QCD thermodynamics at finite baryon density from lattice simulations - a mini review

Attila Pásztor

ELTE Eötvös Loránd University, Budapest

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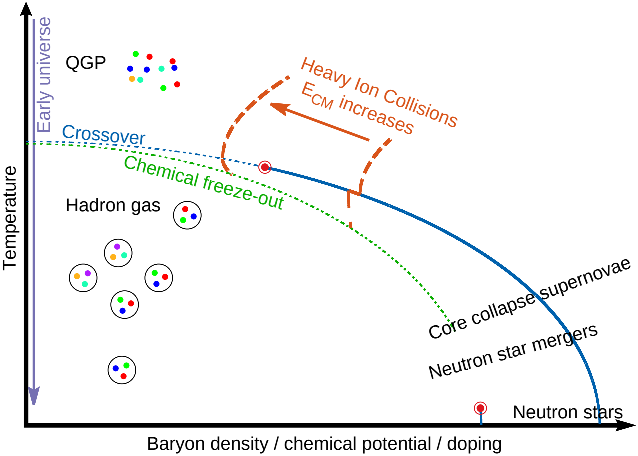
The conjectured phase diagram of QCD



Why is finite μ_B so difficult for the lattice?

Lattice QCD is a set of theoretical and computational techniques to perform the Euclidean path integral:

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\frac{1}{4}\int F_{\mu\nu}F_{\mu\nu} - \int \bar{\psi}(\gamma_\mu \partial_\mu + \gamma_0 \mu + m)\psi}$$

we integrate out the fermions analytically, to get

$$Z = \int \mathcal{D}A_\mu \det M(A_\mu, \mu, m)\psi e^{-\frac{1}{4}\int F_{\mu\nu}F_{\mu\nu}}$$

where M is (a discretized version of) the Dirac-operator. We can simulate this with Monte Carlo techniques if $\det M$ is **real and positive**:

- chemical potential $\mu = 0$
- purely imaginary chemical potentials: $\mathrm{Re}\,\mu = 0$
- isospin chemical potential: $\mu_u = -\mu_d$

Otherwise: **complex action problem**

$\rightarrow$ desperate times, desperate measures

How to avoid the complex action problem?

Simple: simulate a different theory

Imaginary μ method
Simulate at $\mu^2 \leq 0$ where the sign problem is absent to get $\langle O \rangle_{\mu^2 \leq 0}$, then extrapolate to $\mu^2 > 0$.

Taylor method
Simulate at $\mu = 0$ and calculate derivatives like:

$$\frac{\partial^n \langle O \rangle_\mu}{\partial \mu^n}\bigg|_{\mu=0} = \langle \dots \rangle_{\mu=0} + \langle \dots \rangle_{\mu=0} \langle \dots \rangle_{\mu=0} + \dots$$

Reweighting
Simulate an other theory with Boltzmann-weights $w_{\mathrm{simulated}}[U]$ and reconstruct expectation values in the target theory, with (maybe complex) path integral weights $w_{\mathrm{target}}[U]$

$$\langle O \rangle_{target} = \frac{\left\langle O \frac{w_{target}}{w_{simulated}} \right\rangle_{simulated}}{\left\langle \frac{w_{target}}{w_{simulated}} \right\rangle_{simulated}}$$

Common problems of approaches to finite μ

Analytic continuation problem

- Common to the imaginary μ and Taylor methods
- The data used to construct the analytic continuation is different

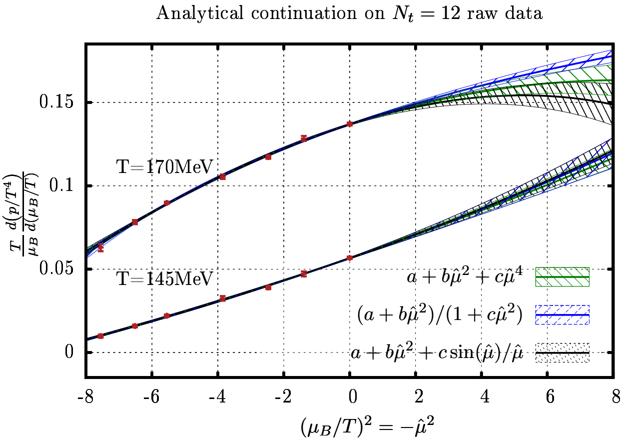
Sign problem

- The complex action problem turn to a sign problem if one uses reweighting: $w_{target}/w_{simulated}$ has fluctuating phases leading to large cancellations.
- The Taylor method also has a remnant: signal to noise ratio gets worse with higher derivatives.

Overlap problem

- Common to the reweighting and Taylor methods
- Insufficient sampling of the tails of the distribution of an observable: always appears for high order cummulants; for heavy tailed distirbutions can appear already for the average

The two uses of imaginary μ simulations



- Numerical differentiation at $\mu = 0$: safe
- Extrapolation: risky
- The hidden third use: understanding

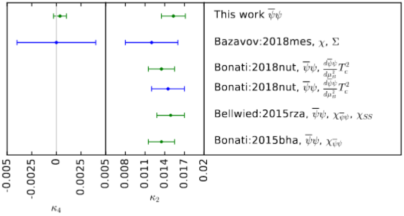
The crossover/transition line $T_c(\mu)$

chiral condensate: $\langle\bar\psi\psi\rangle = \frac{T}{V}\frac{\partial \ln Z}{\partial m_{ud}}$ ch. susceptibility: $\chi = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m_{ud}^2}$

renormalize: $\langle\bar\psi\psi\rangle_R = -\left[\langle\bar\psi\psi\rangle_T - \langle\bar\psi\psi\rangle_0\right]\frac{m_{ud}}{f_\pi^4}$ $\chi_R = \left[\chi_T - \chi_0\right]\frac{m_{ud}^2}{f_\pi^4}$

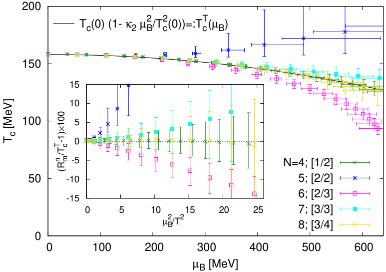
$$\frac{T_c(\mu_B)}{T_c(0)} = 1 - \kappa^2\left(\frac{\mu_B}{T_c(\mu_B)}\right)^2 - \kappa^4\left(\frac{\mu_B}{T_c(\mu_B)}\right)^4 - \ldots$$

Numerical derivative



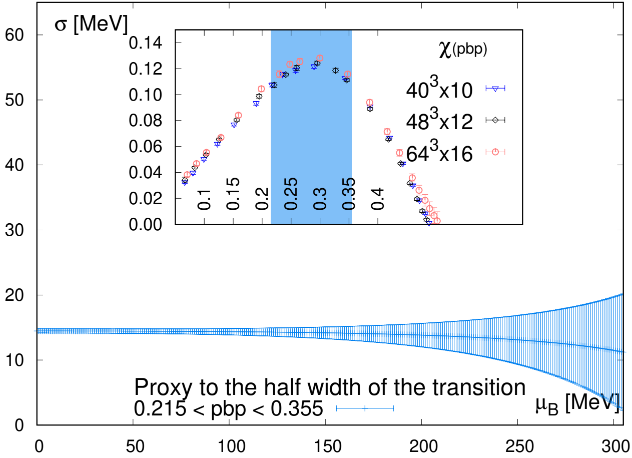
W-B: PRL 125 (2020) 5, 052001; 2002.02821

Extrapolation



Pasztor, Szep, Marko: PRD 103 (2021) 3, 034511; 2010.00394

Looking for criticality with analytic continuation



W-B: PRL 125 (2020) 5, 052001; 2002.02821 [hep-lat]

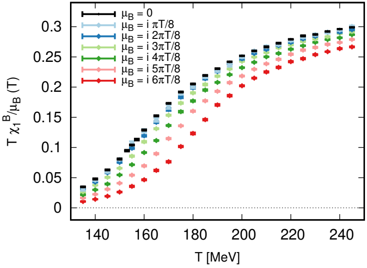
An observation from imaginary μ_B

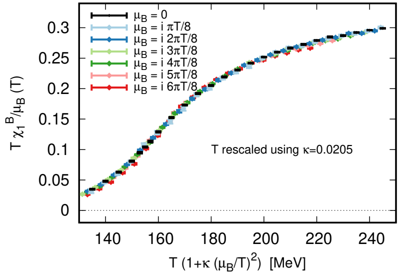
Baryon susceptibilities: $\chi_n^B(T, \hat{\mu}_B) = \frac{\partial^n (p/T^4)}{\partial \hat{\mu}_B^n}$, where $\hat{\mu}_B = \mu_B/T$ Baryon density: $\chi_1^B(T, \hat{\mu}_B) = \frac{\partial (p/T^4)}{\partial \hat{\mu}_B}$

Taylor: $\chi_1^B(T, \hat{\mu}_B) = \hat{\mu}_B \chi_2^B(T, 0) + \frac{\hat{\mu}_B^3}{6} \chi_4^B(T, 0) + \dots$

At imaginary μ_B we *observe* that $\chi_1^B(T, \hat{\mu}_B)$ is to good approx.:

$$\chi_1^B(T, \hat{\mu}_B) \approx \mu_B \, \chi_2^B \left(T \left(1 + \kappa \hat{\mu}_B^2\right), 0\right)$$





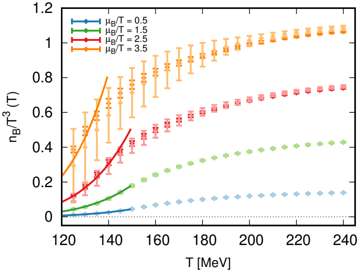
W-B: Phys. Rev. Lett. 126, 232001 (2021); 2102.06660

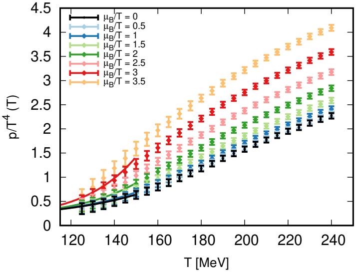
The equation of state at finite (real) μ_B

- The previous observation can be turned into a systematically improvable ansatz:

$$\frac{\chi_1^B(T,\mu_B)}{\mu_B} = \chi_2^B(T',0) \quad T' = T(1 - \kappa_2(T)\hat{\mu}^2 - \kappa_4(T)\hat{\mu}^4 + \dots)$$

- Quite similar to the extrapolation of $T_c(\mu_B)$
- Unlike with the equation of state from the $\mathcal{O}(\mu_B^6)$ Taylor expansions, no pathological (non-monotonic) behavior is present for $\mu_B \leq 2T$





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Reweighting: in general

Target theory: Z_t Simulated theory: Z_s

$$Z_t = \int \mathcal{D}U \; w_t(U) \qquad w_t(U) = detM[U,\mu)e^{-S_g[U]} \in \mathbb{C}$$

$$Z_s = \int \mathcal{D}U \; w_s(U) \qquad w_s(U) > 0$$

$$\frac{Z_t}{Z_s} = \left\langle \frac{w_t}{w_s} \right\rangle_r$$

$$\langle O \rangle_t = \frac{\int \mathcal{D}U \; w_t(U) O(U)}{\int \mathcal{D}U \; w_t(U)} = \frac{\int \mathcal{D}U \; w_s(U) \frac{w_t(U)}{w_s(U)} O(U)}{\int \mathcal{D}U \; w_s(U) \frac{w_t(U)}{w_s(U)}} = \frac{\left\langle \frac{w_t}{w_s} O \right\rangle_s}{\left\langle \frac{w_t}{w_s} \right\rangle_s}$$

Two problems that are exponentially hard in the volume:

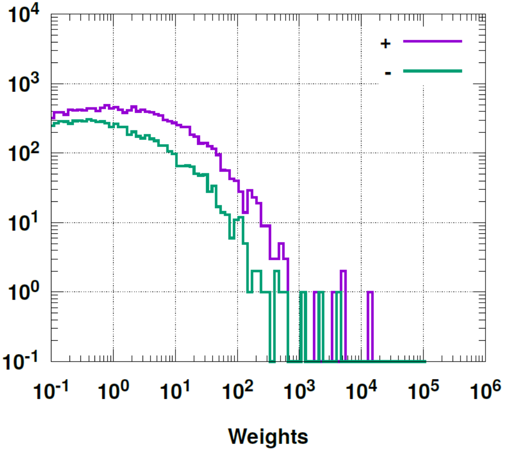
- $\frac{w}{r} \in \mathbb{C} \to$ the complex action problem became a **sign problem**
- Tails of $\rho(\frac{w}{r})$ long $\to$ **overlap problem**

Why does reweighting from $\mu = 0$ fail?

The expectation value of any observable:

$$\langle O \rangle_t = \frac{\left\langle \frac{w_t}{w_s} O \right\rangle_s}{\left\langle \frac{w_t}{w_s} \right\rangle_s}$$

The weights are the $\frac{w_t}{w_s} \propto \frac{\det M(\mu)}{\det M(0)}$. To calculate anything, we need to have control over the observable



The **sign problem is under control, the overlap problem is not:** Giordano, Kapas, Katz, Nogradi, Pasztor; PRD 102, 034503 (2020)

Phase and sign reweighting

A simple way to avoid long tails for the distribution of $\frac{w}{r}$ is to make sure that w/r take values from a compact space.

Phase reweighting

$$\begin{aligned} w_t &= e^{-S_g} \det M = e^{-S_g} |\det M| e^{i\theta} \\ w_s &= e^{-S_g} |\det M| \end{aligned} \quad \Rightarrow \quad \frac{w}{r} = e^{i\theta}$$

Severity of the sign problem: $\left\langle e^{i\theta} \right\rangle_{PQ}$

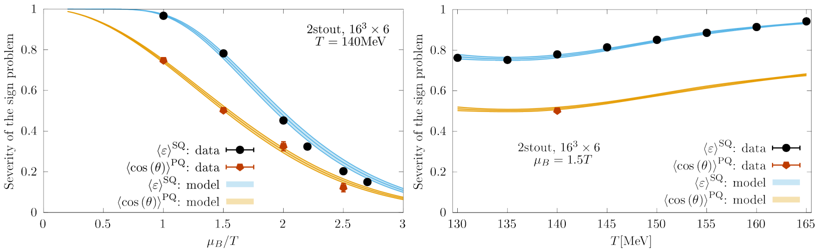
Sign reweighting
A new choice of a theory to reweight to and from:

$$\begin{aligned} w_t &= e^{-S_g} \mathrm{Re} \det \mathrm{M} \\ w_s &= e^{-S_g} |\mathrm{Re} \det \mathrm{M}| \end{aligned} \quad \Rightarrow \quad \frac{w_t}{w_s} = \operatorname{sgn} \cos\theta = \pm 1$$

$\det M \to \mathrm{Re} \det M$ can be done in Z but not in generic expectation values. E.g. things like $\frac{\partial^n \log Z}{\partial \mu_{ud}^n}$, $\frac{\partial^n \log Z}{\partial m_{ud}^n}$ and $\frac{\partial^n \log Z}{\partial \beta^n}$ can be calculated.

Severity of the sign problem: $\langle \pm \rangle_{SQ}$

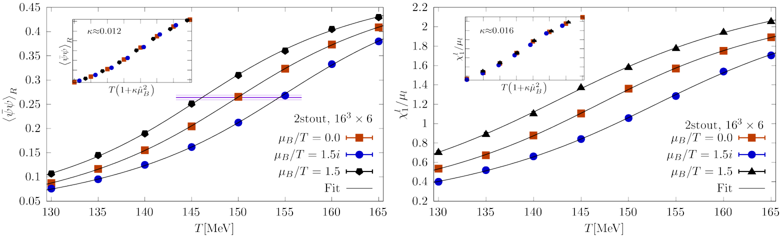
The severity of the sign problem



W-B: 2108.09213 [hep-lat]

- Statistics required $\propto 1/(\text{strength of the sign problem})^2$
- Gaussian model describes simulation data pretty well
- Const. strength of the sign problem for const. $(LT)^3 \left(\frac{\mu_B}{T}\right)^2$ (roughly)
- For $LT = 16/6 \approx 2.7$ ($T = 140\text{MeV} \rightarrow L \approx 4\text{fm}$) the sign problem is managable for the entire RHIC BES range

Temperature scan with sign reweighting



2108.09213 [hep-lat]

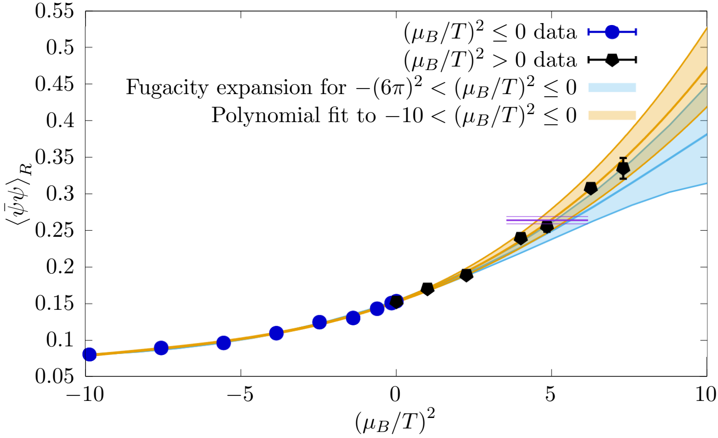
Similar rescalings in the imaginary μ_B direction:

W-B: PRL 126 (2021) 23, 232001;

W-B: PRL 125 (2020) 5, 052001;

Also works at real $\mu_B \rightarrow$ no sign of a strengthening crossover

Chemical potential scan with sign reweighting



2108.09213 [hep-lat]; $T = 140$MeV and $0 \leq \mu_B \leq 380$MeV The direct method penetrates the region where errors from analytic continuation blow up!

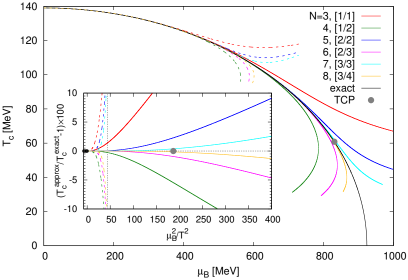
Summary

- The strength of the crossover is approx. const. at small μ
- Many observables (χ_1^B/μ_B, $\langle\psi\rangle_R$, χ_1^S/μ_B) are collapsing sigmoids, when plotted as a fn. of $T(1+\kappa\hat{\mu}^2)$, but κ is different for different observables
- First noted in imaginary μ simulations but also confirmed by recent restuls at real μ: a genuine feature of QCD at small μ_B
- The extrapolation of the crossover line and the equation of state can be pushed farther in μ_B than the extrapolation of measures of the strength of the transition
- The constant strength of the crossover makes it hard to look for criticality with extrapolation methods
- More direct methods are becoming increasingly feasible
- So far give results consistent with extrapolation
- BUT: still lots of room for improvement

Example without noise: a chiral effective model

Chiral limit of the $N_f = 2$ constituent quark-meson model in a leading order large-N expansion. See: Jakovác et al., PLB **582**, 179 (2004).

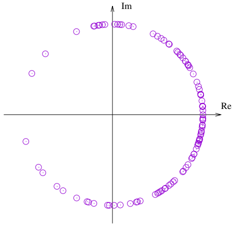
- The model exhibits a **line of second order** phase transitions for $\mu^2 > 0$, which ends in a **tricritical point**.
- Both the transition line and the location of the tricritical point can be determined **analytically**.

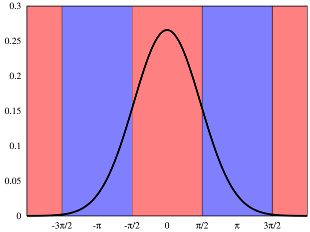


- Alternating convergence of the Padé approximants beyond the radius of convergence of the Taylor series.
- The tricritical point is **not** a special point of the transition line.

How to estimate the severity of the sign problem

Simplification: Assume the phase of $\det M$ to be (a wrapped) Gaussian





Phase Quenched: $\langle \cos\theta \rangle_{PQ} = e^{-\sigma^2(\mu)/2} \approx 1 - \frac{\sigma^2(\mu)}{2}$

Sign Quenched: $\langle \pm \rangle_{SQ} = \frac{\langle \cos\theta \rangle_{PQ}}{\langle |\cos\theta| \rangle_{PQ}} \approx 1 - \frac{4}{\pi} \left(\frac{2\sigma^2(\mu)}{\pi} \right)^{\frac{3}{2}} e^{-\frac{\pi^2}{8\sigma^2(\mu)}}$

Small μ: $\sigma^2(\mu) = -\frac{4}{9}\chi_{11}^{ud}(T)(LT)^3\hat{\mu}_B^2$

Large μ: sign quenched needs a factor of $(\pi/2)^2 \approx 2.5$ less statistics