k}^2 = \frac{gE}{2}$



$\frac{\omega_n^2}{gE}$ as a function of $\mathbf{k}^2$ for $\theta = 0$

Stability of non-Abelian chromoelectric configuration



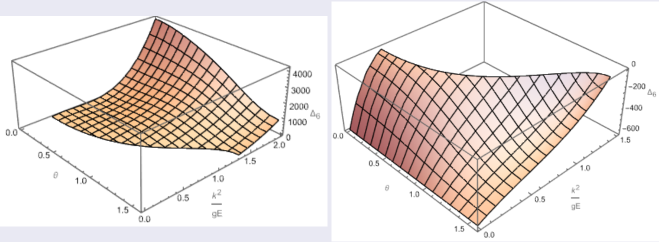
Domain of instability $\omega^2 < 0$

Stability of non-Abelian chromoelectric configuration

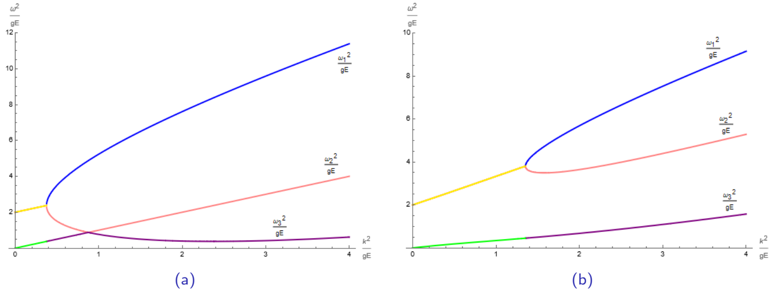
$$\hat{M}_{E_{t,x}} = \begin{bmatrix} \omega^2{+}\mathbf{k}^2 & 2igAk_x & 2igA\omega & 0 & 0 & 0 \\ 2igAk_x & \omega^2{+}\mathbf{k}^2{+}g^2A^2 & 0 & 0 & 0 & 2g^2A^2 \\ 2igA\omega & 0 & \omega^2{+}\mathbf{k}^2 \quad g^2A^2 & 0 & 2g^2A^2 & 0 \\ 0 & 0 & 0 & \omega^2{+}\mathbf{k}^2 & 2igAk_x & 2igA\omega \\ 0 & 0 & 2g^2A^2 & 2igAk_x & \omega^2{+}\mathbf{k}^2{+}g^2A^2 & 0 \\ 0 & 2g^2A^2 & 0 & 2igA\omega & 0 & \omega^2{+}\mathbf{k}^2 \quad g^2A^2 \end{bmatrix}$$

$$\det \hat{M}_{E_{t,x}} = \Big(\quad \omega^6 + (4gE + 3k^2)\omega^4 \quad (7g^2E^2 + 4gE(k^2 \quad k_x^2) + 3k^4)\omega^2 + 3g^2E^2 + 4g^2E^2k_x^2 \quad 4gEk^2k_x^2 + k^6\Big)^2 = 0$$

Discriminant

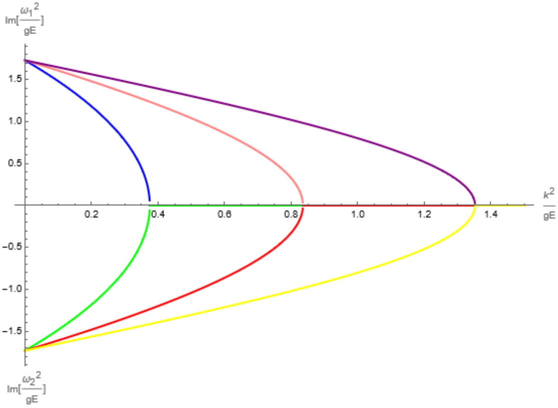


Stability of non-Abelian chromoelectric configuration



$\frac{\omega_n^2}{gE}$ as a function of $\frac{k^2}{gE}$ for θ = 0 (a) and $\theta = \frac{\pi}{2}$ (b)

Stability of non-Abelian chromoelectric configuration



Imaginary part of ω_1^2 and ω_2^2 as a function of k^2 for three values of θ

Summary

We found complete spectra of fluctuation eigenmodes for Abelian and non-Abelian constant and uniform chromoelectric and chromomagnetic fields.

The spectra of Abelian and non-Abelian fields configurations are rather different.

In each case there are unstable modes, which play crucial role of temporal evolution of the system.

Thank you for attention!