

# Instabilities in Classical Chromodynamics

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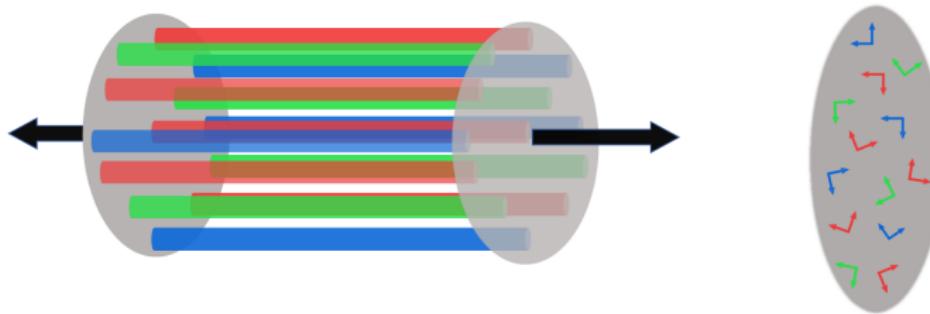
# Overview

- Motivation
- Yang-Mills equations
- Stability of constant fields in classical chromodynamics: Abelian vs. non-Abelian configurations
- Summary

Based on <https://arxiv.org/abs/2111.11396>

## Motivation

- The earliest phase of heavy-ion collisions is described in terms of classical fields.



- It was found, that early configurations are not stable, but the character of the instabilities is not clear (P. Romatschke and R. Venugopalan, Phys. Rev. Lett. 96, 062302 (2006)).
- We want to find a difference between solutions in Abelian and non-Abelian configurations.

# Yang-Mills equations & linearized QCD

## Yang-Mills equations in adjoint representation

$$\partial_\mu F_a^{\mu\nu} + g f^{abc} A_\mu^b F_c^{\mu\nu} = J_a^\nu, \quad F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f^{abc} A_b^\mu A_c^\nu$$

## Yang-Mills equations & linearized QCD

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### Linearized QCD

$$A_a^\mu(t, \mathbf{r}) = \bar{A}_a^\mu(t, \mathbf{r}) + a_a^\mu(t, \mathbf{r}), \quad \text{where } |\bar{A}(t, \mathbf{r})| \gg |a(t, \mathbf{r})|$$

## Yang-Mills equations & linearized QCD

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### Background gauge condition

$$\bar{D}_{ab}^\mu a_\mu^a = \partial^\mu a_\mu^a + g f^{abc} \bar{A}_b^\mu a_\mu^c = 0$$

## Yang-Mills equations & linearized QCD

### Yang-Mills equations in adjoint representation

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### Background gauge condition

$$\bar{D}_{ab}^\mu a_\mu^a = \partial^\mu a_\mu^a + g f^{abc} \bar{A}_b^\mu a_\mu^c = 0$$

### Yang-Mills equations in the background gauge

$$\left[ g^{\mu\nu} (\bar{D}_\rho \bar{D}^\rho)_{ac} + 2 g f^{abc} \bar{F}_b^{\mu\nu} \right] a_\nu^c = J_a^\mu$$

# Stability of Abelian chromomagnetic configuration

S. J. Chang and N. Weiss, Phys. rec. D 20, 869 (1979), P. Sikivie, Phys. Rev. D 22, 877 (1979)

## Constant homogeneous chromomagnetic field

$$\bar{A}_a^\mu(t, \mathbf{r}) = (0, 0, 0, yB)\delta^{a1}$$

Potential  $\bar{A}_a^\mu(t, \mathbf{r})$  satisfies YM equations with vanishing current

The color component  $a_1$  satisfies Abelian equation and decouples from the remaining two components:  $\square a_1^\nu = 0$

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$$a_a^\mu(t, x, y, z) = e^{-i(\omega t - k_x x - k_z z)} a_a^\mu(y)$$

## Mixing

- colors 2 and 3  $\rightarrow T^\pm(y) = a_2^0(y) \pm i a_3^0(y)$ ,
- coordinates  $y$  and  $z$   $\rightarrow U^\pm(y) = Y^+(y) \pm i Z^+(y)$

## Stability of Abelian chromomagnetic configuration

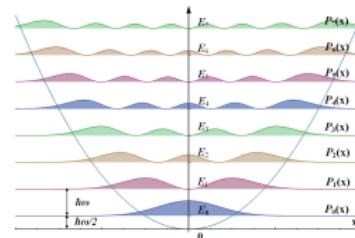
$$\left( -\omega^2 + k_x^2 - \frac{d^2}{dy^2} + (k_z + gBy)^2 \mp 2gB \right) W^\pm = 0$$

## Stability of Abelian chromomagnetic configuration

$$\left( -\omega^2 + k_x^2 - \frac{d^2}{dy^2} + (k_z + gBy)^2 \mp 2gB \right) W^\pm = 0$$

Non-relativistic Schrödinger equation of harmonic oscillator

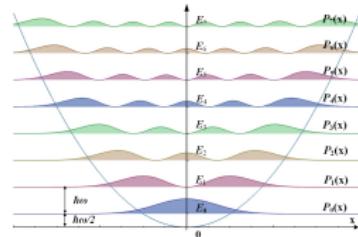
$$\left( -2m\mathcal{E} + m^2\bar{\omega}^2(y_0 - y)^2 - \frac{d^2}{dy^2} \right) \varphi(y) = 0$$



## Stability of Abelian chromomagnetic configuration

$$\left( -\omega^2 + k_x^2 - \frac{d^2}{dy^2} + (k_z + gBy)^2 \mp 2gB \right) W^\pm = 0$$

Non-relativistic Schrödinger equation of harmonic oscillator  
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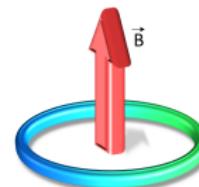


### Unstable solution

$$\omega_-^2 = k_x^2 - gB < 0 \text{ for } gB > k_x^2 \quad \rightarrow \quad a \sim e^{\sqrt{gB - k_x^2}t}$$

Nielsen Olesen instability

The result is purely classical!



# Stability of non-Abelian chromomagnetic configuration

T. N. Tudron, Phys. Rev. D 22, 2566 (1980)

## Constant homogeneous chromomagnetic field

$$\bar{A}_a^\mu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{B/g} \\ 0 & 0 & \sqrt{B/g} & 0 \end{bmatrix}, \quad J_a^\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{gB^3} \\ 0 & 0 & \sqrt{gB^3} & 0 \end{bmatrix}.$$

## Assumption

$$a_a^\mu(t, \mathbf{x}) = a_a^\mu e^{-i(\omega t - \mathbf{kx})}$$

## Matrix equations

12x12 matrix in block form → 2 equal matrices 3x3 and one 6x6

The equations are homogeneous, so the solution exists if matrix's determinant is equal to zero.

## Stability of non-Abelian chromomagnetic configuration

$$\hat{M}_{B_t} = \hat{M}_{B_x} = \begin{bmatrix} -\omega^2 + \mathbf{k}^2 + 2g^2 A^2 & -2igA k_y & 2igA k_z \\ 2igA k_y & -\omega^2 + \mathbf{k}^2 + g^2 A^2 & 0 \\ -2igA k_z & 0 & -\omega^2 + \mathbf{k}^2 + g^2 A^2 \end{bmatrix}$$

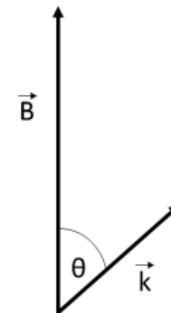
$$\det \hat{M}_{B_t} = (-\omega^2 + \mathbf{k}^2 + g^2 A^2) (\omega^4 - \omega^2 (2\mathbf{k}^2 + g^2 A^2) + g^2 A^2 (3\mathbf{k}^2 - 2k_T^2) + \mathbf{k}^4 + 2g^4 A^4) = 0,$$

$$k_T = \sqrt{k_y^2 + k_z^2}, \quad k_T = k \sin \theta$$

### Solutions

- $\omega^2 = \mathbf{k}^2 + gB$
- $\omega_{\pm}^2 = \mathbf{k}^2 + \frac{3}{2}gB \pm \frac{1}{2}\sqrt{16gBk_T^2 + g^2B^2}$

The solutions are stable.



## Stability of non-Abelian chromomagnetic configuration

$$\hat{M}_{B_{y,z}} = \begin{bmatrix} -\omega^2 + \mathbf{k}^2 + 2g^2 A^2 & -2igAk_y & 2igAk_z & 0 & 0 & 0 \\ 2igAk_y & -\omega^2 + \mathbf{k}^2 + g^2 A^2 & 0 & 0 & 0 & -2g^2 A^2 \\ -2igAk_z & 0 & -\omega^2 + \mathbf{k}^2 + g^2 A^2 & 0 & 2g^2 A^2 & 0 \\ 0 & 0 & 0 & -\omega^2 + \mathbf{k}^2 + 2g^2 A^2 & -2igAk_y & 2igAk_z \\ 0 & 0 & 2g^2 A^2 & 2igAk_y & -\omega^2 + \mathbf{k}^2 + g^2 A^2 & 0 \\ 0 & -2g^2 A^2 & 0 & -2igAk_z & 0 & -\omega^2 + \mathbf{k}^2 + g^2 A^2 \end{bmatrix}$$

$$\begin{aligned} \det \hat{M}_{B_{y,z}} = & \left( -\omega^6 + (3\mathbf{k}^2 + 4gB)\omega^4 - (3\mathbf{k}^4 + 8gB\mathbf{k}^2 - 4gBk_T^2 + g^2 B^2)\omega^2 \right. \\ & \left. + (k^6 + 4gBk^4 + g^2 B^2 k^2 - 4gBk^2 k_T^2 - 4g^2 B^2 k_T^2 - 6g^3 B^3) \right)^2 \end{aligned}$$

## Cubic equation

$$x^3 + a_2x^2 + a_1x + a_0 = 0,$$

where  $a_2, a_1, a_0$  are real numbers.

Character of the equation's roots depends on discriminant's value.

$$\Delta = 18a_0a_1a_2 - 4a_2^3a_0 + a_1^2a_2^2 - 4a_1^3 - 27a_0^2$$

There are three possibilities:

- if  $\Delta > 0$ , the roots are real and distinct;
- if  $\Delta = 0$ , the roots are real and at least two coincide;
- if  $\Delta < 0$ , one root is real and remaining two are complex.

## Cubic equation

For  $\Delta > 0$  the solutions can be written in a Viète's trigonometric form:

$$x_n = 2\sqrt{-\frac{p}{3}} \cos \left[ \frac{1}{3} \arccos \left( \frac{3q}{2p} \sqrt{\frac{-3}{p}} \right) - \frac{2\pi(n-1)}{3} \right] - \frac{a_2}{3},$$

$$\text{where } n = 1, 2, 3 \text{ and } p \equiv \frac{3a_1 - a_2^2}{3}, q \equiv \frac{2a_2^3 - 9a_2a_1 + 27a_0}{27}.$$

When  $\Delta < 0$  the solutions can be found according to Cardano formula.

$$x_1 = -\frac{1}{2}(u+v) + \frac{i\sqrt{3}}{2}(u-v) - \frac{1}{3}a_2,$$

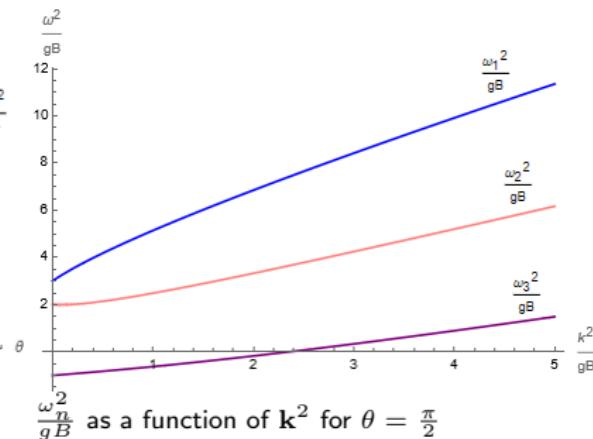
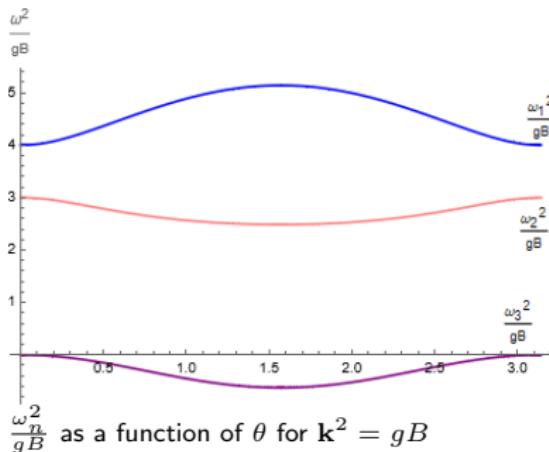
$$x_2 = -\frac{1}{2}(u+v) - \frac{i\sqrt{3}}{2}(u-v) - \frac{1}{3}a_2,$$

$$x_3 = u+v - \frac{1}{3}a_2,$$

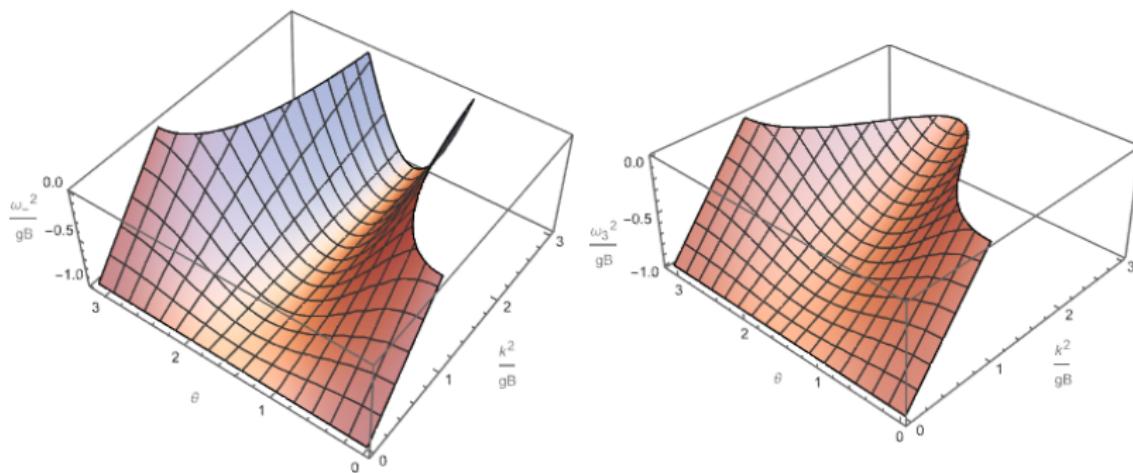
where

$$u \equiv \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \quad v \equiv \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

# Stability of non-Abelian chromomagnetic configuration



## Instability: Abelian vs. non-Abelian chromomagnetic configuration



Domain of instability  $\omega^2 < 0$  of Abelian (left) and non-Abelian (right) configurations

# Stability of Abelian chromoelectric configuration

S. J. Chang and N. Weiss, Phys. Rev. D 20, 869 (1979), P. Sikivie, Phys. Rev. D 22, 877 (1979)

## Constant homogeneous chromoelectric field

$$\bar{A}_a^\mu(t, \mathbf{r}) = (-xE, 0, 0, 0)\delta^{a1}$$

Potential  $\bar{A}_a^\mu(t, \mathbf{r})$  satisfies YM equations with vanishing current

The color component  $a_1$  satisfies Abelian equation and decouples from the remaining two components:  $\square a_1^\nu = 0$

$$a_a^\mu(t, x, y, z) = e^{-i(\omega t - k_y y - k_z z)} a_a^\mu(x)$$

## Mixing

- colors 2 and 3  $\rightarrow T^\pm(x) = a_2^0(x) \pm ia_3^0(x)$ ,
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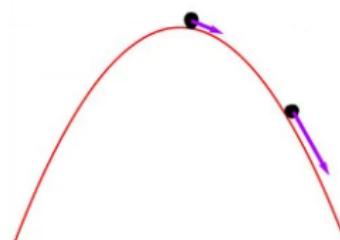
## Stability of Abelian chromoelectric configuration

$$\left( -g^2 x^2 E^2 + k_y^2 + k_z^2 - \frac{d^2}{dx^2} \right) Y^\pm(x) = 0$$

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Coincidence with non-relativistic Schrödinger equation  
of inverted harmonic oscillator



### Solutions

run-away solutions → unstable

## Stability of non-Abelian chromoelectric configuration

### Constant homogeneous chromoelectric field

$$\bar{A}_a^\mu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{E/g} & 0 & 0 & 0 \\ 0 & \sqrt{E/g} & 0 & 0 \end{bmatrix}, \quad J_a^\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{gE^3} & 0 & 0 & 0 \\ 0 & -\sqrt{gE^3} & 0 & 0 \end{bmatrix}.$$

### Assumption

$$a_a^\mu(t, \mathbf{x}) = a_a^\mu e^{-i(\omega t - \mathbf{kx})}$$

### Matrix equations

12x12 matrix in block form  $\longrightarrow$  2 equal matrices 3x3 and one 6x6

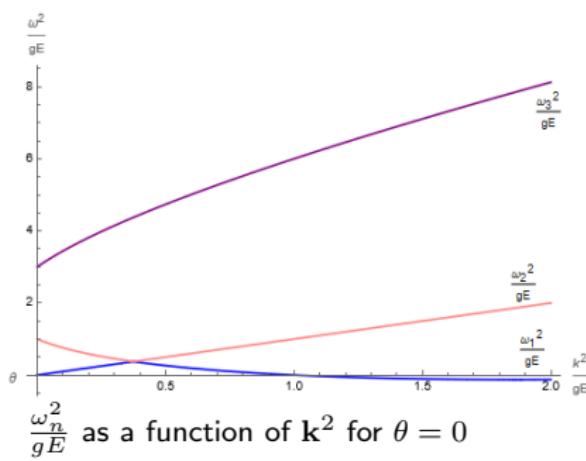
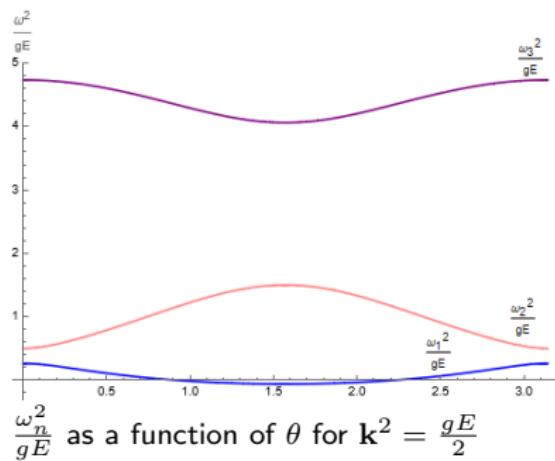
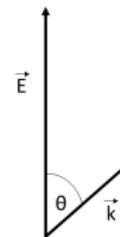
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## Stability of non-Abelian chromoelectric configuration

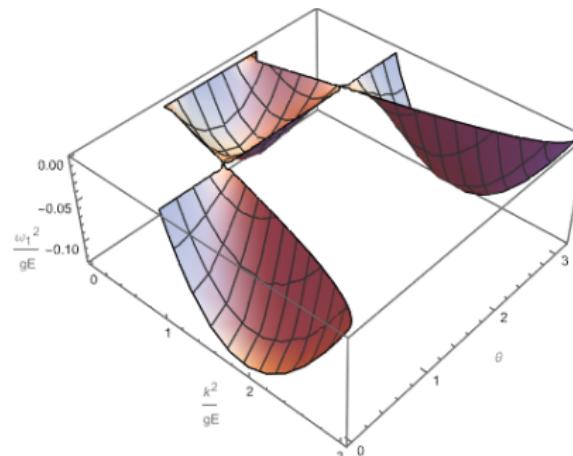
$$\hat{M}_{E_y} = \hat{M}_{E_z} = \begin{bmatrix} -\omega^2 + \mathbf{k}^2 & -2igAk_x & -2igA\omega \\ 2igAk_x & -\omega^2 + \mathbf{k}^2 + g^2A^2 & 0 \\ 2igA\omega & 0 & -\omega^2 + \mathbf{k}^2 - g^2A^2 \end{bmatrix}$$

$$\begin{aligned} \det \hat{M}_{E_y} = \det \hat{M}_{E_z} &= -\omega^6 + (4gE + 3k^2)\omega^4 - (3g^2E^2 + 4gB(k^2 - k_x^2) + 3k^4)\omega^2 \\ &\quad - 4gEk^2k_x^2 + 4g^2E^2k_x^2 - g^2B^2k^2 + k^6 = 0 \end{aligned}$$

## Stability of non-Abelian chromoelectric configuration



## Stability of non-Abelian chromoelectric configuration



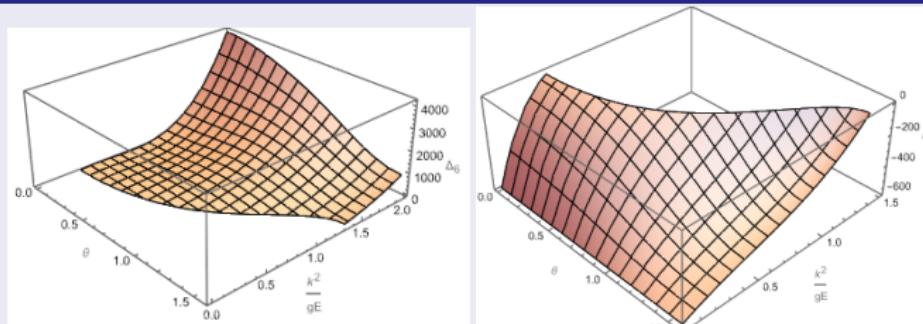
Domain of instability  $\omega^2 < 0$

## Stability of non-Abelian chromoelectric configuration

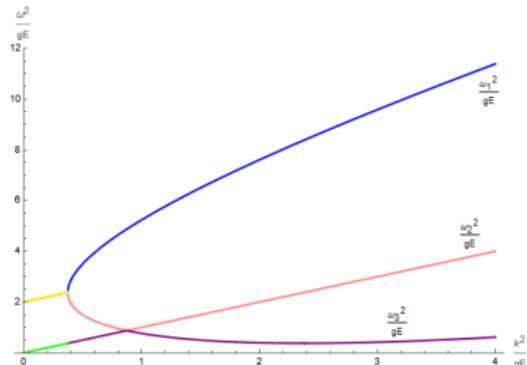
$$\hat{M}_{E_{t,x}} = \begin{bmatrix} -\omega^2 + \mathbf{k}^2 & -2igAk_x & -2igA\omega & 0 & 0 & 0 \\ 2igAk_x & -\omega^2 + \mathbf{k}^2 + g^2A^2 & 0 & 0 & 0 & -2g^2A^2 \\ 2igA\omega & 0 & -\omega^2 + \mathbf{k}^2 - g^2A^2 & 0 & 2g^2A^2 & 0 \\ 0 & 0 & 0 & -\omega^2 + \mathbf{k}^2 & -2igAk_x & -2igA\omega \\ 0 & 0 & -2g^2A^2 & 2igAk_x & -\omega^2 + \mathbf{k}^2 + g^2A^2 & 0 \\ 0 & 2g^2A^2 & 0 & 2igA\omega & 0 & -\omega^2 + \mathbf{k}^2 - g^2A^2 \end{bmatrix}$$

$$\det \hat{M}_{E_{t,x}} = (-\omega^6 + (4gE + 3k^2)\omega^4 - (7g^2E^2 + 4gE(k^2 - k_x^2) + 3k^4)\omega^2 + 3g^2E^2 + 4g^2E^2k_x^2 - 4gEk^2k_x^2 + k^6)^2 = 0$$

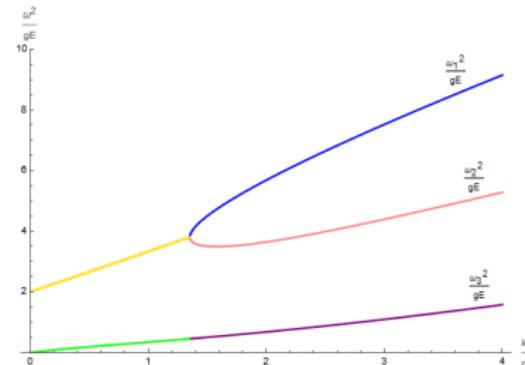
### Discriminant



# Stability of non-Abelian chromoelectric configuration



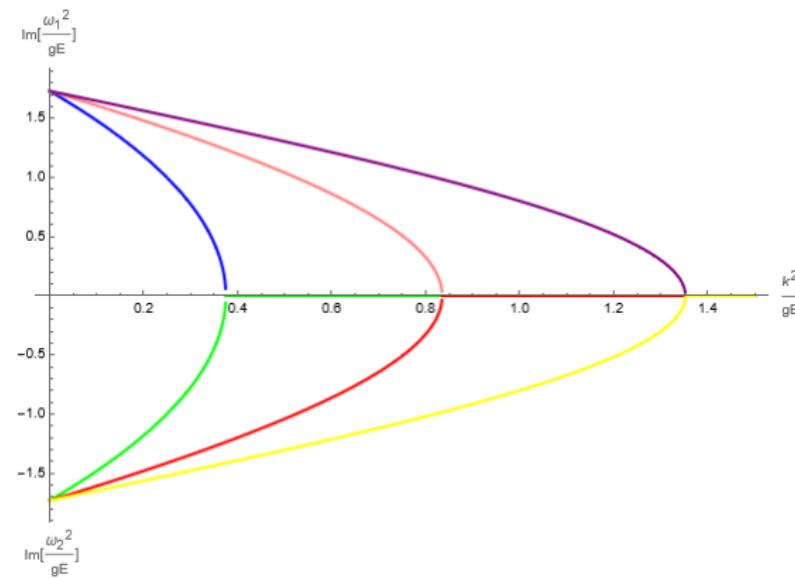
(a)



(b)

$\frac{\omega_n^2}{gE}$  as a function of  $\frac{k^2}{gE}$  for  $\theta = 0$  (a) and  $\theta = \frac{\pi}{2}$  (b)

## Stability of non-Abelian chromoelectric configuration



Imaginary part of  $\omega_1^2$  and  $\omega_2^2$  as a function of  $k^2$  for three values of  $\theta$

## Summary

- We found complete spectra of fluctuation eigenmodes for Abelian and non-Abelian constant and uniform chromoelectric and chromomagnetic fields.
- The spectra of Abelian and non-Abelian fields configurations are rather different.
- In each case there are unstable modes, which play crucial role of temporal evolution of the system.

*Thank you for attention!*